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Common Fixed Point Results for a Pair of Suzuki Type Mappings in b-Complete b-Metric Spaces via Triangular β-Admissible Function

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ABSTRACT: In this manuscript, we explore the presence and uniqueness of coincidence and common fixed point for two self maps in the context of b-complete b-metric space via triangular β-admissible function. Also, several results within the frame of b-metric space endowed with graph and partial order b-metric space are deduced from our main results. An illustrative example is given to demonstrate the main results.

Key Words: Triangular β-admissible map, b-complete b-metric space, coincident point, common fixed point.

Contents

1. Introduction

Fixed point theory is an entrancing subject, with tremendous number of utilizations in different field of mathematics and may be perceived as one of thrust areas of investigation in non linear analysis. The origin of fixed point theory lies in the method of successive approximations to establish the existence and uniqueness of solutions of differential and integral equations. One of the earliest and most important results in fixed point theory is Banach contraction principle which states that every contraction mapping defined on a complete metric space possesses a unique fixed point. Due to its applications in many disciplines within mathematics and outside it, several authors have improved, generalized and extended this principle in nonlinear analysis (e.g. [7], [11], [12], [13]). In 2011, Dukic et al. [6] established fixed point results for Geraghty contractive function in different metric spaces. In 2020, Bota et al.[4] established coupled fixed point results in the framework of b-metric space. Subsequently, Arora et al.[1] conferred common fixed point results for modified β-admissible contraction in the edge of metric space. Recently Arora et al. [2] explore the presence and uniqueness of common fixed point for two pair of functions by utilizing the perception of CLR property in the framework of b-metric spaces. Afterwards, Jain and Kaur [8] established some fixed point results for new contractive mappings in b-metric spaces.

In this paper, we established fixed point results by making combinations of a pair of mappings in the context of b-complete b-metric space by virtue of triangular β-admissible function. As application, various analogous results in fixed point theory are easily deduced for one function in the frame of metric and b-metric spaces.

Now, we present the significant definitions and theorems which are favourable in the proof of our sequel. In 1993, Czerwik suggest the new perception of metric space known as a b-metric space as pursues:

Definition 1.1 *[5] Let $\mathcal{H}$ be a nonempty set and let $s \geq 1$ be a real number. A function $\sigma : \mathcal{H}\times\mathcal{H} \to \mathcal{R}_+$ is called a b-metric provided that, for all $\Omega, \mho, \wp \in \mathcal{H}$,*
(i)$\sigma(\Omega, \mho) = 0$ *if and only if* $\Omega = \mho$*;*
(ii)$\sigma(\Omega, \mho) = \sigma(\mho, \Omega)$*;*
(iii)$\sigma(\Omega, \wp) \leq s[\sigma(\Omega, \mho) + \sigma(\mho, \wp)]$.
A pair $(\mathcal{H}, \sigma)$ is called a b-metric space. It is clear that b-metric space is an extension of usual metric space.

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Definition 1.2 *[3] Let $\rho \geq 1 \in \mathbb{R}$. A function $\psi : [0,\infty) \to [0,\infty)$ is known as (b)-comparison mapping if following assertions fulfil:*
(i) ψ is monotonic increasing;
(ii) For every $a \in (0,1), \exists\, k_0 \in \mathbb{N}$ and convergent series $\Sigma_{k=1}^{\infty} v_k$ such that $\rho^{k+1}\psi^{k+1}(t) \leq a\rho^k\psi^k(t) + v_k, \forall k \geq k_0$ and $t \in [0,\infty)$.

Definition 1.3 *[10] Let $E : \mathcal{H} \to \mathcal{H}$ and $\beta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$. Then, E is said to be $\beta-$admissible if $\beta(\Omega,\mho) \geq 1 \Rightarrow \beta(E\Omega, E\mho) \geq 1$, for each $\Omega, \mho \in \mathcal{H}$.*

Definition 1.4 *[9] Let $E : \mathcal{H} \to \mathcal{H}$ and $\beta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$. Then, E is said to be triangular $\beta-$admissible function if*
(i) E is $\beta-$admissible map;
(ii)$\beta(\Omega,\wp) \geq 1$ and $\beta(\wp,\mho) \geq 1 \Rightarrow \beta(\Omega,\mho) \geq 1, \forall\, \Omega,\mho \in \mathcal{H}$.

Definition 1.5 *[9] Let $E, S : \mathcal{H} \to \mathcal{H}$ and $\beta : \mathcal{H} \times \mathcal{H} \to [0,+\infty)$. Then, E is said to be triangular $\beta-$admissible function with respect to S if*
(i) $\beta(S\Omega, S\mho) \geq 1 \Rightarrow \beta(E\Omega, E\mho) \geq 1$;
(ii)$\beta(\Omega,\wp) \geq 1$ and $\beta(\wp,\mho) \geq 1 \Rightarrow \beta(\Omega,\mho) \geq 1, \forall\, \Omega,\mho \in \mathcal{H}$.

Lemma 1.1 *[9] Let S be triangular $\beta-$admissible function with regard to E. Imagine that $\exists\, \Omega_0 \in \mathcal{H}$ in a way that $\beta(S\Omega_0, E\Omega_0) \geq 1$. Let $\{\Omega_n\}$ be defined as $S\Omega_n = E^n\Omega_0$. Then, $\beta(S\Omega_p, S\Omega_n) \geq 1, \forall p, n \in \mathbb{N}$.*

We consider the family of mappings $\mathcal{K}$, where $\delta \in \mathcal{K}$ if $\delta : [0,\infty) \longrightarrow [0, \frac{1}{\rho})$ possess the property $\delta(\rho_n) \to \frac{1}{\rho} \Rightarrow \rho_n \to 0$ when $n \to \infty$.

2. Main Results

Theorem 2.1 *Let E and S be two mappings in a b-complete b-metric space $(\mathcal{H},\sigma)$ (with parameter $\rho > 1$) and let E be a triangular β-admissible function with regard to S. Imagine that $\exists\, \delta \in \mathcal{K}$ such that $\forall\, \Omega, \mho \in \mathcal{H}$*

$$\frac{1}{2\rho}\sigma(S\Omega, E\Omega) \leq \sigma(S\Omega, S\mho) \Rightarrow \rho\beta(S\Omega, S\mho)\sigma(E\Omega, E\Omega) \leq \delta(N(S\Omega, S\mho))N(S\Omega, S\mho), \tag{2.1}$$

where

$$N(S\Omega, S\mho) = max\left\{\sigma(S\Omega, S\mho), \frac{\sigma(S\Omega, E\Omega) + \sigma(S\mho, E\mho)}{2\rho}, \frac{\sigma(S\Omega, E\mho) + \sigma(S\mho, E\Omega)}{2\rho}\right\}.$$

Also imagine that followings assertions fulfil:

(i) $\beta(S\Omega_0, E\Omega_0) \geq 1$, for some $\Omega_0 \in \mathcal{H}$;
(ii) If $\beta(S\Omega_n, S\Omega_{n+1}) \geq 1$, for any sequence $\{S\Omega_n\}$ in $\mathcal{H}$ such that $S\Omega_n \to S\Omega$ when $n \to \infty$, then $\beta(S\Omega_n, S\Omega) \geq 1\, \forall\, n \in N \cup \{0\}$.

Then, S and E possess a coincident point.

Proof: Since, $E(\mathcal{H}) \subseteq S(\mathcal{H})$, we can choose a point $\Omega_1 \in \mathcal{H}$, such that $E\Omega_0 = S\Omega_1$. Continuing this process, we can choose Ω_{n+1} in $\mathcal{H}$ such that

$$E\Omega_n = S\Omega_{n+1}. \tag{2.2}$$

Since E is $\beta-$admissible w.r.t S and using (i), we have
$\beta(S\Omega_0, E\Omega_0) = \beta(S\Omega_0, S\Omega_1) \geq 1$,
which indicate that $\beta(E\Omega_0, E\Omega_1) = \beta(S\Omega_1, S\Omega_2) \geq 1$.
Using induction, we acquire

$$\beta(S\Omega_n, S\Omega_{n+1}) \geq 1. \forall n = 0, 1, 2, ... \tag{2.3}$$

If $E\Omega_{n+1} = E\Omega_n$ for some n, then by (2.2), we get that $E\Omega_{n+1} = S\Omega_{n+1}$.
Consequently, E and S possess a coincidence point at $\Omega = \Omega_{n+1}$ and so we have completed the proof. Further, we assume that $\sigma(E\Omega_n, E\Omega_{n+1}) > 0$. With the assistance of (2.1) and (2.3), we acquire

$$\begin{aligned}\sigma(S\Omega_n, S\Omega_{n+1}) &= \sigma(E\Omega_{n-1}, E\Omega_n)\\ &\leq \rho\beta(S\Omega_{n-1}, S\Omega_n)\sigma(E\Omega_{n-1}, E\Omega_n)\\ &\leq \delta(N(S\Omega_{n-1}, S\Omega_n))N(S\Omega_{n-1}, S\Omega_n),\end{aligned} \tag{2.4}$$

where

$$N(S\Omega_{n-1}, S\Omega_n) = max\Big\{\sigma(S\Omega_{n-1}, S\Omega_n), \frac{\sigma(S\Omega_{n-1}, E\Omega_{n-1}) + \sigma(S\Omega_n, E\Omega_n)}{2\rho}, \frac{\sigma(S\Omega_{n-1}, E\Omega_n) + \sigma(S\Omega_n, E\Omega_{n-1})}{2\rho}\Big\}.$$

Using (2.2), we acquire

$$N(S\Omega_{n-1}, S\Omega_n) = max\Big\{\sigma(E\Omega_{n-2}, E\Omega_{n-1}), \frac{\sigma(E\Omega_{n-2}, E\Omega_{n-1}) + \sigma(E\Omega_{n-1}, E\Omega_n)}{2\rho}, \frac{\sigma(E\Omega_{n-2}, E\Omega_n) + \sigma(E\Omega_{n-1}, E\Omega_{n-1})}{2\rho}\Big\}.$$

Therefore,

$$N(S\Omega_{n-1}, S\Omega_n) \leq max\{\sigma(E\Omega_{n-2}, E\Omega_{n-1}), \sigma(E\Omega_{n-1}, E\Omega_n)\}.$$

If

$$max\{\sigma(E\Omega_{n-2}, E\Omega_{n-1}), \sigma(E\Omega_{n-1}, E\Omega_n)\} = \sigma(E\Omega_{n-1}, E\Omega_n),$$

then (2.4) indicates that

$$\begin{aligned}\sigma(E\Omega_{n-1}, E\Omega_n) &\leq \delta(\sigma(E\Omega_{n-1}, E\Omega_n))\sigma(E\Omega_{n-1}, E\Omega_n)\\ &< \frac{1}{\rho}\sigma(E\Omega_{n-1}, E\Omega_n)\\ &< \sigma(E\Omega_{n-1}, E\Omega_n),\end{aligned}$$

which is a counterstatement. Consequently,

$$max\{\sigma(E\Omega_{n-2}, E\Omega_{n-1}), \sigma(E\Omega_{n-1}, E\Omega_n)\} = \sigma(E\Omega_{n-2}, E\Omega_{n-1}).$$

$$\therefore N(S\Omega_{n-1}, S\Omega_n) \leq \sigma(E\Omega_{n-2}, E\Omega_{n-1}).$$

With the aid of (2.4), we acquire

$$\begin{aligned}\sigma(E\Omega_{n-1}, E\Omega_n) &\leq \delta(\sigma(E\Omega_{n-2}, E\Omega_{n-1}))\sigma(E\Omega_{n-2}, E\Omega_{n-1})\\ &< \frac{1}{\rho}\sigma(E\Omega_{n-2}, E\Omega_{n-1})\\ &< \sigma(E\Omega_{n-2}, E\Omega_{n-1}),\end{aligned} \tag{2.5}$$

which establishes that $\{\sigma(E\Omega_{n-1}, E\Omega_n)\}$ is decreasing.
Thus, $\exists\, \lambda \geq 0$ in order that $\lim_{n\to\infty} \sigma(E\Omega_{n-1}, E\Omega_n) = \lambda$. Next, we assert that $\lambda = 0$.
On contrary, we imagine that $\sigma(E\Omega_{n-1}, E\Omega_n) > 0$. Letting $n \to \infty$ in (2.4), we get

$$\frac{\lambda}{\rho} \leq \lim_{n\to\infty} \delta(\sigma(E\Omega_{n-1}, E\Omega_n))\lambda,$$

which indicates that $\sigma(E\Omega_{n-1}, E\Omega_n) \to 0$, a contradiction.
Consequently $\sigma(E\Omega_{n-1}, E\Omega_n) = 0$. Now, we exhibit that $\{S\Omega_n\}$ is a b-Cauchy sequence. On contrary, we imagine that $\{S\Omega_n\}$ is not a b-Cauchy sequence. So, there appear $\kappa > 0$, such that two subsequences $\{S\Omega_{n_q}\}$ and $\{S\Omega_{m_q}\}$ of $\{S\Omega_n\}$ satisfies

$$\sigma(S\Omega_{m_q}, S\Omega_{n_q}) \geq \kappa, \tag{2.6}$$

for $n_q > m_q > q$. Thus,

$$\sigma(S\Omega_{m_q}, S\Omega_{n_q-1}) < \kappa,$$

With the aid of triangle inequality and (2.6), we get

$$\begin{aligned}\kappa &\leq \sigma(S\Omega_{m_q}, S\Omega_{n_q})\\ &\leq \rho\sigma(S\Omega_{m_q}, S\Omega_{m_q+1}) + \rho\sigma(S\Omega_{m_q+1}, S\Omega_{n_q}).\end{aligned}$$

Letting $q \to \infty$, we acquire

$$\frac{\kappa}{\rho} \leq \lim_{q\to\infty} sup\sigma(S\Omega_{m_q+1}, S\Omega_{n_q}).$$

(2.5) indicates that

$$\sigma(S\Omega_n, S\Omega_{n+1}) < \sigma(S\Omega_{n-1}, S\Omega_n). \tag{2.7}$$

Imagine that $\exists\, m_0 \in \mathbb{Z}_+$ having

$$\frac{1}{2\rho}\sigma(S\Omega_{m_{q0}}, E\Omega_{m_{q0}}) > \sigma(S\Omega_{m_{q0}}, S\Omega_{n_{q0}-1})$$

and

$$\frac{1}{2\rho}\sigma(S\Omega_{m_{q0}+1}, E\Omega_{m_{q0}+1}) > \sigma(S\Omega_{m_{q0}+1}, S\Omega_{n_{q0}-1}).$$

With the assistance of (2.7), we acquire

$$\begin{aligned}\sigma(S\Omega_{m_{q0}}, S\Omega_{m_{q0}+1}) &\leq \rho(\sigma(S\Omega_{m_{q0}}, S\Omega_{n_{q0}-1}) + \sigma(S\Omega_{m_{q0}+1}, S\Omega_{n_{q0}-1}))\\ &< \rho\left(\frac{1}{2\rho}\sigma(S\Omega_{m_{q0}}, E\Omega_{m_{q0}}) + \frac{1}{2\rho}\sigma(S\Omega_{m_{q0}+1}, E\Omega_{m_{q0}+1})\right)\\ &= \frac{1}{2}\left(\sigma(S\Omega_{m_{q0}}, S\Omega_{m_{q0}+1}) + \sigma(S\Omega_{m_{q0}+1}, S\Omega_{m_{q0}+2})\right)\\ &\leq \frac{1}{2}\left(\sigma(S\Omega_{m_{q0}}, S\Omega_{m_{q0}+1}) + \sigma(S\Omega_{m_{q0}}, S\Omega_{m_{q0}+1})\right)\\ &= \sigma(S\Omega_{m_{q0}}, S\Omega_{m_{q0}+1}),\end{aligned}$$

which is a contradiction. So, our guesswork is wrong. Consequently,

$$\frac{1}{2\rho}\sigma(S\Omega_{m_q}, E\Omega_{m_q}) \leq \sigma(S\Omega_{m_q}, S\Omega_{n_{q-1}})$$

and

$$\frac{1}{2\rho}\sigma(S\Omega_{m_{q+1}}, E\Omega_{m_{q+1}}) \leq \sigma(S\Omega_{m_{q+1}}, S\Omega_{n_{q-1}}).$$

Let us imagine that

$$\frac{1}{2\rho}\sigma(S\Omega_{m_q}, E\Omega_{m_q}) \leq \sigma(S\Omega_{m_q}, S\Omega_{n_q-1}).$$

With the aid of Lemma 1.1, we acquire

$$\beta(S\Omega_{m_q}, S\Omega_{n_q-1}) \geq 1.$$

Now,

$$\rho.\frac{\kappa}{\rho} \leq \rho. \lim_{q\to\infty} sup\sigma(S\Omega_{m_q+1}, S\Omega_{n_q}) \leq \lim_{q\to\infty} sup\delta(N(S\Omega_{m_q}, S\Omega_{n_q-1})) \lim_{q\to\infty} supN(S\Omega_{m_q}, S\Omega_{n_q-1})$$
$$\leq \kappa \lim_{q\to\infty} sup\delta(N(S\Omega_{m_q}, S\Omega_{n_q-1}))$$

□

Theorem 2.2 *Let E and S be two mappings in a b-complete b-metric space $(\mathcal{H}, \sigma)$ (with parameter $\rho > 1$) and let E be an β-admissible function with regard to S. Imagine that $\exists\,\psi \in \Psi_b$ such that*

$$\frac{1}{2\rho}\sigma(S\Omega, E\Omega) \leq \sigma(S\Omega, S\mho) \Rightarrow \beta(S\Omega, S\mho)\sigma(E\Omega, E\Omega) \leq \psi(N(S\Omega, S\mho)), \tag{2.8}$$

where

$$N(S\Omega, S\mho) = max\left\{\sigma(S\Omega, S\mho), \frac{\sigma(S\Omega, E\Omega) + \sigma(S\mho, E\mho)}{2\rho}, \frac{\sigma(S\Omega, E\mho) + \sigma(S\mho, E\Omega)}{2\rho}\right\}$$

and Ψ_b be the class of all (b)-comparison mappings $\psi : [0, \infty) \to [0, \infty)$. Also imagine that followings assertions fulfil:
(i) $\beta(S\Omega_0, E\Omega_0) \geq 1$, for some $\Omega_0 \in \mathcal{H}$;
(ii) If $\beta(S\Omega_n, S\Omega_{n+1}) \geq 1$, for any sequence $\{S\Omega_n\}$ in $\mathcal{H}$ such that $S\Omega_n \to S\Omega$ when $n \to \infty$, then $\beta(S\Omega_n, S\Omega) \geq 1\,\forall\, n \in N \cup \{0\}$.
Then, S and E possess a coincident point.

In addition to the above assumptions, let $u, v \in C(E, S), \exists\, w \in X$ such that $\beta(Su, Sw) \geq 1$ and $\beta(Sv, Sw) \geq 1$ and E, S commute at their coincident point. Then, E and S possess a unique common fixed point.

Example 2.1 *Let $\mathcal{H} = \mathbf{R^2}$. Define $\beta : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ as*

$$\beta(S\Omega, S\mho) = \begin{cases} 1, & if\, \Omega, \mho \in \{(0,0), (10,0), (0,10), (10,12), (12,10)\} = W \\ 0, & otherwise. \end{cases}$$

We define σ on $\mathcal{H}$ as

$$\sigma((\Omega_1, \Omega_2), (\mho_1, \mho_2)) = (\Omega_1 - \mho_1)^2 + (\Omega_2 - \mho_2)^2.$$

Thus, $(\mathcal{H}, \sigma, 2)$ is a complete b-metric space.
Now, $E, S : \mathcal{H} \to \mathcal{H}$ and $\psi : [0, \infty) \to [0, \infty)$ be defined as

$$E(\Omega_1, \Omega_2) = \begin{cases} (\Omega_1, 0) & if\, \Omega_1 \leq \Omega_2\, and\, \Omega_1, \Omega_2 \in W \\ (0, \Omega_2) & if\, \Omega_1 > \Omega_2\, and\, \Omega_1, \Omega_2 \in W \\ (2\Omega_1^2, 8\Omega_2^3) & if\, \Omega_1.\Omega_2 \in \mathbf{R^2} \setminus W, \end{cases}$$

$$S(\Omega_1, \Omega_2) = \left(\frac{\Omega_1^2}{2}, \frac{\Omega_2^2}{2}\right) \text{ and } \psi(s) = \frac{3}{5}s,$$

$\forall(\Omega_1, \Omega_2) \in \mathbf{R^2}$.
Let us imagine that $\frac{1}{4}\sigma(S\Omega, E\Omega) \leq \sigma(S\Omega, S\mho)\, and\, \beta(S\Omega, S\mho) \geq 1$.
Now, following cases arise:
• *Let $(\Omega, \mho) = ((0, 0), (10, 0))$, then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1 \times \sigma(E\Omega, E\mho) = \sigma(E(0, 0), E(10, 0)) = 0. \tag{2.9}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(0,0), E(10,0)) \\ &= \psi(\sigma(0,0),(50,0)) \\ &= \psi((0-50)^2+(0-0)^2 \\ &= \psi(2500) \\ &= \frac{3}{5}\times 2500 \\ &= 1500. \end{aligned} \tag{2.10}$$

(2.9) and (2.10) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((0,0),(0,10)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(0,0), E(0,10)) = 0. \tag{2.11}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(0,0), E(0,10)) \\ &= \psi(\sigma(0,0),(0,50)) \\ &= \psi((0-0)^2+(0-50)^2 \\ &= \psi(2500) \\ &= \frac{3}{5}\times 2500 \\ &= 1500. \end{aligned} \tag{2.12}$$

(2.11) and (2.12) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((0,0),(10,12)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(0,0), E(10,12)) = \sigma((0,0),(10,0)) = 100. \tag{2.13}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(0,0), E(10,12)) \\ &= \psi(\sigma(0,0),(50,72)) \\ &= \psi((0-50)^2+(0-72)^2 \\ &= \psi(2500+5184) \\ &= \frac{3}{5}\times 7684 \\ &= 4610.4. \end{aligned} \tag{2.14}$$

(2.13) and (2.14) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((0,0),(12,10)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(0,0), E(12,10)) = \sigma((0,0),(0,10)) = 100. \tag{2.15}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(0,0), E(12,10))\\ &= \psi(\sigma(0,0),(72,50))\\ &= \psi((0-72)^2+(0-50)^2\\ &= \psi(5184+2500)\\ &= \frac{3}{5}\times 7684\\ &= 4610.4. \end{aligned} \tag{2.16}$$

(2.15) and (2.16) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((10,0),(0,10)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(10,0), E(0,10)) = \sigma((0,0),(0,0)) = 0. \tag{2.17}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(10,0), E(0,10))\\ &= \psi(\sigma(50,0),(0,50))\\ &= \psi((50-0)^2+(0-50)^2\\ &= \psi(2500+2500)\\ &= \frac{3}{5}\times 5000\\ &= 3000. \end{aligned} \tag{2.18}$$

(2.17) and (2.18) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((10,0),(12,10)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(10,0), E(12,10)) = \sigma((0,0),(0,10)) = 100. \tag{2.19}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(10,0), E(12,10))\\ &= \psi(\sigma(50,0),(72,50))\\ &= \psi((50-72)^2+(0-50)^2\\ &= \psi(484+2500)\\ &= \frac{3}{5}\times 2984\\ &= 1790.4. \end{aligned} \tag{2.20}$$

(2.19) and (2.20) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \leq \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((10,0),(10,12)),$ *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times\sigma(E\Omega, E\mho) = \sigma(E(10,0), E(10,12)) = \sigma((0,0),(10,0)) = 100. \tag{2.21}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(10,0), E(10,12))\\ &= \psi(\sigma(50,0),(50,72))\\ &= \psi((50-50)^2+(0-72)^2\\ &= \psi(5184)\\ &= \frac{3}{5}\times 5184\\ &= 3110.4. \end{aligned} \tag{2.22}$$

(2.21) and (2.22) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \le \psi(N(S\Omega, S\mho).$$

- *Let* $(\Omega, \mho) = ((0,10),(12,10))$, *then*

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) = 1\times \sigma(E\Omega, E\mho) = \sigma(E(0,10), E(12,10)) = \sigma((0,0),(0,10)) = 100. \tag{2.23}$$

Now,

$$\begin{aligned}\psi(\sigma(E\Omega, E\mho) &= \psi(\sigma(E(0,10), E(12,10))\\ &= \psi(\sigma(0,50),(72,50))\\ &= \psi((0-72)^2+(50-50)^2\\ &= \psi(5184)\\ &= \frac{3}{5}\times 5184\\ &= 3110.4. \end{aligned} \tag{2.24}$$

(2.23) and (2.24) indicate that

$$\beta(S\Omega, S\mho)\sigma(E\Omega, E\mho) \le \psi(N(S\Omega, S\mho).$$

Thus, all the conditions of Theorem 2.2 satisfied, which indicates that (0, 0) is common fixed point of E and S.

Consequences

Now, we prove some results for contractive functions on b-metric space endowed with a graph.

Definition 2.1 *Let* (X, σ) *be a b-metric space endowed with graph G. The mapping* $E : X \to X$ *is* $G-\psi$*-Suzuki kind rational contraction with respect to S if*
$(Sx, Sy) \in F(G) \Rightarrow (Ex, Ey) \in F(G)$,

$$\frac{1}{2\rho}\sigma(S\Omega, E\Omega) \le \sigma(S\Omega, S\mho) \Rightarrow \sigma(E\Omega, E\Omega) \le \psi(N(\Omega, \mho))$$

where

$$N(S\Omega, S\mho) = max\left\{\sigma(S\Omega, S\mho), \frac{\sigma(S\Omega, E\Omega)+\sigma(S\mho, E\mho)}{2\rho}, \frac{\sigma(S\Omega, E\mho)+\sigma(S\mho, E\Omega)}{2\rho}\right\},$$

$\forall(\Omega, \mho) \in F(G)$ *and* $\psi \in \Psi_b$.

Theorem 2.3 *Let E and S be two mappings in a b-complete b-metric space* $(\mathcal{H}, \sigma)$ *(with parameter* $\rho > 1$*). Let E be a triangular* β*-admissible function with regard to S and* $E : X \to X$ *is* $G-\psi$*-Suzuki kind rational contraction with respect to S. Suppose that the following conditions hold:*
(i) $\exists\, \Omega_0 \in X$ *such that* $(S\Omega_0, E\Omega_0) \in F(G)$*;*
(ii) If $(S\Omega_n, S\Omega_{n+1}) \in F(G)$ *for any sequence* $\{S\Omega_n\}$ *in* $\mathcal{H}$ *such that* $S\Omega_n \to S\Omega$ *when* $n \to \infty$*, then* $(S\Omega_n, S\Omega) \in F(G)\, \forall\, n \in N \cup \{0\}$.
Then, S and E possess a coincident point.

Proof: We formulize $\beta : X \times X \to [0, \infty)$ as

$$\beta(S\Omega, S\Omega) = \begin{cases} 1, & \text{if } (S\Omega, S\Omega) \in F(G) \\ 0, & otherwise. \end{cases}$$

Clearly, E is triangular β-admissible function with regard to S. Also $E : X \to X$ is $G - \psi$-Suzuki kind rational contraction with respect to S. From (i), $\exists\, \Omega_0 \in X$ such that $(S\Omega_0, E\Omega_0) \in F(G)$. Thus, $\beta(S\Omega_0, E\Omega_0) \geq 1$. Consequently, all assumptions of Theorem 2.2 is fulfilled, which indicates that E and S possess coincident point. □

Definition 2.2 *Let (X, σ) be a b-metric space endowed with graph G. The mapping $E : X \to X$ is G-Suzuki kind rational contraction with respect to S if*
$(Sx, Sy) \in F(G) \Rightarrow (Ex, Ey) \in F(G)$,

$$\frac{1}{2\rho}\sigma(S\Omega, E\Omega) \leq \sigma(S\Omega, S\mho) \Rightarrow \rho\sigma(E\Omega, E\Omega) \leq \delta(N(S\Omega, S\mho))N(S\Omega, S\mho),$$

where

$$N(S\Omega, S\mho) = max\left\{\sigma(S\Omega, S\mho), \frac{\sigma(S\Omega, E\Omega) + \sigma(S\mho, E\mho)}{2\rho}, \frac{\sigma(S\Omega, E\mho) + \sigma(S\mho, E\Omega)}{2\rho}\right\},$$

$\forall (\Omega, \mho) \in F(G)$.

Theorem 2.4 *Let E and S be two mappings in a b-complete b-metric space $(\mathcal{H}, \sigma)$ (with parameter $\rho > 1$). Let E be a triangular β-admissible function with regard to S and $E : X \to X$ is G-Suzuki kind rational contraction with respect to S. Suppose that the following conditions hold:*
(i) $\exists\, \Omega_0 \in X$ such that $(S\Omega_0, E\Omega_0) \in F(G)$;
(ii) If $(S\Omega_n, S\Omega_{n+1}) \in F(G)$ for any sequence $\{S\Omega_n\}$ in $\mathcal{H}$ such that $S\Omega_n \to S\Omega$ when $n \to \infty$, then $(S\Omega_n, S\Omega) \in F(G)\, \forall\, n \in N \cup \{0\}$.
Then, S and E possess a coincident point.

Proof: The result follows from Theorem 2.2. □

Now, we prove contractive mapping on partially ordered b-metric spaces as follows:

Theorem 2.5 *Let E and S be two mappings in an ordered b-complete b-metric space $(\mathcal{H}, \sigma, \preceq)$ (with parameter $\rho > 1$). Imagine that $\exists\, \delta \in \mathcal{K}$ such that $\forall\, \Omega, \mho \in \mathcal{H}$*

$$\frac{1}{2\rho}\sigma(S\Omega, E\Omega) \leq \sigma(S\Omega, S\mho) \Rightarrow \rho\beta(S\Omega, S\mho)\sigma(E\Omega, E\Omega) \leq \delta(N(S\Omega, S\mho))N(S\Omega, S\mho),$$

where

$$N(S\Omega, S\mho) = max\left\{\sigma(S\Omega, S\mho), \frac{\sigma(S\Omega, E\Omega) + \sigma(S\mho, E\mho)}{2\rho}, \frac{\sigma(S\Omega, E\mho) + \sigma(S\mho, E\Omega)}{2\rho}\right\}.$$

Also imagine that followings assertions fulfil:
(i) $\exists\, \Omega_0 \in \mathcal{H}$ such that $S\Omega_0 \preceq E\Omega_0$;
(ii) If $S\Omega_n \preceq S\Omega_{n+1}, \forall n \in \mathbb{N} \cup \{0\}$ such that $S\Omega_n \to S\Omega$, when $n \to \infty$, we have $S\Omega_n \preceq S\Omega$. Then, S and E possess a coincident point.

Proof: We formulize $\beta : X \times X \to [0, \infty)$ as

$$\beta(S\Omega, S\Omega) = \begin{cases} 1, & \text{if } S\Omega \preceq S\mho \\ 0, & otherwise. \end{cases}$$

Clearly, E is triangular β-admissible function with regard to S. Also we can show that all assumptions of Theorem 2.2 are fulfilled, which indicates that E and S possess coincident point. □

3. Conclusion

In this paper, various suzuki kind fixed point theorems for non-linear contractions of a pair of functions are established with the aid of triangular β-admissible function. Also results are proved for two functions in the framework of b-metric space endowed with graph and partially ordered b-metric space. The presented results improve and enhance several existing fixed point results in the literature.

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