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Local Properties of Fourier Series via Deferred Riesz Mean

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ABSTRACT: The convergence of Fourier series of a function at a point depends upon the behaviour of the function in the neighborhood of that point, and it leads to the local property of Fourier series. In the proposed work, we introduce and study the absolute convergence of the deferred Riesz summability mean, and accordingly establish a new theorem on the local property of a factored Fourier series. We also suggest a direction for future researches on this subject, which are based upon the local properties of the Fourier series via basic notions of statistical absolute convergence.

Key Words: Fourier series, deferred Riesz-summability, local property

Contents

1. Introduction and Motivation

Suppose $\sum a_n$ be a given infinite series with sequence of partial sum (s_n). Let (a_n) and (b_n) be the be sequences of non-negative integers such that $a_n < b_n$ and $b_n \to \infty$ as $n \to \infty$, and let (p_n) be the sequence of positive numbers such that

$$P_n = \sum_{i=a_n+1}^{b_n} p_i \qquad (P_{-i} = p_{-i} = 0; i \geqq 1).$$

Now we define the sequence-to-sequence transformation

$$\Omega_n = \sum_{i=a_n+1}^{b_n} p_i s_i,$$

represents the sequence (Ω_n) of the deferred Riesz $(D\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is *deferred Riesz summable* $|D\bar{N}, p_n|_k \quad (k \geqq 1)$ if,

$$\sum_{i=a_n+1}^{b_n} \left(\frac{P_i}{p_i}\right)^{k-1} |\Omega_i(s) - \Omega_{i-1}(s)|^k < \infty. \tag{1.1}$$

Here, we discuss some spacial cases as follows:

(i) If $a_n = 0$ and $b_n = n$ for all values of n, $|D\bar{N}, p_n|_k \quad (k \geqq 1)$-summability reduces to the $|\bar{N}, p_n|_k \quad (k \geqq 1)$-summability;

(ii) If $a_n = 0$, $b_n = n$ and $p_n = 1$ for all values of n, $|D\bar{N}, p_n|_k \quad (k \geqq 1)$-summability reduces to the $|C, 1|_k \quad (k \geqq 1)$-summability;

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(iii) If $p_n = 1$ for all values of n, $|D\bar{N}, p_n|_k$ $(k \geqq 1)$-summability reduces to the deferred Cesàro $|DC, 1|_k$ $(k \geqq 1)$-summability.

We use the following notations throughout the work

$$\Delta c_n = c_n - c_{n+1}$$

and

$$\bar{\Delta} c_{n,v} = c_{nv} - c_{n-1,v}, \quad c_{-1,0} = 0, \quad (n, v = 0, 1, 2, \cdots).$$

A sequence (λ_n) is called a *convex sequence* if,

$$\Delta^2(\lambda_n) \geq 0 \quad (\text{for every } n \in Z_+),$$

where

$$\Delta^2(\lambda_n) = \Delta(\lambda_n) - \Delta(\lambda_{n+1}) \quad \text{and} \quad \Delta(\lambda_n) = \lambda_n - \lambda_{n+1}.$$

Let $f(t) \in L(-\pi, \pi)$ be a 2π-periodic function. Without loss of generality let us consider that $a_0 = 0$ in the Fourier series expansion of $f(t)$ that is,

$$\int_{-\pi}^{\pi} f(t)dt = 0. \tag{1.2}$$

Thus the Fourier series expansion of $f(t)$ becomes

$$f(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t). \tag{1.3}$$

We now recall that the convergence of the Fourier series at $t = x$ is a local property [18] of f, and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f.

In the year 1950, Mohanty [15] has established that $|R, \log(n), 1|$-summability of a factored Fourier series

$$\sum \frac{A_n}{\log(n+1)} \tag{1.4}$$

of a function $f(t)$ at any point $t = x$ is a local property of the generating function $f(t)$; however, the summability $|C, 1|$ of this series does not hold. Subsequently, replacing the series (1.4) by

$$\sum \frac{A_n(t)}{(\log\log(n+1))^{\delta}} \quad (\delta > 1), \tag{1.5}$$

Matsumoto [12] obtained a new result on local property of $|R, p_n, 1|$-summability. Generalizing the above result Bhatt [1] proved the following theorem.

Theorem 1.1. *Let (λ_n) be a convex sequence such that $\sum \frac{\lambda_n}{n}$ is convergent, then $|R, \log(n), 1|$-summability of a factored Fourier series $\sum A_n(t)\lambda_n \log(n)$ at any point $t = x$ is a local property of $f(t)$.*

By replacing the factor $\lambda_n \log(n)$ in a most general form, Mishra [14] proved the following theorem.

Theorem 1.2. *Suppose (p_n) be a sequence satisfying following conditions.*

$$P_n = O(np_n),$$
$$P_n \Delta p_n = O(p_n p_{n+1}).$$

Then the $|\bar{N}, p_n|$-summability of a factored Fourier series.

$$\sum_{n=1}^{\infty} A_n(t)\lambda_n P_n(np_n)^{-1} \tag{1.6}$$

at any point $t = x$ is a local property of $f(t)$, where (λ_n) is a convex sequence.

Replacing $|\bar{N}, p_n|$-summability in Mishra's result, Bor [4] proved a more general form on $|\bar{N}, p_n|_k$-summability method. Quite recently, Bor [5] introduced a result on $|\bar{N}, p_n|_k$-summability of a factored Fourier series at any point $t = x$ as a local property of $f(t)$ under more appropriate conditions.

Motivated essentially by the above-mentioned investigations and studies, we introduce and study absolute convergence of the deferred Riesz summability mean, and accordingly establish a new theorem on local property of factored Fourier series. We also suggest a direction for future researches on this subject, which are based upon the local properties of Fourier series via basic notions of statistical absolute convergence. For some recent research works in this direction, see [2], [6], [7], [8], [9], [10], [11] and [16].

2. Main Result

Theorem 2.1. *Let ϕ_n and φ_n be the sequences of non-negative integers, and let (λ_n) be a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the summability $|D\bar{N}, p_n|_k \quad (k \geqq 1)$ of the series $\sum A_n(t)P_n\lambda_n$ at a point can be ensured by a local property.*

We need the following lemmas for the proof of our Theorem 2.1.

Lemma 2.2. *(see [13]) If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then (λ_n) is a nonnegative monotonic decreasing sequence tending to zero, $P_n\lambda_n = o(1)$ as $n \to \infty$ and $\sum P_n\Delta\lambda_n < \infty$.*

Lemma 2.3. *If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the series $\sum a_n\lambda_nP_n$ is summable $|D\bar{N}, p_n|_k$ $(k \geqq 1)$.*

Proof. Let (T_n) be the sequence of $(D\bar{N}, p_n)$ mean of the series $\sum a_n\lambda_nP_n$. Then by definition, we have

$$\begin{aligned} T_n &= \frac{1}{P_n}\sum_{i=\phi_n+1}^{\varphi_n} p_i \sum_{j=\phi_n+1}^{i} a_j\lambda_jP_j \\ &= \frac{1}{P_n}\sum_{i=\phi_n+1}^{\varphi_n} (P_i - P_{i-1})a_i\lambda_iP_i. \end{aligned}$$

Now,

$$T_n - T_{n-1} = \frac{p_n}{P_nP_{n-1}}\sum_{i=\phi_n+1}^{\varphi_n} P_{i-1}P_ia_i\lambda_i.$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{i=\phi_n+1}^{\varphi_n-1} P_iP_is_i\Delta\lambda_i - \frac{p_n}{P_nP_{n-1}}\sum_{i=\phi_n+1}^{\varphi_n-1} P_is_ip_i\lambda_i \\ &\quad - \frac{p_n}{P_nP_{n-1}}\sum_{i=\phi_n+1}^{\varphi_n-1} P_iP_{i+1}s_i\lambda_{i+1} + s_np_n\lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \quad \text{(say)}. \end{aligned}$$

In view of the Minkowski's inequality for competition of the proof of Lemma 2.2, it is sufficient to show that

$$\sum_{v=\phi_n+1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{v,r}|^k < \infty \quad (r = 1, 2, 3, 4). \tag{2.1}$$

Now, applying Hölder inequality we have

$$
\begin{aligned}
Q_1 &= \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,1}|^k \\
&\leqq \sum_{v=\phi_n+2}^{\varphi_m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{i=\phi_n+1}^{\varphi_n-1} P_i P_i |s_i|^k \Delta\lambda_i \right\} \cdot \left\{ \frac{1}{P_{n-1}} \sum_{i=\phi_n+1}^{\varphi_n-1} P_i P_i \Delta\lambda_i \right\}^{k-1}.
\end{aligned}
$$

Since,

$$
\sum_{i=\phi_n+1}^{\varphi_n-1} P_i P_i \Delta\lambda_i \leqq P_{\varphi_n-1} \sum_{i=\phi_n+1}^{\varphi_n-1} P_i \Delta\lambda_i,
$$

it follows by Lemma 2.2, that

$$
\frac{1}{P_{\varphi_n-1}} \sum_{i=\phi_n+1}^{\varphi_n-1} P_i P_i \Delta\lambda_i \leqq \sum_{i=\phi_n+1}^{\varphi_n-1} P_i \Delta\lambda_i = O(1).
$$

Thus,

$$
\begin{aligned}
Q_1 &= \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,1}|^k \\
&\leqq O(1) \sum_{v=\phi_n+2}^{\varphi_m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{i=\phi_n+1}^{\varphi_n-1} P_i P_i |s_i|^k \Delta\lambda_i \right\} \\
&\leqq O(1) \sum_{i=\phi_n+2}^{\varphi_m} P_i P_i |s_i|^k \Delta\lambda_i \left\{ \sum_{v=i+1}^{\varphi_n+1} \frac{p_n}{P_n P_{n-1}} \right\} \\
&\leqq O(1) \sum_{i=\phi_n+2}^{\varphi_m} P_i \Delta\lambda_i = O(1),
\end{aligned}
$$

by virtue of the hypotheses and Lemma 2.2.

Again,

$$\begin{aligned}
Q_2 &= \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,2}|^k \\
&\leqq \sum_{v=\phi_n+2}^{\varphi_m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{i=\phi_n+1}^{\varphi_n-1} |s_i|^k (P_i\lambda_i)^k p_i \right\} \cdot \left\{ \frac{1}{P_{n-1}} \sum_{i=\phi_n+1}^{\varphi_n-1} p_i \right\}^{k-1} \\
&= O(1) \sum_{v=\phi_n+2}^{\varphi_m+1} \frac{p_n}{P_n P_{n-1}} \sum_{i=\phi_n+1}^{\varphi_n-1} |s_i|^k (P_i\lambda_i)^k p_i \\
&= O(1) \sum_{i=\phi_n+2}^{\varphi_m} |s_i|^k (P_i\lambda_i)^k p_i \sum_{v=i+1}^{\varphi_n+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{i=\phi_n+2}^{\varphi_m} |s_i|^k (P_i\lambda_i)^k \frac{p_i}{P_i} \\
&= O(1) \sum_{i=\phi_n+2}^{\varphi_m} |s_i|^k (P_i\lambda_i)^{k-1} p_i\lambda_i \\
&= O(1) \sum_{i=\phi_n+2}^{\varphi_m} p_i\lambda_i = O(1),
\end{aligned}$$

by virtue of the hypotheses and Lemma 2.2.

Using the fact $P_i < P_{i+1}$, and in the similar lines of Q_1 and Q_2, we have

$$Q_3 = \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,3}|^k = O(1) \sum_{i=\phi_n+2}^{\varphi_m} p_{i+1}\lambda_{i+1} = O(1),$$

and

$$\begin{aligned}
Q_4 &= \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,4}|^k = O(1) \sum_{i=\phi_n+2}^{\varphi_m} |s_i|^k (P_i\lambda_i)^{k-1} p_i\lambda_i \\
&= O(1) \sum_{i=\phi_n+2}^{\varphi_m} p_i\lambda_i = O(1),
\end{aligned}$$

by virtue of the hypotheses and Lemma 2.2.

Therefore, we get that

$$Q_1 + Q_2 + Q_3 + Q_4 = \sum_{v=\phi_n+2}^{\varphi_m+1} \left(\frac{P_v}{p_v}\right)^{k-1} |T_{n,r}|^k = O(1) \quad (r = 1, 2, 3, 4).$$

If we take $\phi_n = 0$ and $\varphi_n = n$, for all values of n in Lemma 2.3, then we get the following corollary.

Corollary 2.4. *If (λ_n) is a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the series $\sum a_n\lambda_n P_n$ is summable $|\bar{N}, p_n|_k$ $(k \geqq 1)$.*

Proof of the Theorem 2.1. Since the convergence of the Fourier series at a point is a local property of its generating function $f(t)$, the theorem follows from the above Lemma 2.3 (see, details in Chapter II of the book, Zygmund [19]).

3. Remarkable Conclusion

In this concluding section of our investigation, we further observe some special cases in view of the of the main Theorem 2.1.

Remark 3.1. *Let (ϕ_n) and (φ_n) be the sequences of non-negative integers such that $(\phi_n) = 0$ and $(\varphi_n) = n$ for all values of n, and let (λ_n) be a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the deferred Cesàro $|DC, p_n|_k$-summability of the series $\sum A_n(t)P_n\lambda_n$ at a point can be ensured by a local property.*

Remark 3.2. *Let (ϕ_n) and (φ_n) be the sequences of non-negative integers and let (p_n) be a sequence of positive numbers. Suppose (λ_n) be a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the Riesz $|\bar{N}, p_n|_k$-summability of the series $\sum A_n(t)P_n\lambda_n$ at a point can be ensured by a local property.*

Remark 3.3. *Let (ϕ_n) and (φ_n) be the sequences of non-negative integers such that $(\phi_n) = 0$ and $(\varphi_n) = n$ and let (p_n) be a sequence of positive numbers with $p_n = 1$. Suppose (λ_n) be a convex sequence such that $\sum p_n\lambda_n$ is convergent, then the Cesàro $|C, 1|_k$-summability of the series $\sum A_n(t)P_n\lambda_n$ at a point can be ensured by a local property.*

Remark 3.4. *Motivated by the recently-published results [3], [9] and [17] the interested reader's attention is drawn toward the possibility of investigating the local properties of Fourier series based on basic notions of statistical absolute convergence via different summability means.*

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