Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–10.
ISSN-0037-8712 in press
doi:10.5269/bspm.62420

On Pencil of Bounded Linear Operators on Non-archimedean Banach Spaces

A. El amrani[1], J. Ettayb[1] and A. Blali [2]

ABSTRACT: In this paper, we introduce and check some properties of pseudospectrum and some approximation of a pencil of bounded linear operators on non-archimedean Banach spaces. Our main result extend some results for a pencil of bounded linear operators on non-archimedean Banach spaces and we give some examples to support our work.

Key Words: Non-archimedean Banach spaces, spectrum, pencil of linear operator, pseudospectrum.

Contents

1. Introduction

Throughout this paper, X is a non-archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $\mathcal{L}(X)$ denotes the set of all bounded linear operators on X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$ (is the unit ball of $\mathbb{Q}_p$). We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$. For more details, we refer to [4] and [8].
Remember that a free Banach space X is a non-archimedean Banach space for which there exists a family $(e_i)_{i\in\mathbb{N}}$ in $X\backslash\{0\}$ such that every element $x \in X$ can be written in the form of a convergent sum $x = \sum_{i\in\mathbb{N}} x_i e_i$, $x_i \in \mathbb{K}$ and $\|x\| = \sup_{i\in\mathbb{N}} |x_i|\|e_i\|$. The family $(e_i)_{i\in\mathbb{N}}$ is called an orthogonal basis. In free Banach space X, each bounded linear operator A on X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{i,j\in\mathbb{N}} a_{i,j} e'_j \otimes e_i, \ and \ \forall j \in \mathbb{N}, \ \lim_{i\to\infty} |a_{i,j}|\|e_i\| = 0,$$

where $(\forall j \in \mathbb{N})$ $e'_j(u) = u_j$ (e'_j is the linear form associated with e_j).
Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i\in\mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j} \frac{|a_{ij}|\|e_i\|}{\|e_j\|}.$$

Also, recall that X is of countable type if it contains a countable set whose linear hull is dense in X. For more details, we refer [4] and [8]. An unbounded linear operator $A : D(A) \subseteq X \to Y$ is said to be closed if for all $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $\|x_n - x\| \to 0$ and $\|Ax_n - y\| \to 0$ as $n \to \infty$, for some $x \in X$ and $y \in Y$, then $x \in D(A)$ and $y = Ax$. The collection of closed linear operators from X into Y is denoted by $\mathcal{C}(X,Y)$. When $X = Y$, $\mathcal{C}(X,X) = \mathcal{C}(X)$. If $A \in \mathcal{L}(X)$ and B is an unbounded linear operator, then $A + B$ is closed if and only if B is closed, for more details, we refer to [4].

2010 *Mathematics Subject Classification:* 47A10, 47S10.
Submitted February 09, 2022. Published August 20, 2022

Definition 1.1. *[4] Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $\mathbb{K}$. We define $\mathbb{E}_\omega$ by*

$$\mathbb{E}_\omega = \{x = (x_i)_i : \forall i \in \mathbb{N},\ x_i \in \mathbb{K},\ \text{and}\ \lim_{i\to\infty} |\omega_i|^{\frac{1}{2}}|x_i| = 0\},$$

and it is equipped with the norm

$$\forall x \in \mathbb{E}_\omega : x = (x_i)_i,\ \|x\| = \sup_{i\in\mathbb{N}}(|\omega_i|^{\frac{1}{2}}|x_i|).$$

Remark 1.2. *[4]*

(i) The space $(\mathbb{E}_\omega,\ \|\cdot\|)$ is a non-archimedean Banach space.

(ii) If

$$\begin{array}{rcl}\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega & \longrightarrow & \mathbb{K} \\ (x,y) & \mapsto & \displaystyle\sum_{i=0}^{\infty} x_i y_i \omega_i,\end{array}$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then, the space $\Big(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle\Big)$ is called a p-adic (or non-archimedean) Hilbert space.

(iii) The orthogonal basis $\{e_i,\ i \in \mathbb{N}\}$ is called the canonical basis of $\mathbb{E}_\omega$, where for all $i \in \mathbb{N}$, $\|e_i\| = |\omega_i|^{\frac{1}{2}}$.

Definition 1.3. *[7] Let X, Y, Z be three non-archimedean Banach spaces over $\mathbb{K}$, let $A \in B(X,Y)$ and $B \in B(X,Z)$. Then A majorizes B, if there exists $M > 0$ such that*

$$\text{for all } x \in X,\ \|Bx\| \leq M\|Ax\|. \tag{1.1}$$

Theorem 1.4. *[7] Assume that, either field $\mathbb{K}$ is spherically complete or both Y and Z are countable type Banach spaces over $\mathbb{K}$. Let $A \in B(X,Y)$ and $B \in B(X,Z)$. Then the statements are equivalent:*

(1) $R(A^*) \subset R(B^*)$;

(2) B majorizes A;

(3) there exists a continuous linear operator $D : R(B) \longrightarrow Y$, such that $A = DB$.

In this paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$Ax = \lambda Bx$$

for $\lambda \in \mathbb{K}$, $x \in X$, and $A, B \in \mathcal{L}(X)$.
Fore more basic concepts of non-archimedean operators theory, we refer to [4]. In [2], the authors extended the notion of pseudospectrum of linear operator A on non-archimedean Banach space X as follows.

Definition 1.5. *[2] Let X be a non-archimedean Banach space over $\mathbb{K}$ and $\varepsilon > 0$. The pseudospectrum of a linear operator A on X is defined by*

$$\sigma_\epsilon(A) = \sigma(A) \cup \{\lambda \in \mathbb{K} :\ \|(\lambda - A)^{-1}\| > \varepsilon^{-1}\},$$

by convention $\|(\lambda - A)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A)$.

2. Main results

We introduce the following definitions.

Definition 2.1. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $A, B \in \mathcal{L}(X)$, the spectrum $\sigma(A, B)$ of a pencil of linear operator (A, B) is defined by*

$$\begin{aligned} \sigma(A,B) &= \{\lambda \in \mathbb{K} : \ A - \lambda B \text{ is not invertible in } \mathcal{L}(X)\}, \\ &= \{\lambda \in \mathbb{K} : \ 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

The resolvent set $\rho(A, B)$ of a pencil of bounded linear operator (A, B) is

$$\rho(A,B) = \{\lambda \in \mathbb{K} : \ R(\lambda, A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(X)\}.$$

$R(\lambda, A, B)$ is called the resolvent of pencil of bounded linear operator (A, B).

Definition 2.2. *Let X be a non-archimedean Banach space over $\mathbb{K}$. Let $A, B \in \mathcal{L}(X)$, the couple (A, B) is said to be regular, if $\rho(A, B) \neq \emptyset$.*

For a regular couple (A, B), we have the following definitions.

Definition 2.3. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(A, B)$ of a pencil of bounded linear operator (A, B) on X is defined by*

$$\sigma_\varepsilon(A,B) = \sigma(A,B) \cup \{\lambda \in \mathbb{K} : \ \|(A - \lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(A, B)$ of a pencil of bounded linear operator (A, B) is defined by

$$\rho_\varepsilon(A,B) = \rho(A,B) \cap \{\lambda \in \mathbb{K} : \ \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(A - \lambda B)^{-1}\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

Definition 2.4. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. The generalized pseudospectrum of a pencil of bounded linear operator (A, B) on X is defined by*

$$\Sigma_\varepsilon(A,B) = \sigma(A,B) \cup \{\lambda \in \mathbb{K} : \ \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}.$$

By convention $\|(A - \lambda B)^{-1}B\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

Remark 2.5. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$. Then*

(i) If $B = I$, then, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$ where $\sigma_\varepsilon(A)$ is the pseudospectrum of A.

(ii) Definition 2.3 is a natural generalization of Definition 1.5.

In the rest of this section, we suppose that (A, B) is regular. The next proposition gives a comparison between $\sigma_\varepsilon(A, B)$ and $\Sigma_\varepsilon(A, B)$.

Proposition 2.6. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$. Then for all $\varepsilon > 0$,*

$$\Sigma_\varepsilon(A,B) \subset \sigma_{\varepsilon\|B\|}(A,B).$$

Proof. Let $\varepsilon > 0$ and $\lambda \in \Sigma_\varepsilon(A, B)$, then $\lambda \in \sigma(A, B)$ and

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(A - \lambda B)^{-1}B\| & (2.1) \\ &\leq \|(A - \lambda B)^{-1}\|\|B\|. & (2.2) \end{aligned}$$

Hence

$$\frac{1}{\|B\|\varepsilon} < \|(A - \lambda B)^{-1}\|.$$

Thus $\lambda \in \sigma_{\varepsilon\|B\|}(A, B)$. Consequently

$$\Sigma_\varepsilon(A,B) \subset \sigma_{\varepsilon\|B\|}(A,B).$$

□

We have the following statements.

Lemma 2.7. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $\|B\| = 1$. Then for all $\varepsilon > 0$,*

$$\Sigma_\varepsilon(A, B) \subset \sigma_\varepsilon(A, B).$$

Theorem 2.8. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B, C \in \mathcal{L}(X)$ such that $C^{-1} \in \mathcal{L}(X)$. Then*

(i) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \Sigma_\varepsilon(CA, CB)$.

(ii) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, C) = \sigma_\varepsilon(C^{-1}A)$. In particular $C = I$, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$.

Proof. (i) For all $\lambda \in \rho(A, B)$, we have $(A - \lambda B)^{-1}C^{-1} = (CA - \lambda CB)^{-1}$. Then, $\sigma(A, B) = \sigma(CA, CB)$. In addition, it is clear that

$$\|(CA - \lambda CB)^{-1}CB\| = \|(A - \lambda B)^{-1}B\|. \tag{2.3}$$

Hence $\lambda \in \Sigma_\varepsilon(A, B)$, if, and only if, $\lambda \in \Sigma_\varepsilon(CA, CB)$.

(ii) Assume that C is invertible, then $(A - \lambda C)^{-1}C = (C^{-1}A - \lambda I)^{-1}$. Then $\lambda \in \Sigma_\varepsilon(A, C)$, if and only if $\lambda \in \sigma_\varepsilon(C^{-1}A)$.

□

Proposition 2.9. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$. For all $\varepsilon > 0$, we have*

(i) $\sigma(A, B) = \bigcap_{\varepsilon>0} \Sigma_\varepsilon(A, B)$.

(ii) If $0 < \varepsilon_1 \leq \varepsilon_2$, then $\sigma(A, B) \subset \Sigma_{\varepsilon_1}(A, B) \subset \Sigma_{\varepsilon_2}(A, B)$.

Proof. (i) By Definition 2.4, we have for all $\varepsilon > 0$, $\sigma(A, B) \subset \Sigma_\varepsilon(A, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Sigma_\varepsilon(A, B)$, then for all $\varepsilon > 0$, $\lambda \in \Sigma_\varepsilon(A, B)$. If $\lambda \notin \sigma(A, B)$, then $\lambda \in \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|(A - \lambda B)^{-1}B\| = \infty$. Thus $\lambda \in \sigma(A, B)$.

(ii) For $0 < \varepsilon_1 \leq \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A, B)$, then $\|(A - \lambda B)^{-1}B\| > \varepsilon_1^{-1} \geq \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A, B)$.

□

Proposition 2.10. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let X be a free Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$ be analytic operator with compact spectrum $\sigma(A) \neq \emptyset$, then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B, C)$. In addition we have, for all $\varepsilon > 0$, $\Sigma_\varepsilon(B, C) = \sigma_\varepsilon(A)$.*

Proof. Let $\alpha \in \rho(A)$, we set $C = (A - \alpha I)^{-1}$ and $B = A(A - \alpha I)^{-1}$. Then

$$\begin{aligned}
\lambda \in \rho(A) &\iff (A - \lambda I)^{-1} \in \mathcal{L}(X) \\
&\iff (A - \lambda I)(A - \alpha I)^{-1} \in \mathcal{L}(X) \\
&\iff A(A - \alpha I)^{-1} - \lambda(A - \alpha I)^{-1} \in \mathcal{L}(X) \\
&\iff B - \lambda C \in \mathcal{L}(X) \\
&\iff (B - \lambda C)^{-1} \in \mathcal{L}(X) \\
&\iff \lambda \in \rho(B, C).
\end{aligned}$$

Thus, $\sigma(A) = \sigma(B, C)$, let $\varepsilon > 0$, $z \in \sigma_\varepsilon(A)$, then $z \in \sigma(A)$ and

$$\begin{aligned}
\frac{1}{\varepsilon} &< \|(A - zI)^{-1}\|, \\
&= \|(B - zC)^{-1}C\|.
\end{aligned}$$

Thus $\sigma_\varepsilon(A) = \Sigma_\varepsilon(B, C)$.

□

Proposition 2.11. *Let X be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{C}(X)$ with $\rho(A) \neq \emptyset$, then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B, C)$.*

Proof. Similar to the proof of Proposition 2.10. □

We have the following examples.

Example 2.12. *Let $\mathbb{K} = \mathbb{Q}_p$. Let 2×2 square matrix A and B over $\mathbb{Q}_p \times \mathbb{Q}_p$ and $a, b, c, d \in \mathbb{Q}_p^*$ such that $a \neq b$ and $c \neq d$. Then:*

(i) If

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = -\lambda a$, then $\sigma(A, B) = \{0\}$. Simple calculation, we get

$$(A - \lambda B)^{-1} B = \begin{pmatrix} \frac{-1}{\lambda} & 0 \\ 0 & 0 \end{pmatrix},$$

thus, for all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \{0\} \cup \{\lambda \in \mathbb{Q}_p : |\lambda|_p < \varepsilon\}$.

(ii) If

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

Note that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = (a - \lambda c)(b - \lambda d)$, then $\sigma(A, B) = \{\frac{a}{c}, \frac{b}{d}\}$ and

$$\|(A - \lambda B)^{-1} B\| = \max\left\{\frac{|c|_p}{|a - \lambda c|_p}, \frac{|d|_p}{|b - \lambda d|_p}\right\}.$$

Hence, the generalized pseudospectrum of (A, B) is

$$\Sigma_\varepsilon(A, B) = \{\frac{a}{c}, \frac{b}{d}\} \cup \left\{\lambda \in \mathbb{Q}_p : \max\left\{\frac{1}{|ac^{-1} - \lambda|_p}, \frac{1}{|bd^{-1} - \lambda|_p}\right\} > \frac{1}{\varepsilon}\right\}.$$

Example 2.13. *Let $A, B \in \mathcal{L}(\mathbb{E}_\omega)$ be two diagonal operators such that for all $i \in \mathbb{N}$, $Ae_i = a_i e_i$ and $Be_i = b_i e_i$, where $a_i, b_i \in \mathbb{Q}_p$ and B is invertible and $\sup_{i \in \mathbb{N}} |a_i|_p$ and $\sup_{i \in \mathbb{N}} |b_i|_p$ are finite and $0 < \inf_{i \in \mathbb{N}} |b_i|_p \leq \sup_{i \in \mathbb{N}} |b_i|_p \leq 1$. It is easy to see that*

$$\sigma(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i - \lambda b_i| = 0\} = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0\}$$

and for all $\lambda \in \rho(A, B)$, we have

$$\begin{aligned} \|(A - \lambda B)^{-1} B\| &= \sup_{i \in \mathbb{N}} \frac{\|(A - \lambda B)^{-1} B e_i\|}{\|e_i\|} \\ &= \sup_{i \in \mathbb{N}} \left|\frac{b_i}{a_i - \lambda b_i}\right| \\ &= \frac{1}{\inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda|}. \end{aligned}$$

Hence,

$$\left\{\lambda \in \mathbb{Q}_p : \|(A - \lambda B)^{-1} B\| > \frac{1}{\varepsilon}\right\} = \left\{\lambda \in \mathbb{Q}_p : \frac{1}{\inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda|} > \frac{1}{\varepsilon}\right\}.$$

Consequently,

$$\Sigma_\varepsilon(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0\} \cup \left\{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| < \varepsilon\right\}.$$

We have the following results.

Theorem 2.14. *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in \mathcal{C}(X, Y)$, $B \in \mathcal{L}(X, Y)$ and $\lambda \in \mathbb{K}$. If $A - \lambda B$ is one-to-one and onto, then $(A - \lambda B)^{-1}$ is closed linear operator.*

Proof. Let $\lambda \in \mathbb{K}$ and a sequence $(y_n)_n \subset Y$ such that y_n converges to y in Y and $(A - \lambda B)^{-1} y_n$ converges to x in X. Setting $x_n = (A - \lambda B)^{-1} y_n$, then $x_n \in D(A)$ and $y_n = (A - \lambda B)x_n \in Y$. Since $(A - \lambda B)x_n \to y$ and $x_n \to x$ and $A - \lambda B$ is a closed linear operator wich implies that $x \in D(A)$ and $y = (A - \lambda B)x$, then $y \in Y$ and $x = (A - \lambda B)^{-1} y$. Thus $(A - \lambda B)^{-1}$ is closed linear operator. □

Corollary 2.15. *Let X, Y be two non-Archimedean Banach spaces over $\mathbb{K}$. Let A be a linear operator from X into Y and B be a non null bounded linear operator from X into Y. If A is a non closed operator, then $\sigma(A, B) = \mathbb{K}$.*

Proof. Let A be a linear operator which is not closed. We argue by contradiction. Suppose that $\rho(A, B)$ is not empty, then there exists $\lambda \in \mathbb{K}$ such that $\lambda \in \rho(A, B)$, consequently, $(A - \lambda B)^{-1}$ is a bounded operator. Hence, $A - \lambda B$ is a closed operator. In addition, we can write $A = A - \lambda B + \lambda B$. We conclude that A is a closed operator, which is a contradiction. □

Corollary 2.16. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $\sigma(A, B) = \mathbb{K}$, then A is not invertible.*

Proposition 2.17. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ and $B^{-1} \in \mathcal{L}(X)$, then $\sigma_p(B^{-1}A) = \sigma_p(A, B)$.*

Proof. Let $\lambda \in \sigma_p(A, B)$, then there exists $x \in X \backslash \{0\}$ such that $Ax = \lambda Bx$. Since $B^{-1} \in \mathcal{L}(X)$, we have $B^{-1}Ax = \lambda x$, thus $\lambda \in \sigma_p(B^{-1}A)$, therefore $\sigma_p(A, B) \subseteq \sigma_p(B^{-1}A)$. Similarly, we obtain $\sigma_p(B^{-1}A) \subseteq \sigma_p(A, B)$. Thus $\sigma_p(A, B) = \sigma_p(B^{-1}A)$. □

Theorem 2.18. *Let $A, B \in \mathcal{L}(\mathbb{K}^n)$. If A is invertible and $A^{-1}B$ or BA^{-1} is nilpotent, then $\sigma(A, B) = \emptyset$.*

Proof. Assume that A is invertible and $A^{-1}B$ or BA^{-1} is nilpotent, then for all $\lambda \in \mathbb{K}$, $I - \lambda A^{-1}B$ or $I - \lambda BA^{-1}$ is invertible, hence for all $\lambda \in \mathbb{K}$, $(A - \lambda B)^{-1}$ exists in $\mathcal{L}(\mathbb{K}^n)$. Thus, $\sigma(A, B) = \emptyset$. □

Theorem 2.19. *Let X be a Banach space of countable type over $\mathbb{Q}_p$, let $A, B \in \mathcal{L}(X)$ such that B majorizes A and B is not invertible, then $\sigma(A, B) = \mathbb{Q}_p$.*

Proof. If B majorizes A. From Theorem 1.4, there exists a continuous linear operator $D : R(B) \longrightarrow X$ such that $A = DB$. then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B = (D - \lambda)B$, since B is not invertible then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B$ is not invertible. Thus, $\sigma(A, B) = \mathbb{Q}_p$. □

Proposition 2.20. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $0 \in \rho(A) \cap \rho(B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$.*

Proof. From $A - \lambda B = -\lambda B(A^{-1} - \lambda^{-1}B^{-1})A$, we obtain that $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$.
□

Proposition 2.21. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ with $\sigma(A, B) \neq \emptyset$. If $\mu \in \rho(A, B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda - \mu} \in \sigma((A - \mu B)^{-1}B)$.*

Proof. Let $A, B \in \mathcal{L}(X)$ and $\mu \in \rho(A, B)$. For $\lambda \in \mathbb{K}$ with $\lambda \neq \mu$, we have

$$A - \lambda B = (A - \mu B)[(A - \mu B)^{-1}B - (\lambda - \mu)^{-1}](\mu - \lambda).$$

Hence $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda - \mu} \in \sigma((A - \mu B)^{-1}B)$. □

3. Non-archimedean generalized spectrum approximation

In [1], the authors extended the following definitions to non-archimedean case.

Definition 3.1. *[1] Let X be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$.*

(1) A sequence (A_n) of bounded linear operators on X is said to be norm convergent to A, denoted by $A_n \to A$, if $\lim\limits_{n\to\infty} \|A_n - A\| = 0$.

(2) A sequence (A_n) of bounded linear operators on X is said to be pointwise convergent to A, denoted by $A_n \xrightarrow{p} A$, if for all $x \in X$, $\lim\limits_{n\to\infty} \|A_n x - Ax\| = 0$.

Definition 3.2. *[1] Let X be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$. A sequence (A_n) of bounded linear operators on X is said to be ν-convergent to A, denoted by $A_n \xrightarrow{\nu} A$, if*

(1) $(\|A_n\|)$ is bounded,

(2) $\|(A_n - A)A\| \to 0$ as $n \to \infty$, and

(3) $\|(A_n - A)A_n\| \to 0$ as $n \to \infty$.

Definition 3.3. *[1] Let X be a non-archimedean Banach space over a locally compact filed $\mathbb{K}$ and let $A \in \mathcal{L}(X)$. A sequence (A_n) of bounded linear operators on X is said to be convergent to A in the collectively compact convergence, denoted by $A_n \xrightarrow{c.c} A$, if $A_n \xrightarrow{p} A$, and for some positive integer N,*

$$\bigcup_{n\geq N} \{(A_n - A)x \ : x \in X, \ \|x\| \leq 1\}$$

has compact closure of X.

We have the following results.

Proposition 3.4. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \to A$ or $B_n \to B$, then for any $C \in B(X)$, we have*

$$\|(A_n - A)C(B_n - B)\| \to 0.$$

Proof. Since $A_n \to A$ or $B_n \to B$, then for any $C \in B(X)$, we have

$$\|(A_n - A)C(B_n - B)\| \leq \|(A_n - A)\|\|C\|\|(B_n - B)\| \to 0.$$

□

Proposition 3.5. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \xrightarrow{\nu} A$, $B_n \xrightarrow{\nu} B$ and $0 \in \rho(A) \cap \rho(B)$, then for all $\lambda \in \mathbb{K}$, we have*

$$A_n - \lambda B_n \to A - \lambda B.$$

Proof. Suppose that $A_n \xrightarrow{\nu} A$, $B_n \xrightarrow{\nu} B$, $0 \in \rho(A) \cap \rho(B)$, and for all $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} \|\Big(A_n - \lambda B_n\Big) - \Big(A - \lambda B\Big)\| &\leq \max\Big\{\|\Big(A_n - A\Big)\|; |\lambda|\|\Big(B_n - B\Big)\|\Big\} \\ &\to 0. \end{aligned}$$

Since,

$$\begin{aligned} \|\Big(A_n - A\Big)\| &= \|\Big(A_n - A\Big)AA^{-1}\| \\ &\leq \|\Big(A_n - A\Big)A\|\|A^{-1}\| \\ &\to 0. \end{aligned}$$

□

Similarly, we obtain $\|\Big(B_n - B\Big)\| \to 0$.

Proposition 3.6. *Let X be a non-archimedean Banach space over $\mathbb{Q}_p$ such that $\|X\| \subseteq |\mathbb{Q}_p|$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \overset{p}{\to} A$ and $B_n \overset{cc}{\to} B$, then for any $C \in \mathcal{L}(X)$, we have*

$$\|(A_n - A)C(B_n - B)\| \to 0.$$

Proof. Since $A_n \overset{p}{\to} A$ and $B_n \overset{cc}{\to} B$, and $\|X\| \subseteq |\mathbb{Q}_p|$, hence $A_n \overset{p}{\to} A$ and $B_n \overset{p}{\to} B$ and $\mathrm{C}\Big(\bigcup\limits_{n\geq N}\{(B_n - B)x \ : x \in X, \ \|x\| \leq 1\}\Big)$ has compact closure of X. Then

$$\|(A_n - A)C(B_n - B)\| \to 0.$$

□

The aim of the following results is to discuss the spectrum of a sequence of a pencil of linear operators in a non-archimedean Banach space.

Theorem 3.7. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let (A_n) be a sequence of bounded linear operators on X and $A \in \mathcal{L}(X)$. If $A_n \to A$, then there exists $N \in \mathbb{N}$, we have*

$$for\ all\ n \geq N, \ \sigma(A_n) \subset \sigma(A).$$

Proof. Let $\lambda \in \rho(A)$. Then for all $n \in \mathbb{N}$, we have

$$\lambda I - A_n = (\lambda I - A)\Big(I + (\lambda I - A)^{-1}(A - A_n)\Big).$$

Since $A_n \to A$ Then, $\lim\limits_{n\to\infty} \|A_n - A\| = 0$, hence for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|A_n - A\| < \varepsilon$. In particular, for $\varepsilon = \|(\lambda - A)^{-1}\|^{-1}$, we have

$$for\ all\ n \geq N, \ \|A_n - A\| < \|(\lambda - A)^{-1}\|^{-1}.$$

Thus, for all $n \geq N$, we have

$$\begin{array}{rcl} \|(\lambda I - A)^{-1}(A - A_n)\| & \leq & \|(\lambda I - A)^{-1}\|\|(A - A_n)\| \\ & < & 1. \end{array}$$

Then for all $n \geq N$, $\Big(I + (\lambda I - A)^{-1}(A - A_n)\Big)^{-1} \in \mathcal{L}(X)$, hence for all $n \geq N$, $(\lambda - A_n)^{-1} \in \mathcal{L}(X)$. Thus, for all $n \geq N$, $\lambda \in \rho(A_n)$. □

We have the following proposition.

Proposition 3.8. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let $A, B, C, D \in \mathcal{L}(X)$ such that $\rho(A, C) \neq \emptyset$. For all $z \in \rho(A, C)$ such that $\|R(z, A, C)[(A - B) - z(C - D)]\| < 1$, we have $z \in \rho(B, D)$.*

Proof. Since, for all $z \in \rho(A, C)$ such that $\|R(z, A, C)[(A - B) - z(C - D)]\| < 1$ and

$$B - zD = (A - zC)\Big[I - R(z, A, C)\Big((A - B) - z(C - D)\Big)\Big].$$

Hence $(B - zD)$ is invertible and $(B - zD)^{-1} \in \mathcal{L}(X)$. Thus, $z \in \rho(B, D)$. □

Theorem 3.9. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let (A_n) and (B_n) be a sequences of bounded linear operators on X and $A, B \in \mathcal{L}(X)$. If $A_n \to A$ and $B_n \to B$, then there exists $N \in \mathbb{N}$, we have*

$$for\ all\ n \geq N, \ \sigma(A_n, B_n) \subset \sigma(A, B).$$

Proof. Let $\lambda \in \rho(A, B)$. Then for all $n \in \mathbb{N}$, we can write

$$A_n - \lambda B_n = \Big(I - (E_n - \lambda F_n)\Big)(A - \lambda B),$$

where $E_n = (A - A_n)R(\lambda,\ A,\ B)$ and $F_n = (B - B_n)R(\lambda,\ A,\ B)$. Since $A_n \to A$ and $B_n \to B$, then for all $\lambda \in \rho(A,\ B)$, $\lim\limits_{n\to\infty} \|E_n - \lambda F_n\| = 0$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|E_n - \lambda F_n\| < 1$. Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$(I - (E_n - \lambda F_n))^{-1} \in \mathcal{L}(X).$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $A_n - \lambda B_n$ is invertible $\mathcal{L}(X)$. Then, for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. □

Theorem 3.10. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let (A_n), (B_n) be a sequences of bounded linear operators on X and $A,\ B \in \mathcal{L}(X)$. If $A_n \to A$ and $B_n \to B$, then for all $\lambda \in \rho(A, B)$, $(A_n - \lambda B_n)^{-1} \to (A - \lambda B)^{-1}$.*

Proof. Let $\lambda \in \rho(A,\ B)$. Since $A_n \to A$ and $B_n \to B$, then by using Theorem 3.9, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. Then for all $n \geq N$,

$$\|(A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1}\| \tag{3.1}$$

$$\begin{aligned}
(3.1) &= \|(A_n - \lambda B_n)^{-1}\Big((A - \lambda B) - (A_n - \lambda B_n)\Big)(A - \lambda B)^{-1}\| \\
&= \|(A_n - \lambda B_n)^{-1}\Big((A - A_n) - \lambda(B - B_n)\Big)(A - \lambda B)^{-1}\| \\
&\leq \max\Big\{\|(A - A_n)\|,\ |\lambda|\|(B - B_n)\|\Big\}\|(A_n - \lambda B_n)^{-1}\|\|(A - \lambda B)^{-1}\|
\end{aligned}$$

Since $A_n \to A$ and $B_n \to B$, and $\|(A_n - \lambda B_n)^{-1}\|\||A - \lambda B)^{-1}\| < \infty$. Thus,

$$\lim_{n\to\infty} \|(A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1}\| = 0.$$

□

Proposition 3.11. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let (A_n) be a sequence of bounded linear operators on X and $A \in \mathcal{L}(X)$. If $A_n \to A$, then $A_n \xrightarrow{\nu} A$.*

Proof. Ovbious. □

Theorem 3.12. *Let X be a non-archimedean Banach space over $\mathbb{K}$, let (A_n), (B_n) be a sequence of bounded linear operators on X and $A,\ B \in \mathcal{L}(X)$. If $A_n \to A$ and $B_n \to B$, then $(A_n - \lambda B_n)^{-1} \xrightarrow{\nu} (A - \lambda B)^{-1}$ for all $\lambda \in \rho(A, B)$.*

Proof. It suffices to apply Theorem 3.10 and Proposition 3.11. □

References

1. A. Ammar, A. Bouchekouaa, A. Jeribi, *Some approximation results in a non-Archimedean Banach space*, faac **12** (1) (2020), 33-50.
2. A. Ammar, A. Bouchekouaa, A. Jeribi, *Pseudospectra in a Non-Archimedean Banach Space and Essential Pseudospectra in E_ω*, Filomat **33**, no 12 (2019), 3961-3976.
3. P. M. Anselone, *Collectively compact operator approximation theory and applications to integral equations*, 1971.
4. T. Diagana, F. Ramaroson, *Non-archimedean Operators Theory*, Springer, 2016.
5. G. Karishna Kumar, S. H. Lui, *Pseudospectrum and condition spectrum*, Operators and Matrices (2015), 121-145.

6. A. Khellaf, H. Guebbai, S. Lemita, Z. Aissaoui, *On the Pseudo-spectrum of Operator Pencils*, Asian European Journal of Mathematics, 2019.

7. W. PengHui, Z. Xu, *Range inclusion of operators on non-archimedean Banach space*, Science China Mathematics, Vol. **53** No. 12 (2010), 3215-3224.

8. A. C. M. van Rooij, *Non-Archimedean functional analysis*, Monographs and Textbooks in Pure and Applied Math., **51**. Marcel Dekker, Inc., New York, 1978.

9. L. N. Trefethen, M. Embree, *Spectra and pseudospectra. The behavior of nonnormal matrices and operators*, Princeton University Press, Princeton, 2005.

[1] *Department of Mathematics and Computer Science,*
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Fez, Morocco.
E-mail address: abdelkhalek.elamrani@usmba.ac.ma; jawad.ettayb@gmail.com

and

[2] *Department of Mathematics,*
Sidi Mohamed Ben Abdellah University,
ENS B. P. 5206 Bensouda-Fez, Morocco.
E-mail address: aziz.blali@usmba.ac.ma