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# Hankel Determinant for the Class of Bounded Turning Functions Associated with Generalized Telephone Numbers*

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ABSTRACT: In the present paper making use of subordination between two analytic functions, certain subclass of bounded turning functions associated with generalized telephone numbers is introduced. The few coefficients bounds are obtained through series expansion and later used to investigate the optimum bounds of the Hankel determinant of order three for above defined class.

Key Words: Analytic function, subordination, bounded turning function, Hankel determinant.

## Contents



## 1. Introduction

Let $\mathcal{A}$ represent the class of all normalized analytic functions defined in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ which is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta). \tag{1.1}$$

As usual, we denote $\mathcal{S}$, the subclass of $\mathcal{A}$ which are univalent in $\Delta$. Many subclasses of the class $\mathcal{S}$ can be governed by imposing certain geometric and analytic conditions upon the function in the class $\mathcal{S}$. The well-known familiar classes are the class of starlike, convex and bounded turning functions defined in terms of subordination respectively as:

$$\begin{aligned} \mathcal{S}^* : &= \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \Delta) \right\}, \\ \mathcal{C} : &= \left\{ f \in \mathcal{S} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in \Delta) \right\}, \\ \mathcal{R} : &= \{ f \in \mathcal{S} : f'(z) \prec \phi(z) \qquad (z \in \Delta) \}, \end{aligned} \tag{1.2}$$

where $\phi(z) = \frac{1+z}{1-z}$ and '$\prec$' denote subordination between two analytic functions.
By imposing suitable functions on the right hand side of (1.2) we obtain different subclasses of the class $\mathcal{S}$ having interesting geometrical properties (see [5,6,8,11,14,17,25,26]).

For given two analytic functions $f$ and $g$, one will say $f$ is subordinate to $g$ written as $f(z) \prec g(z)$ if there exists an analytic functions $\omega(z)$ satisfying the conditions of Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z))$.
One of the most attracting area of modern Geometric Function Theory is the coefficient estimates for various subclasses of analytic functions by means of Hankel determinant. Hankel determinant is very

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useful in studying singularities and power series with integral coefficients.
For given parameters $q,\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, the Hankel determinant $H_{q,n}(f)$ for function $f$ of the form (1.1) was introduced by Pommerenke (see [22,23]) as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2}\cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q}\cdots & a_{n+2q-2} \end{vmatrix} \qquad (\ with\ \ a_1 = 1).$$

For various values of $q$ and $n$ , we obtain Hankel determinant for different orders. When $q = 2$ and $n = 1$,

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

is popularly known as classical Fekete-Szegö functional. The sharp bound of the determinant $|a_2a_4 - a_3^2|$ for the class $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ were investigated by Janteng et. al. (see [9,10]). The determinant

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_5(a_3 - a_2^2) - a_4(a_4 - a_2a_3) + a_3(a_2a_4 - a_3^2) \tag{1.3}$$

is known as third order Hankel determinant. Babalola [3] was the first person to obtained the upper bound of $|H_{3,1}(f)|$ for the class of $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$. Later several researchers added their works in that direction with different perspective and their results obtained are available in literature (see [1,2,10,11,13,15,16,18,20,21,27]).

Bednarz and Wolowiec-Musial [4] introduced a new generalized telephone number given by the recurrence relation as follows:

$$T_\vartheta(n) = T_\vartheta(n-1) + \vartheta(n-1)T_\vartheta(n-2) \quad (n \geq 2, \quad \vartheta \geq 1)$$

with initial condition $T_\vartheta(0) = T_\vartheta(1) = 1$. For $\vartheta = 1$, we get the classical telephone number $T_n$(see [7]). $T_\vartheta(n)$ for some values of $n$ are: $T_\vartheta(0) = T_\vartheta(1) = 1$, $T_\vartheta(2) = 1 + \vartheta$, $T_\vartheta(3) = 1 + 3\vartheta$, $T_\vartheta(4) = 3\vartheta^2 + 6\vartheta + 1$, $T_\vartheta(5) = 15\vartheta^2 + 10\vartheta + 1$ and so on. The exponential generating function and the summation formula for the generalized telephone number $T_\vartheta(n)$ is given by:

$$e^{x+\frac{\vartheta x^2}{2}} = \sum_{n=0}^{\infty} T_\vartheta(n)\frac{x^n}{n!} \quad (\vartheta \geq 1).$$

For purpose of our investigation we consider the function $\psi(z) = e^{z+\frac{\vartheta z^2}{2}}$ with its domain of definition in the open unit disk $\Delta$. Recently Murugusundaramoorthy and Vijaya [19] defined a generalized class of starlike functions subordinated to telephone numbers to originate certain initial Taylor coefficient estimates and Fekete–Szegö inequality. Motivated by recent works of Deniz [7] and Murugusundaramoorthy and Vijaya [19] in this paper, we investigate few coefficients bounds and using this to determine the optimum bounds of third Hankel determinant $|H_{3,1}(f)|$ for the following subclass of $\mathcal{S}$ given in Definition1.1.

**Definition 1.1** *A function $f \in \mathcal{A}$ of the form (1.1) is said to be in the class $\mathcal{R}_T^\vartheta(\psi)$ if the following subordination condition:*

$$f'(z) \prec \psi(z) \quad (z \in \Delta) \tag{1.4}$$

*holds.*

In special case, when $\vartheta = 1$ we have $\mathcal{R}_T^\vartheta(\psi) = \mathcal{R}_T(\psi)$.

## 2. Preliminaries

Let $\mathcal{P}$ denote the family of all functions $q$ which are analytic in $\Delta$ with $\Re q(z) > 0$ and has the following form:

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \quad (z \in \Delta). \tag{2.1}$$

We need the following lemmas in order to derive our results.

**Lemma 2.1** *: If $q \in \mathcal{P}$ and is of the form (2.1), then*

$$\mid q_n \mid \quad \leq 2 \text{ for } n \geq 1, \tag{2.2}$$

$$|q_{n+k} - \delta q_n q_k| \leq \begin{cases} 2 & 0 \leq \delta \leq 1 \\ 2|2\delta - 1| & elsewhere. \end{cases} \tag{2.3}$$

$$|q_n q_m - q_l q_k| \leq 4 \quad for \quad n + m = l + k \tag{2.4}$$

$$|q_{n+2k} - \mu q_n q_k^2| \leq 2(1 + 2\mu) \quad for \quad \mu \in \mathbb{R}, \tag{2.5}$$

$$|q_2 - \frac{q_1}{2}| \leq 2 - \frac{|q_1|^2}{2}, \tag{2.6}$$

*and for any complex number $\lambda$, we have*

$$|q_2 - \lambda q_1^2| \leq 2 \max\{1, |2\lambda - 1|\}, \tag{2.7}$$

where results in (2.2)-(2.6) are taken from [24] and for the inequality (2.7) is obtained from [12].

**Lemma 2.2** *(see [2]): Let $q \in \mathcal{P}$ and has of the form (2.1). Then*

$$|Jq_1^3 - Kq_1q_2 + Lq_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|. \tag{2.8}$$

## 3. Coefficients Estimate and Fekete-Szegö Inequality

In this section, we obtain upper bounds for the coefficients $a_i, \ i = 2, 3, 4, 5$ and Fekete-Szegö inequality for the function $f$ of the form (1.1) for the class $\mathcal{R}_T^{\vartheta}(\psi)$.

**Theorem 3.1** *: Let the function $f$ of the form (1.1) be in the class $\mathcal{R}_T^{\vartheta}(\psi)$ $(\vartheta \geq 1)$. Then*

$$|a_2| \leq \frac{1}{2}, \tag{3.1}$$

$$|a_3| \leq \frac{1 + \vartheta}{6}, \tag{3.2}$$

$$|a_4| \leq \frac{15\vartheta - 1}{48}, \tag{3.3}$$

$$|a_5| \leq \frac{3\vartheta^2 + 78\vartheta - 23}{120}. \tag{3.4}$$

*The first two estimates are the best possible.*

**Proof** Let the function $f(z)$ of the form (1.1) belongs to the function class $\mathcal{R}_T^{\vartheta}(\psi)$. Then by Definition 1.1 there exists an analytic function $\nu(z)$ with $\nu(0)=0$ and $|\nu(z)|<1$ in $\Delta$ such that

$$f'(z)=\psi[\nu(z)]=e^{\nu(z)+\frac{\vartheta\nu^2(z)}{2}} \quad (z\in\Delta). \tag{3.5}$$

From (1.1) we have

$$f'(z)=1+2a_2z+3a_3z^2+4a_4z^3+\cdots. \tag{3.6}$$

Define the function $q(z)$ by

$$q(z)=\frac{1+\nu(z)}{1-\nu(z)}=1+q_1z+q_2z^2+\cdots \quad (z\in\Delta). \tag{3.7}$$

Clearly, $q$ is analytic in $\Delta$ with $q(0)=1$ and $\Re q(z)>0$ in $\Delta$. It follows from (3.7) that

$$\begin{aligned}\nu(z)&=\frac{q(z)-1}{q(z)+1}\\ &=\frac{q_1z+q_2z^2+q_3z^3+\cdots}{2+q_1z+q_2z^2++q_3z^3+\cdots}\\ &=\frac{q_1}{2}z+\left(\frac{q_2}{2}-\frac{q_1^2}{4}\right)z^2+\left(\frac{q_3}{2}-\frac{q_1q_2}{2}+\frac{q_1^3}{8}\right)z^3+\left(\frac{q_4}{2}-\frac{q_1q_3}{2}+\frac{3q_1^2q_2}{8}-\frac{q_2^2}{4}-\frac{q_1^4}{16}\right)z^4+\cdots\end{aligned} \tag{3.8}$$

Using (3.8) in the right hand side of (3.5) we obtain

$$\begin{aligned}\psi[\nu(z)] &= e^{\nu(z)+\frac{\vartheta\nu^2(z)}{2}}\\ &= 1+\frac{q_1}{2}z+\frac{1}{8}[(\vartheta-1)q_1^2+4q_2]z^2\\ &+ \frac{1}{48}[(1-3\vartheta)q_1^3+12(\vartheta-1)q_1q_2+24q_3]z^3\\ &+ \frac{1}{384}[(1+6\vartheta+3\vartheta^2)q_1^4+24(1-3\vartheta)q_1^2q_2+96(\vartheta-1)q_1q_3+48(\vartheta-1)q_2^2+192q_4]z^4+\cdots.\end{aligned} \tag{3.9}$$

Using (3.6) and (3.9) in (3.5) and then comparing the coefficients of $z,\ z^2,\ z^3$ and $z^4$ on both sides we get

$$a_2 = \frac{q_1}{4}, \tag{3.10}$$

$$a_3 = \frac{1}{24}[(\vartheta-1)q_1^2+4q_2], \tag{3.11}$$

$$a_4 = \frac{1}{192}[(1-3\vartheta)q_1^3+12(\vartheta-1)q_1q_2+24q_3], \tag{3.12}$$

and

$$a_5=\frac{1}{1920}[(1+6\vartheta+3\vartheta^2)q_1^4+24(1-3\vartheta)q_1^2q_2+96(\vartheta-1)q_1q_3+48(\vartheta-1)q_2^2+192q_4]. \tag{3.13}$$

The bounds on $a_i$ $i=2,\ 3,\ 4,\ 5$ can be obtained by using triangle inequality and Lemma 2.1 and 2.2 for the class $\mathcal{P}$ as follows:

$$|a_2|\le\frac{1}{2},$$

$$|a_3|\le\frac{1}{24}[4(\vartheta-1)+8]=\frac{1+\vartheta}{6}.$$

Rearranging the terms in $a_4$, it gives

$$|a_4| = \frac{1}{192}\left|12(\vartheta-1)q_1q_2 + 24\left(q_3 - \frac{3\vartheta-1}{24}q_1^3\right)\right|. \tag{3.14}$$

Application of triangle inequality along with the relation 2.5 of Lemma 2.1 give

$$|a_4| \leq \frac{1}{192}[48(\vartheta-1) + 4(11+3\vartheta)] = \frac{15\vartheta-1}{48}. \tag{3.15}$$

The bound for $a_5$ can be obtained by virtue of triangle inequality and relation (2.2) of Lemma 2.1.

The first two bounds are sharp for the function $f : \Delta \longrightarrow \mathbb{C}$ given by

$$\begin{aligned} f(z) &= \int_0^z \psi(t)dt \\ &= \int_0^z e^{t+\frac{\vartheta t^2}{2}}dt \\ &= z + \frac{z^2}{2} + \frac{1+\vartheta}{6}z^3 + \frac{1+3\vartheta}{24}z^4 + \frac{3\vartheta^2+6\vartheta+1}{120}z^5 + \cdots. \end{aligned}$$

This completes the proof of Theorem 3.1.

The coefficient bounds for the class $\mathcal{R}_T^1(\psi) = \mathcal{R}_T(\psi)$ is given by the following corollary.

**Corollary 3.1** *Let $f \in \mathcal{A}$ be in the class $\mathcal{R}_T(\psi)$. Then*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{3}, \quad |a_4| \leq \frac{7}{24}, \quad |a_5| \leq \frac{29}{60}.$$

*The first two bounds are sharp.*

**Theorem 3.2** *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{R}_T^\vartheta(\psi)$. Then for any complex number $\mu$, we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}\max\left\{1, \left|\frac{3\mu - 2(1+\vartheta)}{4}\right|\right\}. \tag{3.16}$$

**Proof:** By virtue of (3.10) and (3.11) we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{24}[(\vartheta-1)q_1^2 + 4q_2] - \mu\frac{q_1^2}{16} \\ &= \frac{1}{6}\left[q_2 - \frac{3\mu - 2(\vartheta-1)}{8}q_1^2\right] \\ &= \frac{1}{6}[q_2 - \gamma q_1^2], \end{aligned}$$

where

$$\gamma = \frac{2(1-\vartheta) + 3\mu}{8}.$$

Application of relation (2.7) of Lemma 2.1 yields

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}\max\left\{1, \left|\frac{3\mu - 2(1+\vartheta)}{4}\right|\right\}.$$

Letting $\vartheta = 1$ in Theorem 3.2 we obtain the Fekete-Szegö inequality for the class $\mathcal{R}_T(\psi)$.

**Corollary 3.2** *Let $f \in \mathcal{R}_T(\psi)$. Then for any complex number $\mu$ we have*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3} \max\left\{1, \left|\frac{3\mu - 4}{4}\right|\right\}.$$

Theorem 3.2 for $\mu = 1$ gives the following:

**Corollary 3.3** *Let $f \in \mathcal{R}_T^{\vartheta}(\psi)$. Then*

$$|a_3 - a_2^2| \leq \begin{cases} \frac{1}{3} & \vartheta \leq \frac{5}{2} \\ \frac{2\vartheta - 1}{12} & \vartheta > \frac{5}{2} \end{cases}$$

## 4. Hankel Determinant

Now we obtain some of the coefficient functional bounds which will help us in evaluating the third Hankel determinant for the class $\mathcal{R}_T^{\vartheta}(\psi)$.

**Theorem 4.1** *If $f \in \mathcal{A}$ of the form (1.1) and belongs to the class $\mathcal{R}_T^{\vartheta}(\psi)$, then*

$$|a_2 a_3 - a_4| \leq \begin{cases} \frac{7\vartheta - 5}{48} & \vartheta \leq 7 \\ \frac{5\vartheta + 9}{48} & \vartheta > 7. \end{cases} \tag{4.1}$$

**Proof:** From the relations (3.10) to (3.12) we have

$$\begin{aligned} a_2 a_3 - a_4 &= \left(\frac{1}{96}(\vartheta - 1) - \frac{1 - 3\vartheta}{192}\right) q_1^3 + \left(\frac{1}{24} - \frac{(\vartheta - 1)}{16}\right) q_1 q_2 - \frac{1}{8} q_3 \\ &= \frac{5\vartheta - 3}{192} q_1^3 + \frac{5 - 3\vartheta}{48} q_1 q_2 - \frac{1}{8} q_3. \end{aligned} \tag{4.2}$$

An application of Lemma 2.2 to relation (4.2) gives

$$\begin{aligned} |a_2 a_3 - a_4| &\leq 2\left|\frac{5\vartheta - 3}{192}\right| + 2\left|\frac{3\vartheta - 5}{48} - \frac{5\vartheta - 3}{96}\right| + 2\left|\frac{5\vartheta - 3}{192} - \frac{3\vartheta - 5}{48} - \frac{1}{8}\right| \\ &= \frac{5\vartheta - 3}{96} + \frac{|\vartheta - 7|}{48} + \frac{7\vartheta + 7}{96} \\ &= \begin{cases} \frac{5\vartheta + 9}{48} & \vartheta \leq 7 \\ \frac{7\vartheta - 5}{48} & \vartheta \geq 7. \end{cases} \end{aligned}$$

This proves the assertion (4.1) of Theorem 4.1

**Theorem 4.2** *If the function $f(z) \in \mathcal{A}$ is of the form (1.1) and belongs to the class $\mathcal{R}_T^{\vartheta}(\psi)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{8\vartheta^2 - 2\vartheta + 55}{288}.$$

**Proof:** Putting the values of $a_2, a_3$ and $a_4$ from (3.10), (3.11) and (3.12) in the expression $(a_2 a_4 - a_3^2)$ and after simplification we get

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \left|\left(\frac{1 - 3\vartheta}{768} - \frac{(\vartheta - 1)^2}{576}\right) q_1^4 + \left(\frac{12(\vartheta - 1)}{768} - \frac{8(\vartheta - 1)}{576}\right) q_1^2 q_2 + \frac{24}{768} q_1 q_3 - \frac{16}{576} q_2^2\right| \\ &= \left|\frac{-(4\vartheta^2 + \vartheta + 1)}{2304} q_1^4 + \frac{\vartheta - 1}{576} q_1^2 q_2 + \frac{1}{32} q_1 q_3 - \frac{1}{36} q_2^2\right| \\ &= \left|\frac{-(4\vartheta^2 + \vartheta + 1)}{2304} q_1^4 + \frac{\vartheta - 1}{576} q_1^2 q_2 - \frac{1}{384} q_1 q_3 + \frac{13}{384} q_1 q_3 - \frac{1}{36} q_2^2\right| \\ &= \left|-q_1\left(\frac{4\vartheta^2 + \vartheta + 1}{2304} q_1^3 - \frac{\vartheta - 1}{576} q_1 q_2 + \frac{1}{384} q_3\right) + \frac{13}{384}(q_1 q_3 - q_2^2) + \frac{7}{1152} q_2^2\right|. \end{aligned}$$

Applications of Lemma 2.1 and 2.2 and triangle inequality give

$$\begin{aligned}|a_2a_4-a_3^2| &\le |q_1|\left|\frac{4\vartheta^2+\vartheta+1}{2304}q_1^3-\frac{\vartheta-1}{576}q_1q_2+\frac{1}{384}q_3\right|+\frac{13}{384}|q_1q_3-q_2^2|+\frac{7}{1152}|q_2|^2\\ &=2\left[2\left(\frac{4\vartheta^2+\vartheta+1}{2304}\right)+2\left(\frac{\vartheta-1}{576}-\frac{4\vartheta^2+\vartheta+1}{1152}\right)+2\left(\frac{4\vartheta^2+\vartheta+1}{2304}-\frac{\vartheta-1}{576}+\frac{1}{384}\right)\right]\\ &+\frac{13}{384}(4)+\frac{7}{1152}(4)=\frac{8\vartheta^2-2\vartheta+55}{288}.\end{aligned}$$

This complete the proof of Theorem 4.2

**Theorem 4.3** *Let $f\in\mathcal{A}$ be in the class $\mathcal{R}_T^{\vartheta}(\psi)$. Then*

$$|H_{3,1}(f)|\le\begin{cases}\frac{160\vartheta^3+1533\vartheta^2+10498\vartheta-1243}{34560} & 1\le\vartheta\le\frac{5}{2}\\ \frac{304\vartheta^3+4917\vartheta^2+34\vartheta+1517}{34560} & \frac{5}{2}\le\vartheta\le 7,\\ \frac{304\vartheta^3-5367\vartheta^2-3146\vartheta+1727}{34560} & \vartheta>7.\end{cases}$$

**Proof:** Third order Hankel determinant from relation (1.3) can be written as

$$H_{3,1}(f)=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2).$$

Using triangle inequality and making use of bounds from the Theorem 3.1,4.1, 4.2 and Corollary 3.5 we obtain

$$\begin{aligned}|H_{3,1}(f)| &\le |a_3||a_2a_4-a_3^2|+|a_4||a_4-a_2a_3|+|a_5||a_3-a_2^2|\\ &=\left(\frac{1+\vartheta}{6}\right)\left(\frac{8\vartheta^2-2\vartheta+55}{288}\right)+\frac{15\vartheta-1}{48}\begin{cases}\frac{5\vartheta+9}{48} & \vartheta\le 7\\ \frac{7\vartheta-5}{48} & \vartheta>7\end{cases}\\ &+\frac{3\vartheta^2+78\vartheta-23}{120}\begin{cases}\frac{1}{3} & \vartheta\le\frac{5}{2}\\ \frac{2\vartheta-1}{12} & \vartheta>\frac{5}{2}\end{cases}\\ &=\frac{8\vartheta^3+6\vartheta^2+53\vartheta+55}{1728}+\begin{cases}\frac{75\vartheta^2+130\vartheta-9}{2304} & \vartheta\le 7\\ \frac{105\vartheta^2-82\vartheta+5}{2304} & \vartheta>7\end{cases}\\ &+\begin{cases}\frac{3\vartheta^2+78\vartheta-23}{360} & \vartheta\le\frac{5}{2}\\ \frac{6\vartheta^3+153\vartheta^2-124\vartheta+23}{1440} & \vartheta>\frac{5}{2}\end{cases}\end{aligned}$$

$$=\begin{cases}\frac{8\vartheta^3+6\vartheta^2+53\vartheta+55}{1728}+\frac{75\vartheta^2+130\vartheta-9}{2304}+\frac{3\vartheta^2+78\vartheta-23}{360} & 1\le\vartheta\le\frac{5}{2},\\ \frac{8\vartheta^3+6\vartheta^3+53\vartheta+55}{1728}+\frac{75\vartheta^2+130\vartheta-9}{2304}+\frac{6\vartheta^3+153\vartheta^2-124\vartheta+23}{1440} & \frac{5}{2}\le\vartheta\le 7\\ \frac{8\vartheta^3+6\vartheta^3+53\vartheta+55}{1728}+\frac{105\vartheta^2-82\vartheta+5}{2304}+\frac{6\vartheta^3+153\vartheta^2-124\vartheta+23}{1440} & \vartheta>7\end{cases}$$

$$=\begin{cases}\frac{160\vartheta^3+1533\vartheta^2+10498\vartheta-1243}{34560} & 1\le\vartheta\le\frac{5}{2}\\ \frac{304\vartheta^3+4917\vartheta^2+34\vartheta+1517}{34560} & \frac{5}{2}\le\vartheta\le 7,\\ \frac{304\vartheta^3-5367\vartheta^2-3146\vartheta+1727}{34560} & \vartheta>7\end{cases}$$

whichcompletes the proof of Theorem 4.3.

**Concluding Remark**: In this article, by making use of subordination the authors have introduced the class of bounded turning functions associated with generalized telephone numbers. We have obtained the sharp bounds for the first two coefficients and upper bound of Hankel determinant of order three for above mentioned class. The results of this paper can be generalized by using (p,q)-calculus which play an vital role in the field of Geometric Function Theory.

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