Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–9.
ISSN-0037-8712 in press
doi:10.5269/bspm.62383

Generalizations of 2-absorbing Primal Ideals in Commutative Rings

Ameer Jaber and Rania Shaqbou'a

ABSTRACT: Let R be a commutative ring with unity $(1 \neq 0)$. A proper ideal of R is an ideal I of R such that $I \neq R$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper ideals of R. In this paper we introduce the concept of a ϕ-2-absorbing primal ideal which is a generalization of a ϕ-primal ideal. An element $a \in R$ is defined to be ϕ-2-absorbing prime to I if for any $r, s, t \in R$ with $rsta \in I \setminus \phi(I)$, then $rs \in I$ or $rt \in I$ or $st \in I$. An element $a \in R$ is not ϕ-2-absorbing prime to I if there exist $r, s, t \in R$, with $rsta \in I \setminus \phi(I)$, such that $rs, rt, st \in R \setminus I$. We denote by $\nu_\phi(I)$ the set of all elements in R that are not ϕ-2-absorbing prime to I. We define a proper ideal I of R to be a ϕ-2-absorbing primal if the set $\nu_\phi(I) \cup \phi(I)$ forms an ideal of R. Many results concerning ϕ-2-absorbing primal ideals and examples of ϕ-2-absorbing primal ideals are given.

Key Words: ϕ-2-absorbing ideal, ϕ-primal ideal.

Contents

1. Introduction

In this paper, we study ϕ-2-absorbing primal ideals in commutative rings with unity, which are generalization of ϕ-primal ideals. Many authors gave a generalization of primal ideals for example in [6] A. Y. Darani defined that if R is a commutative ring with unity and I is a proper ideal from R, then $a \in R$ is ϕ-prime to I if $ra \in I \setminus \phi(I)$, for some $r \in R$, then $r \in I$. Also he defined that $a \in R$ is not ϕ-prime to I if there exists $r \in R \setminus I$ such that $ra \in I \setminus \phi(I)$. Let $S_\phi(I)$ be the set of all elements a in R that are not ϕ-prime to I. In [6] A. Y. Darani defined I to be a ϕ-primal ideal in R if $S_\phi(I) \cup \phi(I)$ forms an ideal in R. The concept of 2-absorbing ideals, which is a generalization of the concept prime ideals, was introduced by Badawi in [3] and studied in [1] and [10]. Also the concept of 2-absorbing primary ideals was introduced by Badawi, Tekir and Yetkin in [5] and the concept of the generalizations of 2-absorbing primary ideals was introduced by Badawi, Tekir, Ugurlu, Ulucak and Celikel in [4]. Moreover the concept of 2- absorbing primal ideals was introduced by A. Jaber and H. Obiedat in [9] and the concept of weakly 2-absorbing primal ideals was introduced by A. Jaber in [8].

Let I be a proper ideal of R, an element $a \in R$ is defined to be 2-absorbing prime (weakly 2-absorbing prime) to I if for any $r, s, t \in R$ with $rsta \in I$ $(0 \neq rsta \in I)$, then $rs \in I$ or $rt \in I$ or $st \in I$. An element $a \in R$ is not 2-absorbing prime (not weakly 2-absorbing prime) to I if there exist $r, s, t \in R$, with $rsta \in I$ $(0 \neq rsta \in I)$, such that $rs, rt, st \in R \setminus I$. Recall from [9,8] that I is a 2-absorbing primal ideal (a weakly 2-absorbing primal ideal) of R if $\nu(I)$ $(\nu_0(I) \cup \{0\})$ forms an ideal of R, where $\nu(I)$ $(\nu_0(I))$ is denoted by the set of all elements in R that are not 2-absorbing prime (not weakly 2-absorbing prime) to I. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper ideals of R. An element $a \in R$ is defined to be ϕ-2-absorbing prime to I if for any $r, s, t \in R$ with $rsta \in I \setminus \phi(I)$, then $rs \in I$ or $rt \in I$ or $st \in I$. In this paper we generalize the idea of weakly 2-absorbing primal ideals to the idea of ϕ-2-absorbing primal ideals as follows: an element $a \in R$ is not ϕ-2-absorbing prime to I if there exist $r, s, t \in R$, with $rsta \in I \setminus \phi(I)$, such that $rs, rt, st \in R \setminus I$. We denote by $\nu_\phi(I)$ the set of

2010 *Mathematics Subject Classification:* 13A15, 13C05.
Submitted February 06, 2022. Published September 17, 2022

all elements in R that are not ϕ-2-absorbing prime to I. In this paper we define a proper ideal I of R to be a ϕ-2-absorbing primal if the set $\nu_\phi(I) \cup \phi(I)$ forms an ideal of R.
In this paper some basic properties of ϕ-2-absorbing primal ideals are studied and classified, and some examples are given. Also the relation between 2-absorbing primal ideals and ϕ-2-absorbing primal ideals are studied.

2. ϕ-2-Absorbing Primal ideals

Let R be a commutative ring with unity $(1 \neq 0)$. Recall that if $\psi_1, \psi_2 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ are functions of ideals of R, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{I}(R)$. In the next example we give some famous functions of ideals $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ and their corresponding ϕ-2-absorbing primal ideals.

Example 2.1.

$\phi_\emptyset$	$\phi_\emptyset(I) = \emptyset \ \ \forall\, I \in \mathfrak{I}(R)$	defines a 2-absorbing primal ideal.
ϕ_0	$\phi_0(I) = \{0\} \ \ \forall\, I \in \mathfrak{I}(R)$	defines a weakly 2-absorbing primal ideal.
ϕ_2	$\phi_2(I) = I^2 \ \ \forall\, I \in \mathfrak{I}(R)$	defines an almost 2-absorbing primal ideal.
ϕ_n	$\phi_n(I) = I^n \ \ \forall\, I \in \mathfrak{I}(R)$	defines an n-almost 2-absorbing primal ideal.
ϕ_ω	$\phi_\omega(I) = \cap_{n=1}^{\infty} I^n \ \ \forall\, I \in \mathfrak{I}(R)$	defines an ω-2-absorbing primal ideal.
ϕ_1	$\phi_1(I) = I \ \ \forall\, I \in \mathfrak{I}(R)$	defines any ideal.

Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Let R be a commutative ring with unity $(1 \neq 0)$ and let I be a proper ideal of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper ideals of R. An element $a \in R$ is ϕ-2-absorbing prime to I if for any $r, s, t \in R$ with $rsta \in I \setminus \phi(I)$, then rs or rt or st is in I. An element $a \in R$ is not ϕ-2-absorbing prime to I if there exist $r, s, t \in R$, with $rsta \in I \setminus \phi(I)$, such that rs, rt and st are in $R \setminus I$. We denote by $\nu_\phi(I)$ the set of all elements in R that are not ϕ-2-absorbing prime to I.
It is clear that every ϕ-primal ideal of a ring R is a ϕ-2-absorbing primal ideal of R. If $R = \mathbb{Z}_{16}$, $I = \{0, 8\}$ and $\phi = \phi_0$. Then one can easily see that $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$ since $2.2.2 \neq 0 \in I$ and $4 \notin I$. So I is a ϕ_0-2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}_{16}$. Also one can easily see that $S_0(I) \cup \{0\} = 2\mathbb{Z}_{16} \neq \nu_0(I) \cup \{0\}$. Therefore, $I = \{0, 8\}$ is a ϕ_0-primal and ϕ_0-2-absorbing primal ideal of $\mathbb{Z}_{16}$ with $S_0(I) \neq \nu_0(I)$. The following are two examples of nonzero ϕ_0-2-absorbing primal ideals that are not ϕ_0-primal ideals.

Example 2.2. (1) Let $R = \mathbb{Z}$ and let $I = 30\mathbb{Z}$. Then I is a $\phi_0 - 2$-absorbing primal ideal of $\mathbb{Z}$ with $\nu_0(I) \cup \{0\} = \mathbb{Z}$, since $(2)(3)(5) = 30 \in I$ and $(2)(3) = 6 \notin I$, $(2)(5) = 10 \notin I$ and $(3)(5) = 15 \notin I$. On the other hand I is not a ϕ_0-primal ideal in $\mathbb{Z}$, because $2, 3 \in S_0(I)$ but $1 \notin S_0(I)$. Note that if $1 \in S_0(I)$, then there exists $r \notin I$ with $1.r = r \in I$, a contradiction.
(2) Let $R = \mathbb{Z}[x, y, z]$ and let $I = xyzR$. Then I is a proper ideal of R and since $xyz \neq 0 \in I$ with xy, xz, and yz are in $R \setminus I$, $\nu_0(I) \cup \{0\} = R$. That is I is a $\phi_0 - 2$-absorbing primal ideal of R.
On the other hand, since $xyz \neq 0 \in I$ and $yz \in R \setminus I$, $x \in S_0(I)$. Similarly, $y \in S_0(I)$. We show that $x + y$ can't be in $S_0(I)$. If there exists $f(x, y, z) \in \mathbb{Z}[x, y, z]$ with $(x + y)f(x, y, z) \neq 0 \in I$, then xyz divides $(x + y)f(x, y, z)$ and since x divides xyz, x divides $(x + y)f(x, y, z)$ but x does not divide $x + y$, so x must divide $f(x, y, z)$. Similarly, y divides $f(x, y, z)$ and z divides $f(x, y, z)$. Therefore, xyz divides $f(x, y, z)$ which implies that $f(x, y, z) \in I$, so $x + y \notin S_0(I)$ and hence I is not a ϕ_0-primal ideal of R.

Theorem 2.3. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function and let I be a proper ideal of R such that I is a ϕ-2-absorbing primal ideal of R with $\nu_\phi(I) \cup \phi(I) \neq R$. Then $\nu_\phi(I) \cup \phi(I)$ is a ϕ-prime ideal of R

Proof. Since $\phi(I) \subseteq \phi(\nu_\phi(I) \cup \phi(I))$, then it is easy to check that

$$\nu_\phi(I) \cup \phi(I) - \phi(\nu_\phi(I) \cup \phi(I)) \subseteq \nu_\phi(I) \cup \phi(I) - \phi(I) = \nu_\phi(I).$$

Now, let $a, b \in R$ such that $ab \in \nu_\phi(I) \cup \phi(I) - \phi(\nu_\phi(I) \cup \phi(I))$, then $ab \in \nu_\phi(I)$. Hence there exist $r, s, t \in R$ with $rst(ab) \in I \setminus \phi(I)$ such that $rs, rt, st \in R \setminus I$. Assume that $a \notin \nu_\phi(I) \cup \phi(I)$. We must show that $b \in \nu_\phi(I) \cup \phi(I)$. Since $r(sb)ta \in I \setminus \phi(I)$ and $a \notin \nu_\phi(I)$, $rsb \in I$ or $rt \in I$ or $sbt \in I$. But

$rt \in R \setminus I$, so we must have that $rsb \in I$ or $sbt \in I$. If $rsb \in I$, then $rsb \in I \setminus \phi(I)$, since $rsb \notin \phi(I)$, hence $b \in \nu_\phi(I) \subseteq \nu_\phi(I) \cup \phi(I)$. Similarly, if $stb \in I$, then $b \in \nu_\phi(I) \subseteq \nu_\phi(I) \cup \phi(I)$. Therefore, $\nu_\phi(I) \cup \phi(I)$ is a ϕ-prime ideal of R. □

For example for $\phi = \phi_0$. Let $I = 4\mathbb{Z}$ be a proper ideal of $\mathbb{Z}$ with $\nu_{\phi_0}(I) \cup \phi_0(I) = 2\mathbb{Z}$. Then I is a ϕ_0-2-absorbing primal ideal of $\mathbb{Z}$ and $\nu_\phi(I) \cup \phi(I) = 2\mathbb{Z}$ is ϕ_0-prime ideal of $\mathbb{Z}$. But if $I = 6\mathbb{Z}$, then I is not a ϕ_0-2-absorbing primal ideal of $\mathbb{Z}$, since $(2)(3) \in I \setminus \phi_0(I)$ and $2, 3 \notin I$, $2, 3 \in \nu_{\phi_0}(I)$. Therefore, if $\nu_{\phi_0}(I) \cup \phi_0(I)$ is an ideal of $\mathbb{Z}$, then $1 \in \nu_{\phi_0}(I)$ which implies that there exist $r, s, t \in \mathbb{Z} \setminus 6\mathbb{Z}$ such that $rst \in 6\mathbb{Z} \setminus \phi_0(6\mathbb{Z})$ with $rs, rt, st \notin 6\mathbb{Z}$, but since 6 divides rst, 2 must divide r or s or t and 3 must divide r or s or t. So 6 must divide rs or st or rt which is a contradiction.

Definition 2.4. *Let R be a commutative ring with unity ($1 \neq 0$) and let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Suppose that I is a proper ideal of R such that I is a ϕ-2-absorbing primal ideal of R. Let $r, s, t \in R$, then (r, s, t) is called a ϕ-triple of I if $rst \in \phi(I)$ with $rs, rt, st \in R \setminus I$.*

The following five results on ϕ-2-absorbing primal ideals over R are generalizations to the results on weakly 2-absorbing primal ideals of R proved in [8].

Theorem 2.5. Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$. If (r, s, t) is a ϕ-triple of I, then
(1) $rsI \subseteq \phi(I)$, $rtI \subseteq \phi(I)$ and $stI \subseteq \phi(I)$;
(2) $rI^2 \subseteq \phi(I)$, $sI^2 \subseteq \phi(I)$ and $tI^2 \subseteq \phi(I)$.

Proof. (1) If $rsI \nsubseteq \phi(I)$, then there exists $a \in I$ such that $rsa \in I \setminus \phi(I)$. So, $rs(t+a) = rst + rsa \in I \setminus \phi(I)$ with $rs, r(t+a), s(t+a) \in R \setminus I$ implies that $1 \in \nu_\phi(I)$, a contradiction. Therefore, $rsI \subseteq \phi(I)$. Similarly, $rtI \subseteq \phi(I)$ and $stI \subseteq \phi(I)$.
(2) Suppose $rI^2 \nsubseteq \phi(I)$. Then there exist $a, b \in I$ such that $rab \notin \phi(I)$. So, $r(s + a)(t + b) = rst + rsb + rat + rab \in I \setminus \phi(I)$, since $rst, rsb, rat \in \phi(I)$, with $r(s + a), r(t + b), (s + a)(t + b) \in R \setminus I$ implies that $1 \in \nu_\phi(I)$, a contradiction. Therefore, $rI^2 \subseteq \phi(I)$. Similarly, $sI^2 \subseteq \phi(I)$ and $tI^2 \subseteq \phi(I)$. □

Let I be a proper ideal of R such that I is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$. If I is a 2-absorbing primal ideal of R with $\nu(I) = R$. Then by using Theorem 2.5, we have the following result.

Theorem 2.6. Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$ such that $\nu(I) = R$. Then $I^3 \subseteq \phi(I)$.

Proof. Since $\nu(I) = R$, $1 \in \nu(I)$. Hence there exist $r, s, t \in R$ with $rst \in \phi(I)$ such that $rs, rt, st \in R \setminus I$. Thus, (r, s, t) is a ϕ-triple of I, since if $rst \in I \setminus \phi(I)$, then $1 \in \nu_\phi(I)$, a contradiction. Suppose that $I^3 \nsubseteq \phi(I)$. Then there exist $a, b, c \in I$ such that $abc \notin \phi(I)$. Since, by Theorem 2.5, $rst, rsc, rbt, rbc, ast, asc, abt \in \phi(I)$, $(r+a)(s+b)(t+c) = rst+rsc+rbt+rbc+ast+asc+abt+abc \in I \setminus \phi(I)$, and since $1 \notin \nu_\phi(I)$, $(r + a)(s + b) \in I$ or $(r + a)(t + c) \in I$ or $(s + b)(t + c) \in I$. Hence we have either $rs \in I$ or $rt \in I$ or $st \in I$, a contradiction. Therefore, $I^3 \subseteq \phi(I)$. □

We recall that the radical of an ideal I in a commutative ring R, denoted by $\mathrm{Rad}(I)$, is defined as

$$\mathrm{Rad}(I) = \{r \in R \ : \ r^n \in I \text{ for some positive integer } n\}.$$

By Theorem 2.6 we have the following result.

Corollary 2.7. Let R be a commutative ring with unity ($1 \neq 0$) and let I be a proper ideal of R. Suppose that I is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$. If $\nu(I) = R$, then $I \subseteq \mathrm{Rad}(\phi(I))$.

Theorem 2.8. Let I be a proper ideal of R. Suppose that I is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$ such that $\nu(I) = R$. Then
(1) If $a \in \mathrm{Rad}(\phi(I))$, then either $a^2 \in I$ or $a^2I \subseteq \phi(I)$ and $aI^2 \subseteq \phi(I)$;
(2) $(\mathrm{Rad}(\phi(I)))^2 I^2 \subseteq \phi(I)$.

Proof. (1) Let $a \in \text{Rad}(\phi(I))$. First, we show that if $a^2I \not\subseteq \phi(I)$, then $a^2 \in I$. Now assume that $a^2I \not\subseteq \phi(I)$. Let $i \in I$ such that $a^2i \notin \phi(I)$ and suppose that $n > 0$ is the smallest positive integer such that $a^n \in \phi(I)$. Then $n \geq 3$ and we have $a^2(i + a^{n-2}) \in I \setminus \phi(I)$, since $1 \notin \nu_\phi(I)$, $a^2 \in I$ or $a^{n-1} \in I$. If $a^2 \in I$, then done. If $a^{n-1} \in I$, then $a^2a^{n-3} \in I \setminus \phi(I)$ again since $1 \notin \nu_\phi(I)$, $a^{n-2} \in I$. Continuing this procedure to arrive at $a^2 \in I$. Therefore for each $a \in \text{Rad}(\phi(I))$ we have either $a^2 \in I$ or $a^2I \subseteq \phi(I)$. Now assume that $b^2 \notin I$ for some $b \in \text{Rad}(\phi(I))$. Then $b^2I \subseteq \phi(I)$. We show that $bI^2 \subseteq \phi(I)$. If $bI^2 \not\subseteq \phi(I)$, then there exist $i_1, i_2 \in I$ such that $bi_1i_2 \notin \phi(I)$. Let $m > 0$ be the smallest positive integer such that $b^m \in \phi(I)$, then $m \geq 3$ since $b^2 \notin I$. Hence $b(b + i_1)(b^{m-2} + i_2) = b^m + b^2i_2 + b^{m-1}i_1 + bi_1i_2 \in I \setminus \phi(I)$ and since $1 \notin \nu_\phi(I)$, $b(b + i_1) \in I$ which implies that $b^2 \in I$ (a contradiction) or $b(b^{m-2} + i_2) \in I$ which implies that $b^{m-1} \in I$ (a contradiction). Therefore, $bI^2 \subseteq \phi(I)$.
(2) Let $r, s \in \text{Rad}(\phi(I))$. If $r^2 \notin I$ or $s^2 \notin I$, then, by (1), $(rs)I^2 \subseteq \phi(I)$. Therefore we may assume that $r^2 \in I$ and $s^2 \in I$. So, $rs(r + s) \in I$. If $(r, s, r + s)$ is a ϕ-triple of I, then, by Theorem 2.5(1), $(rs)I \subseteq \phi(I)$ and hence $(rs)I^2 \subseteq \phi(I)$. If $rs(r + s) \in I \setminus \phi(I)$, then $rs \in I$ since $1 \notin \nu_\phi(I)$. So, by Theorem 2.6, $(rs)I^2 \subseteq I^3 \subseteq \phi(I)$. □

Corollary 2.9. Let R be a commutative ring with unity $(1 \neq 0)$ and let A, B, C be proper ideals of R. Suppose that A, B, C are ϕ-2-absorbing primal ideals of R with $1 \notin \nu_\phi(A) \cup \nu_\phi(B) \cup \nu_\phi(C)$ such that $\nu(A) = \nu(B) = \nu(C) = R$. If $\text{Rad}(\phi(B)) \subseteq \text{Rad}(\phi(A))$ and $\text{Rad}(\phi(C)) \subseteq \text{Rad}(\phi(A))$, then $A^2BC \subseteq \phi(A)$ and $A^2B^2 \subseteq \phi(A)$ and $A^2C^2 \subseteq \phi(A)$.

Proof. By Corollary 2.7, $B \subseteq \text{Rad}(\phi(B))$ and $C \subseteq \text{Rad}(\phi(C))$. Therefore,

$$A^2BC \subseteq A^2(\text{Rad}(\phi(B)))(\text{Rad}(\phi(C))) \subseteq A^2(\text{Rad}(\phi(A)))^2$$

and by Theorem 2.8(2), $A^2(\text{Rad}(\phi(A)))^2 \subseteq \phi(A)$. Also, $A^2B^2 \subseteq A^2(\text{Rad}(\phi(B)))^2 \subseteq A^2(\text{Rad}(\phi(A)))^2$, so again by Theorem 2.8(2), $A^2B^2 \subseteq \phi(A)$. Similarly, $A^2C^2 \subseteq \phi(A)$. □

In the next result we give a condition on a ϕ-2-absorbing primal ideal of R to be 2-absorbing primal ideal of R.

Theorem 2.10. Let R be a commutative ring with unity $(1 \neq 0)$ and let I be a proper ideal of R. If I is a ϕ-2-absorbing primal ideal of R with $I^2 \not\subseteq \phi(I)$, then I is a 2-absorbing primal ideal of R

Proof. If $1 \in \nu(I)$, then $\nu(I) = R$ which implies that I is a 2-absorbing primal ideal of R. Therefore we may assume that $1 \notin \nu(I)$. One can easily get that $\nu_\phi(I) \cup \phi(I)$ is an ideal of R, we show that $\nu(I)$ is an ideal of R by proving that $\nu(I) = \nu_\phi(I) \cup \phi(I)$. It is clear that $\nu_\phi(I) \cup \phi(I) \subseteq \nu(I)$. Conversely, let $a \in \nu(I)$, then there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $(rst)a \in I$. If $(rst)a \notin \phi(I)$, then $a \in \nu_\phi(I)$. So we may assume that $rsta \in \phi(I)$. If $(rst)I \not\subseteq \phi(I)$, then there exists $c \in I$ such that $rstc \notin \phi(I)$. Therefore, $(rst)(a + c) \in I \setminus \phi(I)$ which implies that $a + c \in \nu_\phi(I)$ and hence $a \in \nu_\phi(I)$, since $c \in \nu_\phi(I)$. Therefore we may assume that $(rst)I \subseteq \phi(I)$. If $rst \in I$, then $1 \in \nu(I)$ which is a contradiction. Therefore we may assume that $rst \notin I$. Since $I^2 \not\subseteq \phi(I)$, there exist $x, y \in I$ such that $xy \notin \phi(I)$. Hence, $(a + y)(trs + x) = atrs + ax + ytrs + xy \in I$ with $atrs, ytrs \in \phi(I)$. If $ax + xy \in I \setminus \phi(I)$, and since $trs + x \notin I$, then $a + y \in \nu_\phi(I)$ which implies that $a \in \nu_\phi(I)$, since $y \in \nu_\phi(I)$. But, if $ax + xy \in \phi(I)$, then $ax \in I \setminus \phi(I)$ which implies that $a(x + trs) = ax + atrs \in I \setminus \phi(I)$, so $a \in \nu_\phi(I)$, since $trs + x \notin I$. Thus, $\nu(I) = \nu_\phi(I) \cup \phi(I)$. □

We have to remark that if a proper ideal I of R is a ϕ-2-absorbing primal ideal of R with $I^2 \not\subseteq \phi(I)$, and $1 \notin \nu(I)$, then $\nu_\phi(I) \cup \phi(I)$ i a prime ideal of R since, by Theorem 2.10, $\nu(I) = \nu_\phi(I) \cup \phi(I)$.

We recall that if R and S are are commutative rings with unities and P, Q are ϕ-primal ideals of R, S (respectively), then $P \times S$ and $R \times Q$ are ϕ-primal ideals of $R \times S$.

Theorem 2.11. Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities. Let $\phi = \psi_1 \times \psi_2 : \mathfrak{I}(R \times S) \to \mathfrak{I}(R \times S) \cup \{\emptyset\}$ be any function, where $\psi_1 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$, $\psi_2 : \mathfrak{I}(S) \to \mathfrak{I}(S) \cup \{\emptyset\}$ are any functions such that $\psi_2(S) = S$. Let I be a proper ideal of R with

$I \times S \not\subseteq \mathrm{Rad}(\phi(R \times S))$. Then the following statements are equivalent.
(1) $I \times S$ is a ϕ-2-absorbing primal ideal of $R \times S$;
(2) $I \times S$ is a 2-absorbing primal ideal of $R \times S$;
(3) I is a 2-absorbing primal ideal of R.

Proof. ($1 \to 2$) Since $I \times S \not\subseteq \mathrm{Rad}(\phi(R \times S))$, by Corollary 2.7 $\nu(I \times S) \neq R \times S$. To prove that $I \times S$ is a 2-absorbing primal ideal of $R \times S$ we must show that $\nu(I) = \nu_{\psi_1}(I) \cup \psi_1(I)$. It is clear that $\nu_{\psi_1}(I) \cup \psi_1(I) \subseteq \nu(I)$. Conversely, let $a \in \nu(I)$ and let $(rst)a \in I$ for some $r, s, t \in R$ with $rs, rt, st \in R \backslash I$. Since $1 \notin \nu(I)$ and $rs \notin I$, $rta \in I$ or $sta \in I$. If $rta \in I$, then $ra \in I$ or $ta \in I$ since $1 \notin \nu(I)$ and $rt \notin I$. If $ra \in I$, then $a \in I$ since $r \notin I$ and $1 \notin \nu(I)$. Similarly, if $ta \in I$, then $a \in I$. Also, if $sta \in I$, then $a \in I$. If $a \in \psi_1(I)$, then $a \in \nu_{\psi_1}(I) \cup \psi_1(I)$. But, if $a \in I - \psi_1(I)$ then $a \in \nu_{\psi_1}(I) \subseteq \nu_{\psi_1}(I) \cup \psi_1(I)$, since $I - \psi_1(I) \subseteq \nu_{\psi_1}(I)$. Therefore, $\nu(I) = \nu_{\psi_1}(I) \cup \psi_1(I)$ and hence $\nu(I \times S) = \nu(I) \times S$ is an ideal of $R \times S$ which implies that $I \times S$ is a 2-absorbing primal ideal of $R \times S$.
($2 \to 3$) Since $\nu(I \times S) = \nu(I) \times S$ is a prime ideal of $R \times S$, $\nu(I)$ is a prime ideal of R. So I is a 2-absorbing primal ideal of R.
($3 \to 1$) Because I is a 2-absorbing primal ideal of R, $I \times S$ is a 2-absorbing primal ideal of $R \times S$. As a result, using the same approach as in proof ($1 \to 2$) above, one can easily demonstrate that $\nu(I) = \nu_{\psi_1}(I) \cup \psi_1(I)$. Therefore, $\nu(I \times S) = \nu_\phi(I \times S) \cup \phi(I \times S)$, since $\psi_2(S) = S$. Consequently, $I \times S$ is a ϕ-2-absorbing primal ideal of $R \times S$. □

Theorem 2.12. Let $R \times S$ be a commutative ring with unity, where R, S are commutative rings with unities. Let $\phi = \psi_1 \times \psi_2 : \mathfrak{I}(R \times S) \to \mathfrak{I}(R \times S) \cup \{\emptyset\}$ be any function, where $\psi_1 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$, $\psi_2 : \mathfrak{I}(S) \to \mathfrak{I}(S) \cup \{\emptyset\}$ are any functions. Let $I \neq \psi_1(I)$ be a proper ideal of R and $J \neq \psi_2(J)$ an ideal of S with $I \times J \not\subseteq \mathrm{Rad}(\phi(R \times S))$. Then the following statements are equivalent.
(1) $I \times J$ is a ϕ-2-absorbing primal ideal of $R \times S$;
(2) $J = S$ and I is a 2-absorbing primal ideal of R;
(3) $I \times J$ is a 2-absorbing primal ideal of $R \times S$.

Proof. ($1 \to 2$) Suppose $I \times J$ is a ϕ-2-absorbing primal ideal of $R \times S$. Since $I \times J \not\subseteq \mathrm{Rad}(\phi(R \times S))$ and since $J = S$, I is a 2-absorbing primal ideal of R by Theorem 2.11. We show that the case $J \neq S$ can not be happened. Suppose $J \neq S$, we show that J is a prime ideal in S and I is a prime ideal of R. Since $I \times J \not\subseteq \mathrm{Rad}(\phi(R \times S))$, by Corollary 2.7 $\nu(I \times J) \neq R \times S$. Let $a, b \in S$ such that $ab \in J$ and let $i \in I \setminus \psi_1(I)$. Then $(i, 1)(1, a)(1, b) = (i, ab) \in I \times J \setminus \phi(I \times J)$, since $(1, ab) \notin I \times J$ and since $(1, 1) \notin \nu_\phi(I \times J)$, $(i, a) \in I \times J$ or $(i, b) \in I \times J$ so $a \in J$ or $b \in J$. Thus J is a prime ideal of S. Similarly, let $c, d \in R$ such that $cd \in I$, and let $j \in J \setminus \psi_2(J)$. Then $(c, 1)(d, 1)(1, j) = (cd, j) \in I \times J \setminus \phi(I \times J)$, since $(cd, 1) \notin I \times J$ and since $(1, 1) \notin \nu_\phi(I \times J)$, $(c, j) \in I \times J$ or $(d, j) \in I \times J$ so $c \in I$ or $d \in I$. Hence I is a prime ideal of R. In this case we show that $(1, 1) \in \nu(I \times J)$, which is a contradiction to Corollary 2.7. Now, $(1, 0)(0, 1) \in I \times J$ and $(1, 0) \notin I \times J$, $(0, 1) \notin I \times J$, so $(1, 0), (0, 1) \in \nu(I \times J)$. Therefore, if $\nu(I \times J)$ is an ideal in $R \times S$, then $(1, 1) = (1, 0) + (0, 1) \in \nu(I \times J)$. Therefore the only case of part (2) is that $J = S$ and I is a 2-absorbing primal ideal of R.
($2 \to 3$) If $J = S$ and I is a 2-absorbing primal ideal of R, then $I \times J$ is a 2-absorbing primal ideal of $R \times S$ by Theorem 2.11(2).
($3 \to 1$) Clear from Theorem 2.11 □

3. More Properties of ϕ-2-Absorbing Primal ideals

For a commutative ring R, let $\mathfrak{J}(R)$ denotes the intersection of all maximal ideals of R.

Lemma 3.1. Let R be a commutative ring and $a, b \in \mathfrak{J}(R)$. Then the ideal $I = abR$, where $1 \notin \nu_\phi(I)$, is a ϕ-2-absorbing primal ideal of R if and only if $ab \in \phi(I)$.

Proof. If $ab \in \phi(I)$, then $I = \phi(I)$ is a ϕ-2-absorbing primal ideal of R by definition. If $ab \notin \phi(I)$ with $a, b \notin I$, then $1 \in \nu_\phi(I)$, a contradiction. Therefore, $a \in I$ or $b \in I$. If $a \in I$, then $a = abk$ for some $k \in R$. So, $a(1 - bk) = 0$ and since $bk \in \mathfrak{J}(R)$, $1 - bk$ is a unit in R. Thus, $a(1 - bk) = 0$ implies that $a = 0$ and hence $ab = 0 \in \phi(I)$, a contradiction. Therefore, $I = \phi(I)$. □

We recall that R is defined to be quasi-local ring if R has a unique maximal ideal. If (R, M) is a quasi-local ring, where M is the unique maximal ideal of R, then we have the following two results about a ϕ-2-absorbing primal ideal I of R with $1 \notin \nu_\phi(I)$.

Theorem 3.2. Let (R, M) be a quasi-local ring with $\nu_\phi(I) \neq R$ for all proper ideals I of R. Then every proper ideal I of R is a ϕ-2-absorbing primal if and only if $M^2 \subseteq \phi(I)$.

Proof. Let $a, b \in M$, then $I = abR$ is a ϕ-2-absorbing primal ideal of R with $1 \notin \nu_\phi(I)$, hence, by Lemma 3.1, $M^2 \subseteq \phi(I)$. Conversely, let I be a proper ideal of R with $M^2 \subseteq \phi(I)$. Let $a \in \nu_\phi(I)$. If a is a unit in R, then $1 \in \nu_\phi(I)$, a contradiction. So we may assume that a is not a unit in R. Let $r, s, t, \in R$ with $rsta \in I \setminus \phi(I)$ such that $rs, rt, st \notin I$. If $rst \in I \setminus \phi(I)$, then r or s or t is a unit in R which implies that rs or st or rt is in I, a contradiction. Therefore, $rst \notin I$ and since $rsta \in I \setminus \phi(I)$ and a is not a unit, rst is a unit in R, so $a \in I \setminus \phi(I)$, hence $\nu_\phi(I) \cup \phi(I) = I$ which implies that I is a ϕ-2-absorbing primal ideal of R. $\square$

Corollary 3.3. Let (R, M) be a quasi-local ring with $\nu_\phi(I) \neq R$ for all proper ideals I of R. Then every proper ideal I of R with $M^2 \subseteq \phi(I)$, is a 2-absorbing primal ideal of R.

Proof. Let I be a proper ideal of R with $M^2 \subseteq \phi(I)$, then, by Theorem 3.2, I is a ϕ-2-absorbing primal ideal of R. We show that $\nu(I)$ is an ideal in R. Let a, b be nonzero elements in $\nu(I)$. Then there exist $r, s, t \in R$ with $rs, rt, st \in R \setminus I$ such that $rsta \in I$. If $rsta \in I \setminus \phi(I)$, then, by Theorem 3.2, $a \in I \subseteq M$. Since $rs \notin I$, r or s is a unit in R. Therefore, if $rsta \in \phi(I)$, then $(st)a \in \phi(I)$ or $(rt)a \in \phi(I)$. Say $(st)a \in \phi(I)$ again since $st \notin I$, s or t is a unit in R which implies that $sa \in \phi(I)$ or $ta \in \phi(I)$. Say $ta \in \phi(I)$, hence t is not a unit in R, since $a \in I \setminus \phi(I)$. Therefore if $ta \in \phi(I) \subseteq I \subseteq M$ and a is not a unit in R (if a is a unit in R, then $t \in \phi(I)$ a contradiction), then a must be in M, since M is a prime ideal. Similarly, $b \in M$, so $a + b \in M$. If $t(a + b) \notin \phi(I)$, then $t(a + b) \neq 0$ and hence t is a unit in R since $a + b \in M$, a contradiction. Therefore, $t(a + b) \in \phi(I)$ which implies that $a + b \in \nu(I)$ since $t \notin I$. Hence $\nu(I)$ is an ideal of R. $\square$

Let R be a commutative ring with unity and let J be a proper ideal of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Following of [2], we define $\phi_J : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{I}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi = \phi_\emptyset$).

In the next result we give the condition on a proper ideal I of R such that I/J is a ϕ_J-2-absorbing primal ideal of R/J where J is a proper ideal of R subset of I.

Theorem 3.4. Let I, J be proper ideals of R with $J \subseteq I$. If I is a ϕ-2-absorbing primal ideal of R with $\nu_\phi(J) \subseteq I$. Then I/J is a ϕ_J-2-absorbing primal ideal of R/J.

Proof. To prove this result we must show that $\nu_{\phi_J}(I/J) \cup \phi_J(I/J) = [\nu_\phi(I) \cup J]/J$. Let $a + J \in \nu_{\phi_J}(I/J)$. Then there exist $r+J, s+J, t+J \in R/J$ with $rsta+J \in (I/J) \setminus \phi_J(I/J)$ such that $rs+J, rt+J, st+J \notin I/J$. So $rsta \in I \setminus \phi(I)$, since $rsta \notin J$, with $rs, rt, st \notin I$ hence $a \in \nu_\phi(I)$, therefore, $a + J \in [\nu_\phi(I) \cup J]/J$. Conversely, let $a + J \in [\nu_\phi(I) \cup J]/J$ such that $a + J \notin \phi_J(I/J)$. Then $a \in \nu_\phi(I) - J$. If $a \in I$, then $a + J \in \nu_{\phi_J}(I/J)$. So we may assume that $a \notin I$. Then there exist $r, s, t \in R$ with $rsta \in I \setminus \phi(I)$ such that $rs, rt, st \notin I$. If $rsta \in J \setminus \phi(J)$, then $a \in \nu_\phi(J)$, a contradiction, since $\nu_\phi(J) \subseteq I$ and $a \notin I$. Therefore, $r + J, s + J, t + J \in R/J$ with $rsta + J = (rst + J)(a + J) \in I/J \setminus \phi_J(I/J)$ such that $rs + J, rt + J, st + J \notin I/J$, so $a + J \in \nu_{\phi_J}(I/J)$. Hence $\nu_{\phi_J}(I/J) \cup \phi_J(I/J) = [\nu_\phi(I) \cup J]/J$ which implies that I/J is a ϕ_J-2-absorbing primal ideal of R/J. $\square$

Corollary 3.5. Let R_0 be a subring of R with unity. If I is a ϕ-2-absorbing primal ideal of R, then $I \cap R_0$ is a $\overline{\phi}$-2-absorbing primal ideal of R_0, where $\overline{\phi}(I \cap R_0) = \phi(I) \cap R_0$.

Proof. Clear. $\square$

Let R be a commutative ring with unity $(1 \neq 0)$ and let S be a multiplicative closed proper subset of R with $1 \in S$. We recall that if R is a commutative ring with unity, then $R_S = \{\frac{a}{s} : a \in R , s \in S\}$ is a commutative ring with unity. Also if I is an ideal in R, then I_S is an ideal of R_S, where $I_S = \{\frac{a}{s} : a \in I , s \in S\}$. Moreover, if J is an ideal of R_S, then $J \cap R$ is an ideal R.

Now let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, we define $\phi_S : \mathfrak{I}(R_S) \to \mathfrak{I}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(J \cap R))_S$ for every $J \in \mathfrak{I}(R_S)$. Note that $\phi_S(J) \subseteq J$. Since for $J \in \mathfrak{I}(R_S)$, $\phi(J \cap R) \subseteq J \cap R$ implies $\phi_S(J) \subseteq (J \cap R)_S \subseteq J$.

Lemma 3.6. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-2-absorbing primal ideal of R with $P = \nu_\phi(I) \cup \phi(I)$. Suppose $P \cap S = \emptyset$. If $\dfrac{a}{s} \in I_S - (\phi(I))_S$, then $a \in I$. Moreover, if $(\phi(I))_S \cap R \subseteq I$, then $I = I_S \cap R$.

Proof. Let $\dfrac{a}{s} \in I_S - (\phi(I))_S$, so $\dfrac{a}{s} = \dfrac{b}{t}$ for some $b \in I$ and $t \in S$. In this case $uta = usb \in I$ for some $u \in S$. If $uta \in \phi(I)$, then $\dfrac{a}{s} = \dfrac{uta}{uts} \in (\phi(I))_S$, a contradiction. So, $uta \in I - \phi(I)$. If $a \notin I$, then ut is not a ϕ-2-absorbing prime to I; so $ut \in P \cap S$ which contradicts the hypothesis. Therefore $a \in I$.
For the last part, it is clear that $I \subseteq I_S \cap R$. Now let a be an element in $I_S \cap R$. Then $as \in I$ for some $s \in S$. If $as \notin \phi(I)$ and $a \notin I$, then s is not ϕ-2-absorbing prime to I, so $s \in P \cap S$ a contradiction. Therefore, a must be in I. If $as \in \phi(I)$, then $\dfrac{a}{1} = \dfrac{as}{s} \in (\phi(I))_S$, and so $a \in (\phi(I))_S \cap R$. Thus, $I_S \cap R = I \cup ((\phi(I))_S \cap R) = I$, since $(\phi(I))_S \cap R \subseteq I$. Hence $I = I_S \cap R$. □

Lemma 3.7. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-2-absorbing primal ideal of R with $P \cap S = \emptyset$, where $P = \nu_\phi(I) \cup \phi(I)$. Then $[I_S \cap R] - [\phi_S(I_S) \cap R] \subseteq I - \phi(I)$.

Proof. Let $a \in I_S \cap R$ such that $a \notin (\phi_S(I_S) \cap R)$, then $\dfrac{a}{1} \in I_S - \phi_S(I_S) \subseteq I_S - (\phi(I))_S$ and by Lemma 3.6, $a \in I$. If $a \in \phi(I)$, then $\dfrac{a}{1} \in (\phi(I))_S \subseteq \phi_S(I_S)$ implies that $a \in \phi_S(I_S) \cap R$ a contradiction. Therefore, $a \in I - \phi(I)$. □

Let R be a commutative ring with unity and M an R-module. An element $a \in R$ is called a zero-divisor on M if $am = 0$ for some $m \in M$. We denote by $\mathbf{Z}_R(M)$ the set all zero-divisors of R on M.

Corollary 3.8. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-2-absorbing primal ideal of R with $P \cap S = \emptyset$, where $P = \nu_\phi(I) \cup \phi(I)$. Suppose $S \cap \mathbf{Z}_R(R/\phi(I)) = \emptyset$. If $(\phi(I))_S \cap R \subseteq I$, then $(\nu_\phi(I))_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$.

Proof. By Lemma 3.6, if $((\phi(I))_S \cap R) \subseteq I$, then $I_S \cap R = I$. Let $\frac{x}{s}$ be an element in $(\nu_\phi(I))_S - \phi_S(I_S)$, then $\frac{x}{s} = \frac{y}{t}$, where $y \in \nu_\phi(I)$. If $y \in I$, then $\frac{y}{t} = \frac{x}{s} \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $y \notin I$. If $\frac{y}{1} \in I_S$, then $\frac{y}{t} = \frac{x}{s} \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $\frac{y}{1} \notin I_S$. So, $\frac{y}{1}$ is an element in $(\nu_\phi(I))_S - I_S$ and therefore $uy \in \nu_\phi(I)$ for some $u \in S$ and $uy \notin I$. So there exist $r, s, t \in R - I$ such that $rstuy \in I - \phi(I)$. If $rsty \notin I$, then $u \in \nu_\phi(I) \subseteq P$ a contradiction. Therefore, $rsty \in I - \phi(I)$. So $\frac{r}{1}\frac{s}{1}\frac{t}{1}\frac{y}{1} \in I_S$. If $\frac{r}{1}\frac{s}{1}\frac{t}{1}\frac{y}{1} \in \phi_S(I_S)$, then there exists $v \in S$ with $rstvy \in \phi(I_S \cap R) = \phi(I)$, so $v \in S \cap \mathbf{Z}_R(R/\phi(I))$ a contradiction. Thus $\frac{r}{1}\frac{s}{1}\frac{t}{1}\frac{y}{1} \in I_S - \phi_S(I_S)$ and $\frac{r}{1}\frac{s}{1} \notin I_S$, $\frac{r}{1}\frac{t}{1} \notin I_S$ and $\frac{t}{1}\frac{s}{1} \notin I_S$. So , $\frac{y}{1} \in \nu_{\phi_S}(I_S)$. Hence $\frac{x}{s} = \frac{y}{t} \in \nu_{\phi_S}(I_S)$. □

We recall that if I is a proper ideal in R, then $I \subseteq I_S \cap R$, therefore we may assume that $(\phi(I))_S \subseteq \phi_S(I_S)$.

Under the condition that $(\phi(I))_S \cap R \subseteq I$ for all proper ideals I of R, we have the following Propositions.

Proposition 3.9. Let S be a multiplicative closed subset of R with $1 \in S$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let I be a ϕ-2-absorbing primal ideal of R with $P \cap S = \emptyset$, where $P = \nu_\phi(I) \cup \phi(I)$. Suppose $S \cap \mathbf{Z}_R(R/\phi(I)) = \emptyset$. Then I_S is a ϕ_S-2-absorbing primal ideal of R_S.

Proof. It is well known that if P is a prime ideal in R, then P_S is a ϕ_S-prime ideal of R_S.
To show that I_S is a ϕ_S-2-absorbing primal ideal of R_S, we must prove that $P_S = \nu_{\phi_S}(I_S) \cup \phi_S(I_S)$. Clearly, $\phi_S(I_S) \subseteq P_S$, let $\frac{a}{v}$ be an element in $\nu_{\phi_S}(I_S)$. Then there exists $\frac{r}{u_1}, \frac{s}{u_2}, \frac{t}{u_3} \in R_S - I_S$ such that $\frac{r}{u_1}\frac{s}{u_2} \notin I_S$, $\frac{r}{u_1}\frac{t}{u_3} \notin I_S$ and $\frac{s}{u_2}\frac{t}{u_3} \notin I_S$ and with $(\frac{r}{u_1}) \cdot (\frac{s}{u_2}) \cdot (\frac{t}{u_3}) \cdot (\frac{a}{s}) \in I_S - \phi_S(I_S) \subseteq I_S - (\phi(I))_S$. So $rsta \notin \phi(I)$ and, by Lemma 3.6, $rsta \in I$. Hence, $rsta \in I - \phi(I)$ and $rs \notin I$, $rt \notin I$, and $st \notin I$. Thus $a \in \nu_\phi(I) \subseteq P$ and hence $\frac{a}{s} \in P_S$.
Conversely, let $\frac{a}{s} \in P_S$ such that $\frac{a}{s} \notin \phi_S(I_S)$. Then $a \in P_S \cap R = P$. If $\frac{a}{s} \in I_S$, then $(\frac{1}{1})(\frac{a}{s}) \in I_S - \phi_S(I_S)$, $(\frac{1}{1}) \notin I_S$, so $\frac{a}{s}$ is not ϕ_S-prime to I_S, thus $\frac{a}{s} \in \nu_{\phi_S}(I_S)$. Therefore, we may assume that $\frac{a}{s} \notin I_S$, that is $ta \notin I$ for every $t \in S$. So, $a \notin I$. Therefore, $a \in P - I \subseteq \nu_\phi(I)$. Thus, $\frac{a}{s} \in (\nu_\phi(I))_S - \phi_S(I_S)$. Since, by Corollary 3.8, $(\nu_\phi(I))_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$, $\frac{a}{s} \in \nu_{\phi_S}(I_S)$. □

Proposition 3.10. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, and let J be a ϕ_S-2-absorbing primal ideal of R_S with $Q = \nu_{\phi_S}(J) \cup \phi_S(J)$. Then $Q \cap R$ is a ϕ-prime ideal of R and $J \cap R$ is a ϕ-2-absorbing primal ideal of R with $Q \cap R = \nu_\phi(J \cap R) \cup \phi(J \cap R)$, and with $(Q \cap R) \cap S = \emptyset$, $S \cap \mathbf{Z}_R(R/\phi(J \cap R)) = \emptyset$. Moreover, $J = (J \cap R)_S$.

Proof. To show that $Q \cap R$ is a ϕ-prime ideal of R, it is enough to prove that $J \cap R$ is a ϕ-2-absorbing primal ideal of R with $Q \cap R = \nu_\phi(J \cap R) \cup \phi(J \cap R)$. Then, by using Theorem 2.3, $Q \cap R$ will be a ϕ-prime ideal of R.
Now, to prove that $J \cap R$ is a ϕ-2-absorbing primal ideal of R we must show that $Q \cap R = \nu_\phi(J \cap R) \cup \phi(J \cap R)$. But $\phi(J \cap R) \subseteq J \cap R \subseteq Q \cap R$. Let a be an element in $\nu_\phi(J \cap R)$ with $a \notin \phi(J \cap R)$. Then $\frac{a}{1} \in (\nu_\phi(J \cap R))_S - \phi_S(J)$, since $S \cap \mathbf{Z}_R(R/\phi(I)) = \emptyset$ and, by Corollary 3.8, $(\nu_\phi(J \cap R))_S - \phi_S(J) \subseteq \nu_{\phi_S}(J)$. Thus, $\frac{a}{1} \in \nu_{\phi_S}(J) \subseteq Q$ and hence $a \in Q \cap R$.
Conversely, let a be an element in $Q \cap R$. Then $\frac{a}{1}$ in Q. We may assume that $a \notin \phi(J \cap R)$, since $S \cap \mathbf{Z}_R(R/\phi(J \cap R)) = \emptyset$, $\frac{a}{1} \notin \phi_S(J)$. If $\frac{a}{1} \in J$, then $(\frac{a}{1}) \in J - \phi_S(J)$ and since $\phi(J \cap R) \subseteq \phi_S(J) \cap R$, $a \in (J \cap R) - (\phi_S(J) \cap R) \subseteq (J \cap R) - \phi(J \cap R)$, but $1 \notin J \cap R$, so $a \in \nu_\phi(J \cap R)$. If $\frac{a}{1} \notin J$, then $\frac{a}{1} \in Q - J$ and so $\frac{a}{1} \in \nu_{\phi_S}(J)$. Let $\frac{x}{s}, \frac{y}{r}, \frac{z}{t} \in R_S$ such that $(\frac{x}{s})(\frac{y}{r}) \notin J$, $(\frac{x}{s})(\frac{z}{t}) \notin J$ and $(\frac{y}{r})(\frac{z}{t}) \notin J$, with $(\frac{a}{1})(\frac{x}{s})(\frac{y}{r})(\frac{z}{t}) \in J - \phi_S(J)$. Then $axyz \in (J \cap R) - (\phi_S(J) \cap R) \subseteq (J \cap R) - \phi(J \cap R)$, since $\frac{axyz}{1} \in J$ and $\frac{axyz}{1} \notin \phi_S(J)$, for if $\frac{axyz}{1} \in \phi_S(J)$, then $\frac{axyz}{s} \in \phi_S(J)$, a contradiction. Thus we have $axyz \in (J \cap R) - \phi(J \cap R)$ and $xy, xz, yz \notin J \cap R$, since $(\frac{x}{s})(\frac{y}{r}) \notin J$, $(\frac{x}{s})(\frac{z}{t}) \notin J$ and $(\frac{y}{r})(\frac{z}{t}) \notin J$. Therefore, $a \in \nu_\phi(J \cap R)$ and so $J \cap R$ is a ϕ-2-absorbing primal ideal of R with $Q \cap R = \nu_\phi(J \cap R) \cup \phi(J \cap R)$.
Finally, we show that $J = (J \cap R)_S$. Clearly, $J \subseteq (J \cap R)_S$. Conversely, let $\frac{x}{s}$ be an element in $(J \cap R)_S$. Then $xt \in J \cap R$ for some $t \in S$. Thus, $\frac{xt}{1} \in J$, and hence $(\frac{xt}{1})(\frac{1}{st}) = \frac{x}{s} \in J$. Therefore, $J = (J \cap R)_S$. □

Under the condition that $(\phi(I))_S \cap R \subseteq I$ for all proper ideals I of R and by using Propositions 3.9 and 3.10 we have the following main result.

Corollary 3.11. Let R be a commutative ring with unity. Let S be a multiplicative closed subset of R such that $1 \in S$. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. Then there is one-to-one correspondence between the ϕ-2-absorbing primal ideals I of R and ϕ_S-2-absorbing primal ideals I_S of R_S with $S \cap \mathbf{Z}_R(R/\phi(I)) = \emptyset$, $P \cap S = \emptyset$ where $P = \nu_\phi(I) \cup \phi(I)$. ■

Acknowledgments

We thank the referee for his valuable suggestions.

4. Bibliography

References

1. D. Anderson, A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, 39, 1646-1672, (2011).
2. D. Anderson, M. Bataineh, *Generalizations of prime ideals*, Comm. in Algebra 36, 686-696, (2008).
3. A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75, 417-429, (2007).
4. A. Badawi, U. Tekir, E. A. Ugurlu, G. Ulucak, E. Y. Celikel, *Generalizations of 2-absorbing primary ideals of commutative rings*, Turkish J. of Math., 40, 703-717, (2016).

5. A. Badawi, U. Tekir, E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., 51(4), 1163-1173, (2014).

6. Y. Darani, *Generalizations of primal ideals in commutative rings*, MATEMATIQKI VESNIK, 64(1), 25-31, (2012).

7. L. Fuchs, *On primal ideals*, Amer. Math. Soc. 1, 1-6, (1950).

8. A. Jaber, *Properties of weakly 2-absorbing primal ideals*, Italian Journal of pure and applied mathematics, 47, 609-619, (2022).

9. A. Jaber, H. Obiedat, *On 2-absorbing primal ideals*, Far East Journal of Mathematical Sciences, 103(1), 53-66, (2018).

10. S. Payrovi and S. Babaei, *On the 2-absorbing ideals*, Int. Math. Forum, 7(6), 265-271, (2012).

Ameer Jaber,
Department of Mathematics, Faculty of Science,
The Hashemite University, Zarqa, Jordan.
E-mail address: ameerj@hu.edu.jo

and

Rania Shaqbou'a,
Department of Mathematics, Faculty of Science,
The Hashemite University, Zarqa, Jordan.
E-mail address: Rania.Shaqboua@hu.edu.jo