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Weak and Renormalized Solutions for Anisotropic Neumann Problems with Degenerate Coercivity

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abstract: In this work, we study the following quasilinear Neumann boundary-value problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i(a_i(x,u,\nabla u)) + |u|^{p_0-2}u = f(x,u,\nabla u) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot \eta_i = g(x) & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $I\!R^N$, $(N \geq 2)$ and η_i denotes the unit outward normal to $\partial\Omega$. We prove the existence of a weak solution for $f \in L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$ and the existence of renormalized solutions for L^1-data f and g. The functional setting involves anisotropic Sobolev spaces with constant exponents.

Key Words: Weak solutions, renormalized solutions, nonlinear elliptic problem, anisotropic Sobolev spaces, Neumann boundary condition.

Contents

1. Introduction

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest has turned towards anisotropic elliptic equations. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. Much less is known about anisotropic elliptic problems like problems with degenerate coercivity, for example let us cite only some recent works on this field [1,2,8,12]. In [8], Boccardo et al. have studied some quasilinear elliptic problems with degenerate coercivity of the type

$$\begin{cases} -\text{div}(A(x,u)\nabla u) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where f is assumed to be a measurable function in $L^m(\Omega)$ with $m \geq 1$, they have proved existence of solutions under various assumptions on the datum f.
In [2] a more general problem than the one studied in [8] is handled: the authors prove existence of solutions for the nonlinear and noncoercive elliptic problem

$$\begin{cases} -\text{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

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where $f \in L^m(\Omega)$ and $0 < \theta < 1$,.
On the other hand, in the framework of the Lorentz space we find that Guibé et al. have studied in [12] a class of nonlinear and noncoercive elliptic problems whose prototype is

$$\begin{cases} -\Delta_p u - \operatorname{div}(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{in} \quad \partial\Omega, \end{cases}$$

with μ is a radon measure with bounded variation on Ω. They have proved the existence of renormalized solutions in the case $0 \leq \gamma \leq p-1$ and $0 \leq \lambda \leq p-1$ with $c(x)$ and $b(x)$ belong to some appropriate Lorentz spaces, (see also [14]).
Recently, Akdim et al. have studied in [1] the existence of entropy solutions to the obstacle problem associated with the equation having degenerate coercivity, whose prototype is given by:

$$\begin{cases} -\operatorname{div}(b(|u|)|\nabla u|^{p-2}\nabla u) + d(|u|)|\nabla u|^p = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $b(\cdot)$ and $d(\cdot)$ are some positive decreasing function such that $\dfrac{d(|.|)}{b(|.|)} \in L^1(\Omega) \cap L^\infty(\Omega)$ and $f(x,s)$ satisfying some growth condition.
Our aim in this work is to prove the existence of renormalized solutions for the following quasilinear anisotropic Neumann problem having a degenerate coercivity:

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^i(a_i(x,u,\nabla u)) + |u|^{p_0-2}u = f(x,u,\nabla u) \quad \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x,u,\nabla u).n_i = g(x) \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $I\!R^N$, $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$, $(n_1, n_2, \ldots, n_N)$ is the outer unit normal vector on $\partial\Omega$, $g \in L^1(\partial\Omega)$, $a_i(x,u,\nabla u)$ is a non-coercive Leray Lions operator and $f(x,u,\nabla u)$ is a Carathéodory function which verify only some growth condition.
The main feature of our problem (1.1) is that we can not apply the standard Leray-Lions Theorem due to the the absence of coercivity. Besides of this, we have the term of the right-hand side which has an important influence in choosing p_0 and thus we haven't existence of weak solution. To overcome this problem, we prove existence of weak solution in case where right hand side $f(x,u,\nabla u)$ is assumed to be in $L^\infty(\Omega)$ and $g \in L^\infty(\partial\Omega)$.
This paper is organized as follows : In section 2, we will present some definitions and properties of the anisotropic Sobolev spaces. The section 3 is divided into two essential parts, in the first one, we present assumptions under which our problem has at least one weak solution where g is assumed to be in $L^\infty(\partial\Omega)$ and $|f(x,s,\xi)| \leq C_0$ for any $x \in \Omega$ and $(s,\xi) \in I\!R \times I\!R^N$. In the second part of section 3, we will prove the existence of renormalized solutions for our nonlinear and noncoercive problem in the case of $g \in L^1(\partial\Omega)$ and under some growth condition on the Carathéodory function $f(x,s,\xi)$. We conclude this paper by giving an example.

2. Preliminaries

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$.
Let $p_1, \ldots, p_N$ be N real constant numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$.
We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad \underline{p}^+ = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows :

$$W^{1,\vec{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\big(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\big)$ is a separable and reflexive Banach space (cf. [16]).
Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

Proposition 2.1. *(cf. [15])*
Let $u \in W^{1,\vec{p}}(\Omega)$, we have
(i) *Poincaré Wirtinger inequality: there exists a constant $C_p > 0$, such that*

$$\|u - med(u)\|_{L^{p_i}(\Omega)} \leq C_p \, \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \dots, N.$$

with

$$med(u) = \frac{1}{|\Omega|} \int_\Omega |u| \, dx.$$

(ii) *Sobolev inequality : there exists an other constant $C_s > 0$, such that*

$$\|u - med(u)\|_q \leq \frac{C_s}{N} \sum_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{p_i},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N, \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 2.2. *(cf. [11]) Let Ω be a bounded open set in $I\!R^N$ $(N \geq 2)$, we set*

$$s = \max(q, \; \underline{p}^+),$$

then, we have the following embedding :

- *if $\overline{p} < N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of this lemma follows from the Proposition 2.1.

Definition 2.3. *Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by*

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto I\!R \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.4. *Let $u \in \mathcal{T}^{1,\vec{p}}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto I\!R$ such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u| < k\}} \qquad \text{a.e.} \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

The proof of the Proposition 2.4 follows the usual techniques developed in [7] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [4,6,9,10].
We introduce the set $\mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$ as a subset of $\mathcal{T}^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace may be defined (see also [3] for the case of constant exponent). More precisely, $\mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$ is the set of function u in $\mathcal{T}^{1,\vec{p}}(\Omega)$, such that : there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega)$ and a measurable function v on $\partial\Omega$ verifying

(a) $u_n \longrightarrow u$ a.e. in Ω,

(b) $D^i T_k(u_n) \longrightarrow D^i T_k(u)$ in $L^1(\Omega)$ for every $k > 0$.

(c) $u_n \longrightarrow v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [3].
Let $u \in W^{1,\vec{p}}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $\tau(u)$.
For any $u \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $tr(u)$ or u, the operator $tr(\cdot)$ satisfied the following properties

(i) if $u \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$.

(ii) if $\varphi \in W^{1,\vec{p}}(\Omega)$, then, for any $u \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$, we have $u - \varphi \in \mathcal{T}_{tr}^{1,\vec{p}}(\Omega)$ and $tr(u-\varphi) = tr(u) - \tau(\varphi)$.

In the case where $u \in W^{1,\vec{p}}(\Omega)$, $tr(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}}(\Omega) \subset \mathcal{T}_{tr}^{1,\vec{p}}(\Omega) \subset \mathcal{T}^{1,\vec{p}}(\Omega).$$

Lemma 2.5. *(see [13], Theorem 13.47) Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) *$u_n \to u$ a.e. in Ω,*

(ii) *$u_n \geq 0$ and $u \geq 0$ a.e. in Ω,*

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ in $L^1(\Omega)$.

3. Main results

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we assume that the vector $\vec{p} = (1, p_1, \ldots, p_N)$ satisfying the requirements that $1 < p_i < \infty$ for $i = 1, \ldots, N$, and let $p_0 > 1$.
Let A be a Leray-Lions operator acted from $W_0^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u)$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ are Carathéodory functions, for $i = 1, \ldots, N$, (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) that satisfy the following conditions :

$$|a_i(x,s,\xi)| \leq \beta(d_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1}) \qquad \text{for} \quad i = 1, \ldots, N, \tag{3.1}$$

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi'_i) > 0 \quad \text{for} \quad \xi_i \neq \xi'_i, \tag{3.2}$$

for almost every $x \in \Omega$ and all (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$. The nonnegative functions $d_i(\cdot)$ are assumed to be in $L^{p'_i}(\Omega)$ for $i = 1, \ldots, N$, where β is a strictly positive real constant.

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \qquad \text{for} \quad i = 1, \ldots, N, \tag{3.3}$$

such that $b(|\cdot|) : I\!R^+ \mapsto I\!R^+$ is a decreasing function that belongs to $L^1(R) \cap L^\infty(R)$, and there exists a positive constant b_0 which verifying

$$\frac{b_0}{(1+|s|)^\lambda} \leq b(|s|) \qquad \text{for any} \quad s \in I\!R. \tag{3.4}$$

where $0 < \lambda < p_0 - 1$.
As a consequence of (3.2) and the continuity of the function $a_i(x, s, \cdot)$ with respect to ξ, we have

$$a_i(x, s, 0) = 0.$$

We are going now to recall the following technical Lemma, useful to prove our main results.

Lemma 3.1. *(see [5]) Assuming that (3.1) − (3.3) hold true, and let $(u_n)_{n\in I\!N}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,\vec{p}}(\Omega)$ and*

$$\begin{aligned}&\int_\Omega (|u_n|^{\underline{p}-2}u_n - |u|^{\underline{p}-2}u)(u_n - u)\, dx \\ &\quad + \sum_{i=1}^N \int_\Omega (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u)\, dx \to 0,\end{aligned} \tag{3.5}$$

then $u_n \to u$ in $W^{1,\vec{p}}(\Omega)$ for a subsequence.

3.1. Existence of weak solutions for $L^\infty-$data

We consider the quasilinear anisotropic elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^i a_i(x, T_n(u), \nabla u) + |u|^{p_0-2}u = F(x, u, \nabla u) & \text{on} \quad \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x, u, \nabla u).n_i = G(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{3.6}$$

with

$$G(x) \in L^\infty(\partial\Omega) \qquad \text{and} \qquad |F(x, s, \xi)| \leq C_0 \qquad \text{for any} \quad x \in \Omega \quad \text{and} \quad (s, \xi) \in I\!R \times I\!R^N, \tag{3.7}$$

with C_0 is a positive constant.

Definition 3.2. *A measurable function u is called weak solution for the quasilinear anisotropic elliptic equation (3.6), if $u \in W^{1,\vec{p}}(\Omega)$, $|u|^{p_0} \in L^1(\Omega)$, and u verifies the following equality*

$$\sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u) D^i v\, dx + \int_\Omega |u|^{p_0-2}uv\, dx = \int_{\partial\Omega} Gv\, d\sigma + \int_\Omega F(x, u, \nabla u)v\, dx, \tag{3.8}$$

for every $v \in W^{1,\vec{p}}(\Omega)$.

Theorem 3.3. *Assuming that (3.1) − (3.4) and (3.7) hold true. Then, there exists at least one weak solution for the quasilinear elliptic problem (3.6).*

Proof of Theorem 3.3

Step 1: Approximate problem.
We consider the approximate problem :

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^i a_i(x, T_n(u_m), \nabla u_m) + |T_m(u_m)|^{p_0-2}T_m(u_m) + \frac{1}{m}|u_m|^{\underline{p}-2}u_m = F(x, u_m, \nabla u_m) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x, T_n(u_m), \nabla u_m).n_i = G(x) \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{3.9}$$

We define the operators A_m and H from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$ by

$$\langle A_m u, v\rangle = \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,dx + \int_\Omega |T_m(u)|^{p_0-2}T_m(u)v\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,dx,$$

and

$$\langle Hu, v\rangle = \int_\Omega F(x,u,\nabla u)v\,dx + \int_{\partial\Omega} G\;v\,d\sigma,$$

for any $u,v\in W^{1,\vec{p}}(\Omega)$.

Lemma 3.4. *The operator $B_m = A_m - H$ acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$, is bounded and pseudo-monotone. Moreover, B_m is coercive in the following sense*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \textit{as} \quad \|v\|_{1,\vec{p}}\to\infty \quad \textit{for} \quad v\in W^{1,\vec{p}}(\Omega).$$

Using the Hölder's inequality and the growth condition (3.1), we can show that the operator A_m is bounded, and since

$$\begin{aligned} |<Hu,v>| &= \Big|\int_\Omega F(x,u,\nabla u)v\,dx + \int_{\partial\Omega} G\;v\,d\sigma\Big| \\ &\le \int_\Omega |F(x,u,\nabla u)|\;|v|\,dx + \int_{\partial\Omega}|G|\;|v|\,d\sigma \\ &\le C_0\|v\|_{L^1(\Omega)} + \|G\|_{L^\infty(\partial\Omega)}\|v\|_{L^1(\partial\Omega)} \\ &\le C_1\|v\|_{1,\vec{p}} \qquad\qquad \text{for any } u,v\in W^{1,\vec{p}}(\Omega). \end{aligned} \tag{3.10}$$

We conclude that the operator B_m is bounded. For coercivity, we have

$$\begin{aligned} \langle B_m u, u\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i u\,dx + \int_\Omega |T_m(u)|^{p_0-1}|u|\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx \\ &\quad - \int_\Omega F(x,u,\nabla u)u\,dx - \int_{\partial\Omega} G\;u\,d\sigma \\ &\ge \sum_{i=1}^{N}\int_\Omega b(|T_n(u)|)|D^i u|^{p_i}\,dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,dx - C_0\int_\Omega |u|\,dx - \|G\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}|u|\,d\sigma \\ &\ge \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i}\,dx + \frac{\|u\|^{\underline{p}}_{L^1(\Omega)}}{m\|1\|^{\underline{p}}_{L^{\underline{p}'}(\Omega)}} - C_3\|u\|_{1,\vec{p}} \\ &\ge C_4\|u\|^{\underline{p}}_{1,\vec{p}} - C_5\|u\|_{1,\vec{p}} - C_6, \end{aligned}$$

thus we obtain that

$$\frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p}}} \ge \frac{C_5\|u\|^{\underline{p}}_{1,\vec{p}} - C_6\|u\|_{1,\vec{p}}}{\|u\|_{1,\vec{p}}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1,\vec{p}}\to\infty.$$

It remains to show that B_m is pseudo-monotone. Let $(u_k)_{k\in I\!N}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } \; W^{1,\vec{p}}(\Omega), \\ B_m u_k \rightharpoonup \chi_m & \text{in } \; (W^{1,\vec{p}}(\Omega))', \\ \limsup\limits_{k\to\infty}\langle B_m u_k, u_k\rangle \le \langle \chi_m, u\rangle. \end{cases} \tag{3.11}$$

We will show that

$$\chi_m = B_m u \quad \text{and} \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle\chi_m, u\rangle \;\text{ as }\; k\to+\infty.$$

In view of the compact embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ and $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^1(\partial\Omega)$, then there exists a subsequence still denoted $(u_k)_{k\in\mathbb{N}^*}$ such that $u_k \to u$ in $L^{\underline{p}}(\Omega)$ and $u_k \to u$ in $L^1(\partial\Omega)$.
As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}}(\Omega)$, using the growth condition (3.1), it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}^*}$ is bounded in $L^{p_i'}(\Omega)$, and there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \text{ weakly in } \quad L^{p_i'}(\Omega) \quad \text{as} \quad k \to \infty. \tag{3.12}$$

Similarly, since $(F(x, u_k, \nabla u_k))_{k\in\mathbb{N}^*}$ is bounded in $L^{\underline{p}'}(\Omega)$, then there exists a function $\psi \in L^{\underline{p}'}(\Omega)$, such that

$$F(x, u_k, \nabla u_k) \rightharpoonup \psi \qquad \text{weakly in} \quad L^{\underline{p}'}(\Omega). \tag{3.13}$$

Moreover, since $u_k \to u$ a.e. in Ω, and in view of Lebesgue dominated convergence theorem, it follows that

$$|T_m(u_k)|^{p_0-2}T_m(u_k) \longrightarrow |T_m(u)|^{p_0-2}T_m(u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega), \tag{3.14}$$

and

$$\frac{1}{m}|u_k|^{\underline{p}-2}u_k \longrightarrow \frac{1}{m}|u|^{\underline{p}-2}u \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{3.15}$$

Thus, for any $v \in W^{1,\vec{p}}(\Omega)$ we have

$$\begin{aligned}
\langle \chi_m, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k\to\infty} \int_\Omega |T_m(u_k)|^{p_0-2}T_m(u_k) v \, dx \\
&\quad + \lim_{k\to\infty} \frac{1}{m} \int_\Omega |u_k|^{\underline{p}-2} u_k v \, dx - \lim_{k\to\infty} \int_\Omega F(x, u_k, \nabla u_k) v \, dx - \lim_{k\to\infty} \int_{\partial\Omega} G \, v \, d\sigma \\
&= \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i v \, dx + \int_\Omega |T_m(u)|^{p_0-2}T_m(u) v \, dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2} u v \, dx \\
&\quad - \int_\Omega \psi v \, dx - \int_{\partial\Omega} G v \, d\sigma.
\end{aligned} \tag{3.16}$$

By relations (3.11) and (3.16), we conclude that

$$\begin{aligned}
\limsup_{k\to\infty} \langle B_m(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big(\int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_\Omega |T_m(u_k)|^{p_0-1}|u_k| \, dx \\
&\quad + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}} \, dx - \int_\Omega F(x, u_k, \nabla u_k) u_k \, dx - \lim_{k\to\infty} \int_{\partial\Omega} G \, u_k \, d\sigma \Big) \\
&\leq \sum_{i=1}^{N} \int_\Omega \varphi_i \, D^i u \, dx + \int_\Omega |T_m(u)|^{p_0-1}|u| \, dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}} \, dx \\
&\quad - \int_\Omega \psi u \, dx - \int_{\partial\Omega} G u \, d\sigma.
\end{aligned} \tag{3.17}$$

Thanks to (3.13) − (3.15) we have

$$\int_\Omega |T_m(u_k)|^{p_0-1}|u_k| \, dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}} dx \longrightarrow \int_\Omega |T_m(u)|^{p_0-1}|u| \, dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}} dx \qquad \text{as} \quad k \to \infty, \tag{3.18}$$

and

$$\int_\Omega F(x, u_k, \nabla u_k) u_k \, dx \longrightarrow \int_\Omega \psi u \, dx \qquad \text{as} \quad k \to \infty, \tag{3.19}$$

and since $G \in L^\infty(\partial\Omega)$ then

$$\int_{\partial\Omega} G \, u_k \, d\sigma \longrightarrow \int_{\partial\Omega} G u \, d\sigma \qquad \text{as} \quad k \to \infty. \tag{3.20}$$

It follows that

$$\limsup_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \le \sum_{i=1}^N\int_\Omega \varphi_i\ D^iu\,dx. \tag{3.21}$$

On the other hand, by condition (3.3), we have

$$\sum_{i=1}^N\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\ge 0, \tag{3.22}$$

then

$$\begin{aligned}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \quad &\ge \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu\,dx\\ &\quad+\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx.\end{aligned}$$

Using Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ in $L^{p_i}(\Omega)$, thus $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega)$, and using (3.12) we get

$$\liminf_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx \ge \sum_{i=1}^N\int_\Omega \varphi_i\ D^iu\,dx. \tag{3.23}$$

Having in mind (3.21), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx = \sum_{i=1}^N\int_\Omega \varphi_i\ D^iu\,dx. \tag{3.24}$$

Therefore, having in mind (3.18), (3.19) and (3.20) we obtain

$$\langle B_mu_k,u_k\rangle \longrightarrow \langle \chi_m,u\rangle \quad \text{as} \quad k\to\infty. \tag{3.25}$$

On the other hand, thanks to (3.24) we can show that

$$\lim_{k\to\infty}\sum_{i=1}^N\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx = 0.$$

We have $u_k\to u$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\begin{aligned}&\int_\Omega (|u_n|^{\underline{p}-2}u_n-|u|^{\underline{p}-2}u)(u_k-u)\,dx\\ &\quad+\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u)(D^iu_n-D^iu)\,dx\to 0.\end{aligned} \tag{3.26}$$

According to Lemma 3.1, we get

$$u_k\to u \quad \text{in} \quad W^{1,\vec{p}}(\Omega) \qquad \text{and} \qquad D^iu_k\to D^iu \quad \text{a.e. in} \quad \Omega.$$

Thanks to (3.1) and (3.7) and the convergence almost everywhere, we conclude that

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{in} \quad L^{p_i'}(\Omega) \quad \text{for} \quad i=1,\dots,N$$

and

$$F(x,u_k,\nabla u_k)\rightharpoonup F(x,u,\nabla u) \quad \text{in} \quad L^{\underline{p}'}(\Omega),$$

according to (3.14) and (3.15), we obtain $\chi_m=B_mu$ and the proof of Lemma 3.4 is complete.

By Lemma 3.4, there exists at least one weak solution $u_m \in W^{1,\vec{p}}(\Omega)$ of the problem (3.9), i.e.

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i v\, dx + \int_\Omega |T_m(u_m)|^{p_0-2} T_m(u_m) v\, dx \\ &+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2} u_m v\, dx = \int_\Omega F(x, u_m, \nabla u_m) v\, dx + \int_{\partial\Omega} G\, v\, d\sigma, \end{aligned} \tag{3.27}$$

for any $v \in W^{1,\vec{p}}(\Omega)$, we refer the reader to (cf. [15], Theorem 8.2).

Step 2: Weak convergence of the sequence $(u_m)_m$.
By using $u_m \in W^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (3.9), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i u_m\, dx + \int_\Omega |T_m(u_m)|^{p_0-1}\, |u_m|\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\, dx \\ &= \int_\Omega F(x, u_m, \nabla u_m) u_m\, dx + \int_{\partial\Omega} G\, u_m\, d\sigma. \end{aligned} \tag{3.28}$$

From (3.3) and (3.7), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega b(|T_n(u_m)|) |D^i u_m|^{p_i}\, dx + \int_\Omega |T_m(u_m)|^{p_0-1}\, |u_m|\, dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\, dx \\ &\leq \int_\Omega |F(x, u_m, \nabla u_m)|\, |u_m|\, dx + \int_{\partial\Omega} |G(x)|\, |u_m|\, d\sigma \\ &\leq C_0 \int_\Omega |u_m|\, dx + \|G\|_{L^\infty(\Omega)} \int_{\partial\Omega} |u_m|\, d\sigma \\ &\leq C_1 \|u_m\|_{1,1} \\ &= C_1\Big(\int_\Omega |u_m|\, dx + \sum_{i=1}^{N} \int_\Omega |D^i u_m|\, dx \Big), \end{aligned} \tag{3.29}$$

with C_1 is a constant that doesn't depend on m and n. We set $0 < b_n = \min\limits_{|s|\leq n} b(s)$, thanks to Young's inequality we get

$$\begin{aligned} &b_n \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i}\, dx + \int_{\{|u_n|\leq m\}} |u_m|^{p_0}\, dx + m^{p_0-1} \int_{\{|u_n|>m\}} |u_m|\, dx \\ &\leq C_2 + \frac{1}{2}\int_{\{|u_n|\leq m\}} |u_m|^{p_0}\, dx + C_1 \int_{\{|u_n|>m\}} |u_m|\, dx + \frac{b_n}{2} \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i}\, dx. \end{aligned} \tag{3.30}$$

By taking $m \geq 1$ large enough $\left(\text{for example } \dfrac{m^{p_0-1}}{2} > C_1\right)$, we conclude that

$$\begin{aligned} &\frac{b_n}{2} \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i}\, dx + \frac{1}{2}\int_\Omega |u_m|\, dx \\ &\leq \frac{b_n}{2} \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i}\, dx + \frac{1}{2}\int_{\{|u_n|\leq m\}} |u_m|^{p_0}\, dx + \frac{m^{p_0-1}}{2}\int_{\{|u_n|>m\}} |u_m|\, dx + C_3 \\ &\leq C_4. \end{aligned} \tag{3.31}$$

Hence

$$\begin{aligned} \|u_m\|_{1,\vec{p}} &= \|u_m\|_{1,1} + \sum_{i=1}^{N} \|D^i u_m\|_{L^{p_i}(\Omega)} \\ &\leq \|u_m\|_{L^1(\Omega)} + C_5 \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i}\, dx \\ &\leq C_6, \end{aligned} \tag{3.32}$$

with C_6 is a constant that doesn't depend on m. Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(u_m)_m$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ u_m \longrightarrow u & \text{strongly in } \; L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega, \\ u_m \longrightarrow u & \text{strongly in } \; L^{1}(\partial\Omega) \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \partial\Omega. \end{cases} \tag{3.33}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{3.34}$$

Moreover, in view of (3.29) and (3.32) we conclude that $(T_m(u_m))_m$ is bounded in $L^{p_0}(\Omega)$, and since $T_m(u_m) \to u$ almost everywhere in Ω, we deduce that

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{p_0}(\Omega). \tag{3.35}$$

Step 3 : The convergence almost everywhere of the gradient.
By taking $u_m - u$ as a test function in the approximated problem (3.9) we obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)(D^iu_m - D^iu)\,dx + \int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)\,(u_m-u)\,dx \\ &+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m-u)\,dx = \int_\Omega F(x,u_m,\nabla u_m)(u_m-u)\,dx + \int_{\partial\Omega} G(u_m-u)\,d\sigma, \end{aligned} \tag{3.36}$$

it follows that

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega \left(a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u)\right)(D^iu_m - D^iu)\,dx \\ &+\int_\Omega \left(|T_m(u_m)|^{p_0-2}T_m(u_m) - |T_m(u)|^{p_0-2}T_m(u)\right)(u_m-u)\,dx \\ &\le \sum_{i=1}^{N}\int_\Omega |a_i(x,T_n(u_m),\nabla u)|\,|D^iu_m - D^iu|\,dx + \int_\Omega |T_m(u)|^{p_0-1}\,|u_m-u|\,dx \\ &+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}\,|u_m-u|\,dx + \int_\Omega |F(x,u_m,\nabla u_m)|\,|u_m-u|\,dx + \int_{\partial\Omega}|G|\,|u_m-u|\,d\sigma. \end{aligned} \tag{3.37}$$

For the first term on the right-hand side of (3.37), we have $T_n(u_m) \to T_n(u)$ strongly in $L^{p_i}(\Omega)$ then

$$|a_i(x,T_n(u_m),\nabla u)| \longrightarrow |a_i(x,T_n(u),\nabla u)| \quad \text{strongly in} \quad L^{p_i'}(\Omega),$$

and since $D^iu_m \rightharpoonup D^iu$ weakly in $L^{p_i}(\Omega)$, we can write

$$\int_\Omega |a_i(x,T_n(u_m),\nabla u)|\,|D^iu_m - D^iu|\,dx \longrightarrow 0 \qquad \text{for any} \quad i=1,\dots,N. \tag{3.38}$$

Concerning the second and third terms on the right-hand side of (3.37), by (3.33) and (3.34), we have

$$\int_\Omega |T_m(u)|^{p_0-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty, \tag{3.39}$$

and

$$\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \tag{3.40}$$

For the two last terms on the right-hand side of (3.37). The sequence $(F(x,u_m,\nabla u_m))_m$ is bounded in $L^{\underline{p}'}(\Omega)$ then there exists a measurable function $\phi \in L^{\underline{p}'}(\Omega)$ such that

$$F(x,u_m,\nabla u_m) \rightharpoonup \phi \qquad \text{weakly in} \quad L^{\underline{p}'}(\Omega),$$

and since $u_m \to u$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\int_\Omega |F(x,u_m,\nabla u_m)|\,|u_m-u|\,dx \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \tag{3.41}$$

Also, we have $G \in L^\infty(\partial\Omega)$ and $u_m \to u$ strongly in $L^1(\partial\Omega)$, then

$$\int_{\partial\Omega} |G|\,|u_m-u|\,d\sigma \longrightarrow 0 \qquad \text{as} \quad m\to\infty. \tag{3.42}$$

By combining (3.37) and (3.39) – (3.42), it follows that

$$\sum_{i=1}^N \int_\Omega \left(a_i(x,T_n(u_m),\nabla u_m) - a_i(x,T_n(u_m),\nabla u)\right)(D^iu_m - D^iu)\,dx \longrightarrow 0 \quad \text{as} \quad m\to\infty \tag{3.43}$$

and since $u_m \to u$ in $L^{\underline{p}}(\Omega)$. Thus, applying Lemma 3.1, we obtain

$$\begin{cases} u_m \to u & \text{strongly in} \quad W^{1,\vec{p}}(\Omega), \\ D^iu_m \to D^iu & \text{a.e. in} \quad \Omega \quad \text{for} \quad i=1,\dots,N. \end{cases} \tag{3.44}$$

We conclude that $a_i(x,T_n(u_n),\nabla u_n) \longrightarrow a_i(x,T_n(u),\nabla u)$ and $F(x,u_m,\nabla u_m) \longrightarrow F(x,u,\nabla u)$ almost everywhere in Ω, hence

$$a_i(x,T_n(u_m),\nabla u_m) \rightharpoonup a_i(x,T_n(u),\nabla u) \qquad \text{weakly in} \quad L^{p_i}(\Omega) \quad \text{for} \quad i=1,\dots,N. \tag{3.45}$$

$$F(x,u_m,\nabla u_m) \rightharpoonup F(x,u,\nabla u) \qquad \text{weakly in} \quad L^{\underline{p}'}(\Omega). \tag{3.46}$$

Thanks to (3.9), we have for any $v \in W^{1,\vec{p}}(\Omega)$

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^iv\,dx + \int_\Omega |T_m(u_m)|^{p_0-2}T_m(u_m)v\,dx \\ &+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m v\,dx = \int_\Omega F(x,u_m,\nabla u_m)v\,dx + \int_{\partial\Omega} G\;v\,d\sigma. \end{aligned} \tag{3.47}$$

In view of (3.34), (3.35) and (3.45) – (3.46), by letting m tends to infinity we conclude that

$$\sum_{i=1}^N \int_\Omega a_i(x,T_n(u),\nabla u)D^iv\,dx + \int_\Omega |u|^{p_0-2}uv\,dx = \int_\Omega F(x,u,\nabla u)v\,dx + \int_{\partial\Omega} G\;v\,d\sigma. \tag{3.48}$$

This concludes the proof of theorem 3.3.

3.2. Existence of renormalized solutions for L^1–data

Let Ω be a bounded open subset of $I\!R^N$ ($N \ge 2$). We set

$$p_0 > \frac{p_i(2+\lambda)-q_i}{p_i-q_i} > 1 \qquad \text{and} \qquad 0 \le q_i < p_i - 1 \qquad \text{for} \quad i=0,1,\dots,N.$$

Now, we consider the quasilinear anisotropic elliptic problem

$$\begin{cases} Au + |u|^{p_0-2}u = f(x,u,\nabla u) & \text{on} \qquad \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x,u,\nabla u).n_i = g(x) & \text{in} \qquad \partial\Omega, \end{cases} \tag{3.49}$$

where the term on the right-hand side $f(x,s,\xi)$ is a Carathéodory function which verify only the following growth condition :

$$|f(x,s,\xi)| \le |f_0(x)| + c_0(x)|s|^{q_0} + \sum_{i=1}^N c_i(x)|\xi_i|^{q_i}, \tag{3.50}$$

for a.e. $x \in \Omega$ and any $(s,\xi) \in I\!R \times I\!R^N$, with f_0 is assumed to be in $L^1(\Omega)$, such that $c_0 \in L^{\frac{p_0-1}{p_0-q_0-1}}(\Omega)$ and

$$c_i \in L^{r_i}(\Omega) \qquad \text{with} \qquad r_i > \frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+1)} \qquad \text{for} \quad i=1,\dots,N,$$

where the data $g(\cdot)$ is assumed to be in $L^1(\partial\Omega)$.

Definition 3.5. *A measurable function u is called renormalized solution for the quasilinear anisotropic elliptic equation (3.49), If $u \in \mathcal{T}^{1,\vec{p}}_{tr}(\Omega)$, $|u|^{p_0-2}u \in L^1(\Omega)$, $f(x,u,\nabla u) \in L^1(\Omega)$, and*

$$\lim_{h\to\infty} \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}} a_i(x,u,\nabla u)D^i u\,dx = 0, \tag{3.51}$$

such that u verifies the following equality

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)(S'(u)\varphi D^i u + S(u)D^i\varphi)\,dx + \int_\Omega |u|^{p_0-2}uS(u)\varphi\,dx \\ = \int_{\partial\Omega} gS(u)\varphi\,d\sigma + \int_\Omega f(x,u,\nabla u)S(u)\varphi\,dx,\end{aligned} \tag{3.52}$$

for every $\varphi \in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ and for any smooth function $S(\cdot) \in W^{1,\infty}(\Omega)$ with a compact support.

The main result of this paper is to prove the following existence theorem.

Theorem 3.6. *Assuming that (3.1) – (3.4) and (3.50) hold true. Then, there exists at least one renormalized solution for the noncoercive quasilinear elliptic problem (3.49).*

In the next sections, we denote by C_0, C_1, C_2, ... some real constants that doesn't depend on n and k.

Proof of Theorem 3.6

Step 1: Approximate problem.
Let g_n be a bounded sequence in $L^\infty(\partial\Omega)\cap L^1(\partial\Omega)$ (for example $g_n = T_n(g)$), such that

$$g_n \to g \qquad \text{strongly in} \qquad L^1(\partial\Omega).$$

We consider the approximate problem :

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + |u_n|^{p_0-2}u_n = f_n(x,u_n,\nabla u_n) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,T_n(u_n),\nabla u_n).n_i = g_n(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{3.53}$$

with $f_n(x,s,\xi) = T_n(f(x,s,\xi))$.
Using Theorem 3.3, there exists at least one weak solution $u_n \in W^{1,\vec{p}}(\Omega)$ of the problem (3.53), i.e.

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i v\,dx + \int_\Omega |u_n|^{p_0-2}u_n v\,dx = \int_\Omega f_n(x,u_n,\nabla u_n)v\,dx + \int_{\partial\Omega} g_n\, v\,d\sigma, \tag{3.54}$$

for any $v \in W^{1,\vec{p}}(\Omega)$, such that $|u_n| \in L^{p_0}(\Omega)$.

Step 2: Weak convergence of truncations.
Let $k \geq 1$, and choosing $1 < \delta$ small enough such that

$$p_0 > \frac{p_i(1+\delta+\lambda) - q_i}{p_i - q_i} \quad \text{and} \quad \frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i) - p_i(\lambda+\delta)} \leq r_i \quad \text{for} \quad i = 1, \dots, N.$$

We set $H(s) = \displaystyle\int_0^s \frac{1}{(1+|\tau|)^\delta} d\tau$ then $0 < H(\infty) < \infty$.
By taking $\varphi(u_n) = T_k(u_n)(1+|T_k(u_n)|)^\lambda e^{H(|u_n|)}$ as a test function for the approximate problem (3.53), we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \varphi(u_n)\, dx + \int_\Omega |u_n|^{p_0-2} u_n \varphi(u_n)\, dx \\
&= \int_\Omega f_n(x, u_n, \nabla u_n) \varphi(u_n)\, dx + \int_{\partial\Omega} g_n(x) \varphi(u_n)\, dx,
\end{aligned} \tag{3.55}$$

it follows that

$$\begin{aligned}
&\sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n (1+|T_k(u_n)|)^\lambda e^{H(|u_n|)} dx \\
&+\lambda \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n (1+|T_k(u_n)|)^{\lambda-1} |T_k(u_n)| e^{H(|u_n|)} dx \\
&+ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n \frac{|T_k(u_n)|(1+|T_k(u_n)|)^\lambda}{(1+|u_n|)^\delta} e^{H(|u_n|)} dx \\
&+ \int_\Omega |u_n|^{p_0-1}\ |T_k(u_n)|(1+|T_k(u_n)|)^\lambda e^{H(|u_n|)}\, dx \\
&= \int_\Omega f_n(x, u_n, \nabla u_n) T_k(u_n)(1+|T_k(u_n)|)^\lambda e^{H(|u_n|)}\, dx + \int_{\partial\Omega} g_n(x) T_k(u_n)(1+|T_k(u_n)|)^\lambda e^{H(|u_n|)} d\sigma.
\end{aligned} \tag{3.56}$$

By (3.4), we have

$$\begin{aligned}
&b_0 \sum_{i=1}^N \int_{\{|u_n| \leq k\}} |D^i T_k(u_n)|^{p_i}\, dx + \lambda b_0 \sum_{i=1}^N \int_{\{|u_n| \leq k\}} \frac{|D^i T_k(u_n)|^{p_i}}{(1+|T_k(u_n)|)} |T_k(u_n)|\, dx \\
&+ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n \frac{|T_k(u_n)|}{(1+|u_n|)^\delta} (1+|T_k(u_n)|)^\lambda\, dx \\
&+ \int_\Omega |u_n|^{p_0-1}\ |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx \\
&\leq e^{H(\infty)} \int_\Omega |f_0(x)| T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx + e^{H(\infty)} \int_\Omega c_0(x)|u_n|^{q_0} |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx \\
&+e^{H(\infty)} \sum_{i=1}^N \int_\Omega c_i(x)|D^i u_n|^{q_i} |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx + e^{H(\infty)} \int_{\partial\Omega} |g_n(x)|\ |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, d\sigma,
\end{aligned} \tag{3.57}$$

we obtain

$$\begin{aligned}
&b_0 \sum_{i=1}^N \int_{\{|u_n| \leq k\}} |D^i T_k(u_n)|^{p_i}\, dx + \lambda b_0 \sum_{i=1}^N \int_{\{|u_n| \leq k\}} \frac{|D^i T_k u_n|^{p_i}}{1+|T_k(u_n)|} |T_k(u_n) dx \\
&+b_0 \sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i} |T_k(u_n)|}{(1+|u_n|)^{\lambda+\delta}} (1+|T_k(u_n)|)^\lambda\, dx + \int_\Omega |u_n|^{p_0-1}\ |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx \\
&\leq k(1+k)^\lambda e^{H(\infty)} (\|f_0\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) + e^{H(\infty)} \int_\Omega |c_0(x)||u_n|^{q_0} |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx \\
&+e^{H(\infty)} \sum_{i=1}^N \int_\Omega c_i(x)|D^i u_n|^{q_i} |T_k(u_n)|(1+|T_k(u_n)|)^\lambda\, dx.
\end{aligned} \tag{3.58}$$

By the Young inequality, we get

$$\begin{aligned} & e^{H(|u_n|)}\int_\Omega |c_0(x)|u_n|^{q_0}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \leq C_0 \int_\Omega |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx + \frac{1}{2}\int_\Omega |u_n|^{p_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \leq C_1 k(1+k)^\lambda + \frac{1}{2}\int_\Omega |u_n|^{p_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx. \end{aligned} \tag{3.59}$$

Moreover, we have

$$\begin{aligned} & \sum_{i=1}^N \int_\Omega c_i(x)|D^i u_n|^{q_i}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & = \sum_{i=1}^N \int_\Omega c_i(x)(1+|u_n|)^{\frac{q_i(\lambda+\delta)}{p_i}} \frac{|D^i u_n|^{q_i} \ |T_k(u_n)|(1+|T_k(u_n)|)^\lambda}{(1+|u_n|)^{\frac{q_i(\lambda+\delta)}{p_i}}} \, dx \\ & \leq C_2 \sum_{i=1}^N \int_\Omega |c_i(x)|^{\frac{p_i}{p_i-q_i}}(1+|u_n|)^{\frac{p_i(\lambda+\delta)}{p_i-q_i}}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \quad + \frac{b_0}{2}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\delta}}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \leq C_3 \sum_{i=1}^N \int_\Omega |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \quad + \frac{1}{4}\sum_{i=1}^N \int_\Omega |u_n|^{p_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \quad + \frac{b_0}{2}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\delta}}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx + C_4 k(1+k)^\lambda. \end{aligned} \tag{3.60}$$

By combining (3.58) – (3.60), we conclude that

$$\begin{aligned} & b_0 \sum_{i=1}^N \int_{\{|u_n|\leq k\}} |D^i T_k(u_n)|^{p_i} \, dx + \lambda b_0 \sum_{i=1}^N \int_{\{|u_n|\leq k\}} \frac{|D^i T_k(u_n)|^{p_i}}{(1+|T_k(u_n)|)}|T_k(u_n)| dx \\ & + \frac{b_0}{2}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}|T_k(u_n)|}{(1+|u_n|)^{\delta+\lambda}}(1+|T_k(u_n)|)^\lambda \, dx + \frac{1}{4}\int_\Omega |u_n|^{p_0-1}|T_k(u_n)|(1+|T_k(u_n)|)^\lambda \, dx \\ & \leq C_5 k(1+k)^\lambda, \end{aligned} \tag{3.61}$$

with C_5 is a positive constant non depending on k and n, it follows that

$$\sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i} \, dx \leq C_6 k(1+k)^\lambda. \tag{3.62}$$

Moreover, we have

$$\sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}} \, dx + \int_{\{|u_n|>k\}} |u_n|^{p_0-1} \, dx \leq C_7. \tag{3.63}$$

On the one hand, we have

$$\begin{aligned}
\|T_k(u_n)\|_{1,\vec{p}} &= \|T_k(u_n)\|_{1,1} + \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{p_i} \\
&= \int_\Omega |T_k(u_n)|\,dx + \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|\,dx + \sum_{i=1}^{N} \Big(\int_\Omega |D^i T_k(u_n)|^{p_i}\,dx\Big)^{\frac{1}{p_i}} \\
&\leq k \cdot \mathrm{meas}(\Omega) + 2\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\,dx + N + N\cdot|\Omega| \\
&\leq C_8 k(1+k)^{\lambda}.
\end{aligned} \tag{3.64}$$

Thus, the sequence $(T_k(u_n))_{n\in N^*}$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_{n\in N^*}$ and $v_k \in W^{1,\vec{p}}(\Omega)$ such that

$$T_k(u_n) \rightharpoonup v_k \qquad \text{weakly in } W^{1,\vec{p}}(\Omega), \tag{3.65}$$

and by compact embedding, we obtain

$$T_k(u_n) \longrightarrow v_k \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{3.66}$$

In view of (3.63), we have

$$\begin{aligned}
k^{p_0-1}\mathrm{meas}(|u_n|>k) &\leq \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_0-1}\,dx \\
&\leq \int_{\{|u_n|>k\}} |u_n|^{p_0-1}\,dx \\
&\leq C_7,
\end{aligned} \tag{3.67}$$

it follows that

$$\mathrm{meas}(\{|u_n|>k\}) \leq \frac{C_7}{k^{p_0-1}} \longrightarrow 0 \quad \text{as} \quad k\to\infty. \tag{3.68}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure.
For all $\lambda > 0$, we have

$$\mathrm{meas}\{|u_n-u_m|>\lambda\} \leq \mathrm{meas}\{|u_n|>k\} + \mathrm{meas}\{|u_m|>k\} + \mathrm{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\}.$$

Let $\varepsilon > 0$, using (3.68) we may choose $k = k(\varepsilon)$ large enough such that

$$\mathrm{meas}\{|u_n|>k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \mathrm{meas}\{|u_m|>k\} \leq \frac{\varepsilon}{3}. \tag{3.69}$$

On the other hand, thanks to (3.66) we have $T_k(u_n) \to v_k$ in $L^1(\Omega)$ and a.e. in Ω. Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure, and for all $k > 0$ and $\varepsilon, \lambda > 0$, there exists $n_0 = n_0(k,\varepsilon,\lambda)$ such that

$$\mathrm{meas}\{|T_k(u_n)-T_k(u_m)|>\lambda\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m,n \geq n_0(k,\varepsilon,\lambda). \tag{3.70}$$

Combining (3.69) – (3.70), we obtain

$$\forall \varepsilon,\lambda>0 \quad \text{there exists} \quad n_0 = n_0(\varepsilon,\lambda) \quad \text{such that} \quad \mathrm{meas}\{|u_n-u_m|>\lambda\} \leq \varepsilon$$

for any $n,m \geq n_0(\varepsilon,\lambda)$. It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases}
T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W^{1,\vec{p}}(\Omega), \\
T_k(u_n) \to T_k(u) & \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \\
T_k(u_n) \to T_k(u) & \text{strongly in} \quad L^1(\partial\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega.
\end{cases} \tag{3.71}$$

Therefore, applying the Lebesgue dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{in} \quad L^{p_i}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \dots, N. \tag{3.72}$$

On the other hand, we have $\Big\|\dfrac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} \longrightarrow 0$ as k tends to infinity, it follows necessary

$$\begin{aligned}
\lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\partial\Omega)} &\le \lim_{k\to\infty} C\Big\|\frac{T_k(u_n)}{k}\Big\|_{W^{1,1}(\Omega)} \\
&\le C \lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C \lim_{k\to\infty} \sum_{i=1}^N \Big\|\frac{D^i T_k(u_n)}{k}\Big\|_{L^1(\Omega)} \\
&\le C \lim_{k\to\infty} \Big\|\frac{T_k(u_n)}{k}\Big\|_{L^1(\Omega)} + C \lim_{k\to\infty} \sum_{i=1}^N \|1\|_{L^{p_i'}(\Omega)}\Big\|\frac{D^i T_k(u_n)}{k}\Big\|_{L^{p_i}(\Omega)}.
\end{aligned}$$

Thus

$$\frac{T_k(u_n)}{k} \rightharpoonup 0 \quad \text{weak} - * \ \text{ in } L^\infty(\partial\Omega). \tag{3.73}$$

Step 3 : Some regularity results.

In this step, we will show that

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x, T_n(u), \nabla u_n) D^i u_n \, dx = 0. \tag{3.74}$$

By taking $\varphi(u_n) = \dfrac{T_h(u_n)}{h} e^{H(|u_n|)} \in W^{1,\vec{p}}(\Omega)$ as a test function in the approximate problem (3.53) we have

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n e^{H(|u_n|)} dx + \frac{1}{h}\sum_{i=1}^N \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n)}{(1+|u_n|)^\delta} D^i u_n T_h(u_n) e^{H(|u_n|)} dx \\
&+ \int_\Omega |u_n|^{p_0-2} u_n \frac{T_h(u_n)}{h} e^{H(|u_n|)} dx \\
&= \frac{1}{h}\int_\Omega f_n(x, u_n, \nabla u_n) T_h(u_n) e^{H(|u_n|)} dx + \frac{1}{h}\int_{\partial\Omega} g_n(x) T_h(u_n) e^{H(|u_n|)} d\sigma.
\end{aligned} \tag{3.75}$$

Thanks to (3.3), (3.4) and (3.50), we obtain

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx \\
&\quad + \frac{b_0}{h}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}} |T_h(u_n)|\, dx + \int_\Omega |u_n|^{p_0-1} \frac{|T_h(u_n)|}{h}\, dx \\
&\le \frac{e^{H(\infty)}}{h} \int_\Omega |f_0(x)||T_h(u_n)|\, dx + \frac{e^{H(\infty)}}{h} \int_{\partial\Omega} |g_n(x)||T_h(u_n)|\, d\sigma \\
&\quad + \frac{e^{H(\infty)}}{h} \int_\Omega c_0(x)|u_n|^{q_0}|T_h(u_n)|\, dx + \frac{e^{H(\infty)}}{h} \sum_{i=1}^N \int_\Omega c_i(x)|D^i u_n|^{q_i}|T_h(u_n)|\, dx.
\end{aligned} \tag{3.76}$$

From the Young inequality, and similarly as in (3.60), we deduce that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx + \frac{b_0}{h}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}} |T_h(u_n)|\, dx + \int_\Omega |u_n|^{p_0-1}\frac{|T_h(u_n)|}{h} dx \\
&\le \frac{e^{H(\infty)}}{h} \int_\Omega |f_0(x)||T_h(u_n)|\, dx + \frac{e^{H(\infty)}}{h} \int_{\partial\Omega} |g_n(x)||T_h(u_n)|\, d\sigma + \frac{C_7}{h}\sum_{i=1}^N \int_\Omega |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}} |T_h(u_n)| dx \\
&+ \frac{C_8}{h}\int_\Omega |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}} |T_h(u_n)|\, dx + \frac{1}{2h}\int_\Omega |u_n|^{p_0-1}|T_h(u_n)|\, dx + \frac{b_0}{2h}\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}} |T_h(u_n)|\, dx.
\end{aligned} \tag{3.77}$$

It follows that

$$
\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx\\
&+\frac{b_0}{2h}\sum_{i=1}^{N}\int_\Omega \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}|T_h(u_n)|\,dx+\frac{1}{2h}\int_\Omega |u_n|^{p_0-1}|T_h(u_n)|\,dx\\
&\le \frac{e^{H(\infty)}}{h}\int_\Omega |f_0(x)||T_h(u_n)|\,dx+\frac{e^{H(\infty)}}{h}\int_{\partial\Omega}|g_n(x)||T_h(u_n)|\,d\sigma\\
&+\frac{C_7}{h}\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}|T_h(u_n)|dx+\frac{C_8}{h}\int_\Omega |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}}|T_h(u_n)|\,dx.
\end{aligned}
\tag{3.78}
$$

For the terms on the right-hand side, we have $meas\{|u_n|\ge h\}\longrightarrow 0$ as $h\longrightarrow\infty$ then $\dfrac{T_h(u_n)}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$, and since $|f_0|\in L^1(\Omega)$ then

$$\int_\Omega |f_0|\frac{T_h(u_n)}{h}dx\longrightarrow 0 \quad as \quad h\longrightarrow\infty. \tag{3.79}$$

Moreover, we have $|c_0|^{\frac{p_0-1}{p_0-q_0-1}}\in L^1(\Omega)$ and $|c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}\in L^1(\Omega)$, it follows that

$$\int_\Omega |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}}\frac{T_h(u_n)}{h}dx\longrightarrow 0 \quad as \quad h\longrightarrow\infty, \tag{3.80}$$

and

$$\sum_{i=1}^{N}\int_\Omega |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}\frac{T_h(u_n)}{h}dx\longrightarrow 0 \quad as \quad h\longrightarrow\infty. \tag{3.81}$$

Similarly, thanks to (3.73) we have $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\partial\Omega)$, and since $g_n\to g$ in $L^1(\partial\Omega)$ we obtain

$$\int_{\partial\Omega}|g_n|\frac{T_h(u_n)}{h}dx\longrightarrow 0 \quad as \quad h\longrightarrow\infty. \tag{3.82}$$

By combining (3.78) and (3.79) – (3.82), we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx=0. \tag{3.83}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}\,dx=0. \tag{3.84}$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}|u_n|^{p_0-1}\,dx=0. \tag{3.85}$$

It follows that : for all $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{|u_n|>h(\eta)\}}|u_n|^{p_0-1}\,dx\le\frac{\eta}{2}. \tag{3.86}$$

On the other hand, for any $\eta>0$ there exists a positive constant $\beta(\eta)>0$ such that

$$\int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\le\frac{\eta}{2} \quad \text{for all } E\subset\Omega \text{ such that meas}(E)\le\beta(\eta). \tag{3.87}$$

By combining (3.86) and (3.87), we conclude that

$$\begin{aligned}\int_E |u_n|^{p_0-1}\,dx &\leq \int_{\{|u_n|>h(\eta)\}} |u_n|^{p_0-1}\,dx + \int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx \\ &\leq \eta \qquad \text{for all} \quad E\subset\Omega \text{ such that } \operatorname{meas}(E)\leq\beta(\eta).\end{aligned} \tag{3.88}$$

It follows that the sequence $(|u_n|^{p_0-2}u_n)_n$ is uniformly equi-integrable, and since

$$|u_n|^{p_0-1} \longrightarrow |u|^{p_0-1} \qquad \text{a.e. in} \quad \Omega.$$

In view of Vitali's theorem, we conclude that

$$|u_n|^{p_0-1} \longrightarrow |u|^{p_0-1} \qquad \text{strongly in} \quad L^1(\Omega). \tag{3.89}$$

Step 4 : The strong convergence of the gradient.
In this step, we will denote by $\varepsilon_i(n)$, $i=0,1,...$ a various functions of real numbers which converges to 0 as n tends to infinity (respectively for $\varepsilon_i(h)$ and $\varepsilon_i(h,n)$).
Let $h\geq k\geq 1$, we define

$$\varphi_h(s) = 1-\frac{|T_{2h}(s)-T_h(s)|}{h} \qquad \text{and} \qquad \psi(s) = s.\exp(\frac{\gamma^2 s^2}{2}),$$

where $\gamma = 3\dfrac{\|b(|\cdot|)\|_{L^\infty(I\!R)}}{b_0}$, note that $\psi'(s)-\gamma|\psi(s)|\geq\dfrac{1}{2} \qquad \forall s\in I\!R.$
Let $\delta>1$ and $H(s)=\displaystyle\int_0^s \frac{1}{(1+|s|)^\delta}\,ds$. By taking $\psi(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}\in W^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (3.53), we have

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx \\ &-\frac{1}{h}\sum_{i=1}^N\int_{\{h\leq|u_n|\leq 2h\}} a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)|\psi(T_k(u_n)-T_k(u))|e^{H(|u_n|)}\,dx \\ &+\sum_{i=1}^N\int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)D^iu_n}{(1+|u_n|)^\delta}\operatorname{sign}(u_n)\varphi_h(u_n)\psi(T_k(u_n)-T_k(u))e^{H(|u_n|)}\,dx \\ &+\int_\Omega |u_n|^{p_0-2}u_n\psi(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}\,dx \\ &\leq\int_\Omega f_n(x,u_n,\nabla u_n)\psi(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}\,dx \\ &+\int_{\partial\Omega} g_n(x)\psi(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}d\sigma.\end{aligned} \tag{3.90}$$

In view of (3.50), we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\
&-\sum_{i=1}^{N}\int_{\{|u_n|>k\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u)\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\
&-\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} \frac{a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)}{(1+|u_n|)^{\delta}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\
&+\sum_{i=1}^{N}\int_{\{|u_n|>k\}} \frac{a_i(x,T_n(u_n),\nabla u_n)D^iu_n}{(1+|u_n|)^{\delta}}\varphi_h(u_n)|\psi(T_k(u_n)-T_k(u))|e^{H(|u_n|)}\,dx\\
&\le e^{H(\infty)}\int_{\Omega}\left(|f_0(x)|+c_0(x)|u_n|^{q_0}+\sum_{i=1}^{N}c_i(x)|D^iu_n|^{q_i}\right)|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\
&+e^{H(\infty)}\int_{\partial\Omega}|g_n(x)|\ |\psi(T_k(u_n)-T_k(u))||\varphi_h(u_n)|d\sigma\\
&+\frac{\psi(2k)e^{H(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le 2h\}} a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx\\
&+e^{H(\infty)}\int_{\Omega}|u_n|^{p_0-1}|\psi(T_k(u_n)-T_k(u))|\ |\varphi_h(u_n)|dx.
\end{aligned}
\tag{3.91}
$$

We have $h\ge k$, then $\varphi_h(u_n)=1$ on the set $\{|u_n|\le k\}$, and in view of (3.3) we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\
&-\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} \frac{a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)}{(1+|u_n|)^{\delta}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\
&+b_0\sum_{i=1}^{N}\int_{\{|u_n|>k\}} \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\
&\le e^{H(\infty)}\sum_{i=1}^{N}\int_{\Omega}c_i(x)|D^iu_n|^{q_i}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\
&+\psi'(2k)e^{H(\infty)}\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\,dx\\
&+\frac{\psi(2k)e^{H(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)\,dx\\
&+e^{H(\infty)}\int_{\Omega}\left(|f_0(x)|+c_0(x)|u_n|^{q_0}+|u_n|^{p_0-1}\right)|\psi(T_k(u_n)-T_k(u))|\,dx\\
&+e^{H(\infty)}\int_{\partial\Omega}|g(x)|\ |\psi(T_k(u_n)-T_k(u))|d\sigma.
\end{aligned}
\tag{3.92}
$$

Concerning the second term on the left-hand side of (3.92), we have $(a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\vartheta_i\in L^{p_i'}(\Omega)$ such that $a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))\rightharpoonup\vartheta_i$ in $L^{p_i'}(\Omega)$ for any $i=1,\dots,N$, we conclude that

$$
\begin{aligned}
\varepsilon_1(n)\ &=\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_k(u)\,dx\\
&\longrightarrow\sum_{i=1}^{N}\int_{\{k<|u|\le 2h\}}\vartheta_i\ D^iT_k(u)\,dx=0\qquad\text{as}\quad n\to\infty.
\end{aligned}
\tag{3.93}
$$

For the third term on the right-hand side of (3.92), in view of (3.82) we have

$$\varepsilon_2(h)=\frac{\psi(2k)e^{H(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h\le|u_n|\le 2h\}}a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty. \tag{3.94}$$

We have $\psi(T_k(u_n)-T_k(u))\rightharpoonup 0$ weak$-\star$ in $L^\infty(\Omega)$, and since $|f_0|$ belongs to $L^1(\Omega)$ with (3.89), we conclude that

$$\varepsilon_3(n)\;=\int_\Omega\left(|f_0(x)|+c_0(x)|u_n|^{q_0}+|u_n|^{p_0-1}\right)|\psi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty. \tag{3.95}$$

Similarly, we have $\psi(T_k(u_n)-T_k(u))\rightharpoonup 0$ weak$-\star$ in $L^\infty(\partial\Omega)$, then

$$\varepsilon_4(n)=\int_{\partial\Omega}|g|\;|\varphi(T_k(u_n)-T_k(u))|\,d\sigma\longrightarrow 0\qquad\text{as}\quad n\to\infty. \tag{3.96}$$

By combining (3.92) and (3.93) – (3.96), we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\&-\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\frac{a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)}{(1+|u_n|)^\delta}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\&+b_0\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\&\le e^{H(\infty)}\sum_{i=1}^{N}\int_\Omega c_i(x)|D^iu_n|^{q_i}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx+\varepsilon_4(n,h).\end{aligned} \tag{3.97}$$

For the first term on the right-hand side of (3.97), in view of Young's inequality, we have

$$\begin{aligned}&\int_\Omega c_i(x)|D^iu_n|^{q_i}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\&=\int_\Omega\frac{c_i(x)|D^iu_n|^{q_i}}{(1+|u_n|)^{\frac{q_i(\delta+\lambda)}{p_i}}}(1+|u_n|)^{\frac{q_i(\delta+\lambda)}{p_i}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\&\le\frac{b_0}{2}\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{(\delta+\lambda)}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\&\quad+C_0\int_\Omega|c_i(x)|^{\frac{p_i}{p_i-q_i}}(1+|u_n|)^{\frac{q_i(\delta+\lambda)}{p_i-q_i}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\&\le\frac{b_0}{2}\int_\Omega\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{(\delta+\lambda)}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)\,dx\\&\quad+C_1\int_\Omega|c_i(x)|^{\frac{(p_0-1)p_i}{(p_0-1)(p_i-q_i)-p_i(\delta+\lambda)}}|\psi(T_k(u_n)-T_k(u))|\,\varphi_h(u_n)dx\\&\quad+C_2\int_\Omega(1+|u_n|)^{p_0-1}|\psi(T_k(u_n)-T_k(u))|\,\varphi_h(u_n)dx.\end{aligned} \tag{3.98}$$

We have $|c_i|^{\frac{(p_0-1)p_i}{(p_0-1)(p_i-q_i)-p_i(\delta+\lambda)}}\in L^1(\Omega)$ and $(1+|u_n|)^{p_0-1}\mapsto(1+|u|)^{p_0-1}$ strongly in $L^1(\Omega)$. thus, similarly as in (3.95) we conclude that the two last terms on the right-hand side of (3.98) tends to 0 as

n tends to infinity. Thus, by combining (3.97) and (3.98), we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\&-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\&+\frac{b_0}{2}\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\&\leq\varepsilon_5(n,h).\end{aligned}\tag{3.99}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\\&\qquad\times\left(\psi'(T_k(u_n)-T_k(u))-|\psi(T_k(u_n)-T_k(u))|\right)\varphi_h(u_n)e^{H(|u_n|)}dx\\&\leq\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))||D^iT_k(u_n)-D^iT_k(u)|\psi'(T_k(u_n)-T_k(u))\varphi_h(u_n)e^{H(|u_n|)}dx\\&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx\\&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)|\psi(T_k(u_n)-T_k(u))|\varphi_h(u_n)e^{H(|u_n|)}\,dx+\varepsilon_5(n,h)\\&\leq(\psi'(2k)+\psi(2k))e^{H(\infty)}\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))||D^iT_k(u_n)-D^iT_k(u)|\,dx\\&\quad+e^{H(\infty)}\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)||\psi(T_k(u_n)-T_k(u))|\,dx+\varepsilon_5(n,h).\end{aligned}\tag{3.100}$$

For the first term on the right-hand side of (3.100), we have $T_k(u_n)\to T_k(u)$ in $L^{p_i}(\Omega)$, then $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$, and since $D^iT_k(u_n)$ tends to $D^iT_k(u)$ weakly in $L^{p_i}(\Omega)$, we conclude that

$$\varepsilon_6(n)\;=\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\,|D^iT_k(u_n)-D^iT_k(u)|\,dx\to0\quad\text{as}\quad n\to\infty.\tag{3.101}$$

Concerning the second term on the right-hand side of (3.100), we have $(a_i(x,T_k(u_n),D^iT_k(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\nu_i\in L^{p_i'}(\Omega)$ such that $|a_i(x,T_k(u_n),D^iT_k(u_n))|\rightharpoonup\nu_i$ weakly in $L^{p_i'}(\Omega)$, and since $|D^iT_k(u)|\;|\psi(T_k(u_n)-T_k(u))|$ tends strongly to 0 in $L^{p_i}(\Omega)$ for any $i=1,\dots,N$, it follows that

$$\varepsilon_7(n)\;=\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)||\psi(T_k(u_n)-T_k(u))|\,dx\longrightarrow0\quad\text{as}\quad n\to\infty,\tag{3.102}$$

By combining (3.100) and (3.101) – (3.102), we conclude that

$$\begin{aligned} 0 \quad &\leq \frac{1}{2}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),D^iT_k(u_n)) - a_i(x,T_k(u_n),D^iT_k(u))\Big)(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &\leq \sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u)) \\ &\qquad\qquad \times\left(\psi'(T_k(u_n)-T_k(u)) - |\psi(T_k(u_n)-T_k(u))|\right)\varphi_h(u_n)e^{H(|u_n|)}dx \\ &\leq \varepsilon_8(n,h) \longrightarrow 0 \qquad \text{as} \qquad n,h\to\infty. \end{aligned} \tag{3.103}$$

therefore, by applying again the Lebesgue dominated convergence theorem, we obtain $T_k(u_n) \to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$. Thus, by letting n then h tend to infinity, we deduce that

$$\begin{aligned} &\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n) - D^iT_k(u))\,dx \\ &+\int_{\Omega}(|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n)-T_k(u))\,dx \to 0 \qquad \text{as} \qquad n\to\infty. \end{aligned} \tag{3.104}$$

Thanks to Lemma 3.1, we conclude that

$$\begin{cases} T_k(u_n)\to T_k(u) & \text{strongly in} \quad W^{1,\vec{p}}(\Omega), \\ D^iu_n \to D^iu & \text{a.e. in} \quad \Omega \quad \text{for} \quad i=1,\ldots,N. \end{cases} \tag{3.105}$$

Moreover, we have $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ tends to $a_i(x,u,\nabla u)D^iu$ almost everywhere in Ω, and in view of Fatou's lemma, we conclude that

$$\begin{aligned} &\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\leq h\}} a_i(x,u,\nabla u)D^iu\,dx \\ &\leq \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx \\ &\leq \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx = 0. \end{aligned} \tag{3.106}$$

Step 5 : The equi-integrability of $(f_n(x,u_n,\nabla u_n))_n$.
In this section, we will prove that

$$f_n(x,u_n,\nabla u_n) \longrightarrow f(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{3.107}$$

According to (3.105) we have

$$f_n(x,u_n,\nabla u_n) \longrightarrow f(x,u,\nabla u) \qquad \text{a.e. in} \quad \Omega. \tag{3.108}$$

By using Vitali's Theorem, it is sufficient to prove that $(f_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable. Indeed, let E be a measurable subset of Ω, we have

$$\int_E |f_n(x,u_n,\nabla u_n)|\,dx \leq \int_{E\cap\{|u_n|\leq h\}}|f_n(x,u_n,\nabla u_n)|\,dx + \int_{\{|u_n|>h\}}|f_n(x,u_n,\nabla u_n)|\,dx. \tag{3.109}$$

On the one hand, in view of Young's inequality we have

$$\begin{aligned} \int_{E\cap\{|u_n|\leq h\}}|f_n(x,u_n,\nabla u_n)|\,dx \quad &\leq \int_E\Big(|f_0(x)| + c_0(x)|T_h(u_n)|^{q_0} + \sum_{i=1}^{N}c_i(x)|D^iT_h(u_n)|^{q_i}\Big)\,dx \\ &\leq \int_E|f_0(x)|\,dx + \int_E|c_0(x)|^{\frac{p_0-1}{p_0-1-q_0}}\,dx + \int_E|T_h(u_n)|^{p_0-1}\,dx \\ &\quad +\sum_{i=1}^{N}\int_E|c_i(x)|^{\frac{p_i}{p_i-q_i}}\,dx + \sum_{i=1}^{N}\int_E|D^iT_h(u_n)|^{p_i}\Big)\,dx, \end{aligned}$$

since $|f_0|$, $|c_0|^{\frac{p_0-1}{p_0-1-q_0}}$ and $|c_i|^{\frac{p_i}{p_i-q_i}}$ belongs to $L^1(\Omega)$ and in view of (3.62) and (3.89), we deduce that

$$\forall \varepsilon > 0, \quad \exists \beta(\varepsilon) > 0 \quad \text{such that} \quad \int_{E\cap\{|u_n|\le h\}} |f_n(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2} \qquad \text{for any} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{3.110}$$

On the other hand, let $1 < \delta$ small enough, and in view of Young's inequality we have

$$\begin{aligned}
&\int_{\{|u_n|>h\}} |f_n(x,u_n,\nabla u_n)|\,dx \\
&\le \int_{\{|u_n|>h\}} \Big(|f_0(x)| + c_0(x)|u_n|^{q_0} + \sum_{i=1}^N c_i(x)|D^i u_n|^{q_i}\Big)\,dx \\
&\le \int_{\{|u_n|>h\}} \Big(|f_0(x)| + |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}} + \sum_{i=1}^N |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}\Big)\,dx \\
&\quad + (N+1)\int_{\{|u_n|>h\}} |u_n|^{p_0-1}\,dx + \sum_{i=1}^N \int_{\{|u_n|>h\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\delta+\lambda}}\,dx.
\end{aligned} \tag{3.111}$$

We have $f_0(\cdot)$, $|c_0(\cdot)|^{\frac{p_0-1}{p_0-q_0-1}}$ and $|c_i(\cdot)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}$ for $i=1,\dots,N$ belongs to $L^1(\Omega)$, and since meas $\{|u_n| > h\}) \to 0$ as $h \to \infty$, then

$$\lim_{h\to\infty} \int_{\{|u_n|>h\}} \Big(|f_0(x)| + |c_0(x)|^{\frac{p_0-1}{p_0-q_0-1}} + \sum_{i=1}^N |c_i(x)|^{\frac{p_i(p_0-1)}{(p_0-1)(p_i-q_i)-p_i(\lambda+\delta)}}\Big)\,dx = 0, \tag{3.112}$$

and thanks to (3.84) and (3.86), we conclude that

$$\forall \varepsilon > 0, \quad \exists h_0 > 0 \quad \text{such that} \quad \int_{\{|u_n|>h\}} |f_n(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2} \qquad \text{for any} \quad h \le h_0. \tag{3.113}$$

Having in mind (3.109) and (3.110), (3.113), we deduce that :

$$\int_E |f_n(x,u_n,\nabla u_n)|\,dx \le \varepsilon \qquad \text{for any} \quad E\subset\Omega \quad \text{with} \quad \text{meas}(E) \le \beta(\varepsilon). \tag{3.114}$$

We conclude that the sequences $(f_n(x,u_n,\nabla u_n))_n$ is a uniformly equi-integrable. Thus, in view of Vitali's Theorem we obtain

$$f_n(x,u_n,\nabla u_n) \longrightarrow f(x,u,\nabla u) \quad \text{in} \quad L^1(\Omega). \tag{3.115}$$

Step 6: Passage to the limit.
Let $\varphi \in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$, and choosing $S(.)$ be a smooth function in $C_0^1(R)$ such $sup(S(.)) \subseteq [-M,M]$, $M \ge 0$. By choosing $S(u_n)\varphi \in W^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ as test function in the approximate problem (3.53), we obtain

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,T(u_n),\nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)\,dx + \int_\Omega |u_n|^{p_0-2}u_n S(u_n)\varphi\,dx \\
&= \int_\Omega f_n(x,u_n,\nabla u_n)S(u_n)\varphi\,dx + \int_{\partial\Omega} g_n S(u_n)\varphi\,d\sigma.
\end{aligned} \tag{3.116}$$

We begin by the first term on the left-hand side (3.116), we have

$$\begin{aligned}
&\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)dx \\
&= \sum_{i=1}^N \int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i\varphi)\,dx.
\end{aligned} \tag{3.117}$$

In view of (3.105), we have $(a_i(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $L^{p'_i}(\Omega)$, and since $a_i(x,T_M(u_n),\nabla T_M(u_n))$ tends to $a_i(x,T_M(u),\nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a_i(x,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a_i(x,T_M(u),\nabla T_M(u)) \qquad \text{in} \quad L^{p'_i}(\Omega),$$

and since $S'(u_n)\varphi D^iT_M(u_n) + S(T_M(u_n))D^i\varphi$ tends strongly to $S'(u)\varphi D^iT_M(u) + S(T_M(u))D^i\varphi$ in $L^{p_i}(\Omega)$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^iu_n + S(u_n)D^i\varphi)\,dx\\
&= \lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\left(S'(u_n)\varphi D^iT_M(u_n) + S(T_M(u_n))D^i\varphi\right)\,dx\\
&= \sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u),\nabla T_M(u))\left(S'(u)\varphi D^iT_M(u) + S(T_M(u))D^i\varphi\right)\,dx\\
&= \sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)\left(S'(u)\varphi D^iu + S(u)D^i\varphi\right)\,dx.
\end{aligned} \tag{3.118}$$

Concerning the second term on the left-hand side of (3.116), we have $S(T_M(u_n))\varphi \to S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$, and thanks to (3.89) and (3.115) we deduce that

$$\begin{aligned}
\lim_{n\to\infty}\int_\Omega |u_n|^{p_0-2}u_nS(u_n)\varphi\,dx &= \lim_{n\to\infty}\int_\Omega |u_n|^{p_0-2}u_nS(T_M(u_n))\varphi\,dx\\
&= \int_\Omega |u|^{p_0-2}uS(T_M(u))\varphi\,dx\\
&= \int_\Omega |u|^{p_0-2}uS(u)\varphi\,dx,
\end{aligned} \tag{3.119}$$

and

$$\lim_{n\to\infty}\int_\Omega f_n(x,u_n,\nabla u_n)S(T_M(u_n))\varphi dx = \int_\Omega f(x,u,\nabla u)S(T_M(u))\varphi dx = \int_\Omega f(x,u,\nabla u)S(u)\varphi dx. \tag{3.120}$$

Similarly, we have $g_n \to g$ strongly in $L^1(\partial\Omega)$ and we since $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\partial\Omega)$ then

$$\lim_{n\to\infty}\int_{\partial\Omega} g_nS(T_M(u_n)\varphi d\sigma = \int_{\partial\Omega} gS(T_M(u))\varphi d\sigma = \int_{\partial\Omega} gS(u)\varphi d\sigma. \tag{3.121}$$

By combining (3.116) and (3.118) – (3.121), we conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)(S'(u)\varphi D^iu + S(u)D^i\varphi)\,dx + \int_\Omega |u|^{p_0-2}uS(u)\varphi\,dx\\
&\quad= \int_\Omega f(x,u,\nabla u)S(u)\varphi\,dx + \int_{\partial\Omega} gS(u)\varphi\,d\sigma,
\end{aligned} \tag{3.122}$$

which complete the prove of the theorem 3.6.

As model example of applications for problem (3.49), we state the following model:

Example 3.7. *Let $0 < \lambda < p-1$, $q_0 = 0$, $q \le p-1$ and $p_0 > \frac{(2+\lambda)p-q}{p-q}$, we consider the noncoercive Neumann elliptic equation*

$$\begin{cases}
-div\left(\dfrac{|\nabla u|^{p-2}\nabla u}{(1+u)^\lambda}\right) + |u|^{p_0-2}u = f + |\nabla u|^q & in \quad \Omega\\
\dfrac{\partial u}{\partial n_i} = 0 \quad on \quad \partial\Omega.
\end{cases} \tag{3.123}$$

In view of theorem 3.6, there exists at least one renormalized solution for the noncoercive quasilinear elliptic problem (3.123). Moreover, we have $|u|^{p_0-1} \in L^1(\Omega)$ and $|\nabla u|^q \in L^1(\Omega)$.

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