On Pencil of Bounded Linear Operators on Non-archimedean Banach Spaces

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Abstract: In this paper, we introduce and check some properties of pseudospectrum and some approximation of a pencil of bounded linear operators on non-archimedean Banach spaces. Our main result extends some results for a pencil of bounded linear operators on non-archimedean Banach spaces and we give some examples to support our work.

Key Words: Non-archimedean Banach spaces, spectrum, pencil of linear operator, pseudospectrum.

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1. Introduction

Throughout this paper, \( X \) is a non-archimedean (n.a) Banach space over a (n.a) non-trivially complete valued field \( \mathbb{K} \) with valuation \(| \cdot |\), \( \mathcal{L}(X) \) denotes the set of all bounded linear operators on \( X \), \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers \((p \geq 2 \text{ being a prime})\) equipped with \( p \)-adic valuation \(| \cdot |_p\), \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic integers of \( \mathbb{Q}_p \) (the unit ball of \( \mathbb{Q}_p \)). We denote the completion of algebraic closure of \( \mathbb{Q}_p \) under the \( p \)-adic valuation \(| \cdot |_p\) by \( \mathbb{C}_p \). For more details, we refer to [4] and [8].

Remember that a free Banach space \( X \) is a non-archimedean Banach space for which there exists a family \((e_i)_{i \in \mathbb{N}} \) in \( X \setminus \{0\} \) such that every element \( x \in X \) can be written in the form of a convergent sum \( x = \sum_{i \in \mathbb{N}} x_i e_i, \ x_i \in \mathbb{K} \) and \( \|x\| = \sup_{i \in \mathbb{N}} \|x_i\| \|e_i\| \). The family \((e_i)_{i \in \mathbb{N}} \) is called an orthogonal basis. In free Banach space \( X \), each bounded linear operator \( A \) on \( X \) can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix \((a_{i,j})_{(i,j) \in \mathbb{N} \times \mathbb{N}}\) with coefficients in \( \mathbb{K} \) such that

\[ A = \sum_{i,j \in \mathbb{N}} a_{i,j} e_j^* \otimes e_i, \text{ and } \forall j \in \mathbb{N}, \lim_{i \to \infty} |a_{i,j}| \|e_i\| = 0, \]

where \((\forall j \in \mathbb{N})\ e_j^*(u) = u_j\ (e_j^* \text{ is the linear form associated with } e_j).\)

Moreover, for each \( j \in \mathbb{N},\ \ A e_j = \sum_{i \in \mathbb{N}} a_{i,j} e_i \) and its norm is defined by

\[ \| A \| = \sup_{i,j} \frac{|a_{i,j}| \|e_i\|}{\|e_j\|}. \]

Also, recall that \( X \) is of countable type if it contains a countable set whose linear hull is dense in \( X \). For more details, we refer to [4] and [8]. An unbounded linear operator \( A : D(A) \subseteq X \to Y \) is said to be closed if for all \((x_n)_{n \in \mathbb{N}} \subset D(A)\) such that \( \|x_n - x\| \to 0 \) and \( \|Ax_n - y\| \to 0 \) as \( n \to \infty \), for some \( x \in X \) and \( y \in Y \), then \( x \in D(A) \) and \( y = Ax \). The collection of closed linear operators from \( X \) into \( Y \) is denoted by \( \mathcal{C}(X, Y) \). When \( X = Y \), \( \mathcal{C}(X, X) = \mathcal{L}(X) \).

If \( A \in \mathcal{L}(X) \) and \( B \) is an unbounded linear operator, then \( A + B \) is closed if and only if \( B \) is closed, for more details, we refer to [4].

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Definition 1.1. [4] Let $\omega = (\omega_i)_i$ be a sequence of non-zero elements of $K$. We define $E_\omega$ by

$$E_\omega = \{ x = (x_i)_i : \forall i \in \mathbb{N}, x_i \in K, \text{ and } \lim_{i \to \infty} |\omega_i|^{\frac{1}{2}} |x_i| = 0 \},$$

and it is equipped with the norm

$$\forall x \in E_\omega : x = (x_i)_i, \| x \| = \sup_{i \in \mathbb{N}} (|\omega_i|^{\frac{1}{2}} |x_i|).$$

Remark 1.2. [4]

(i) The space $(E_\omega, \| \cdot \|)$ is a non-archimedean Banach space.

(ii) If

$$\langle \cdot, \cdot \rangle : E_\omega \times E_\omega \longrightarrow K \quad (x, y) \mapsto \sum_{i=0}^{\infty} x_i y_i \omega_i,$$

where $x = (x_i)_i$ and $y = (y_i)_i$. Then, the space $(E_\omega, \| \cdot \|, \langle \cdot, \cdot \rangle)$ is called a p-adic (or non-archimedean) Hilbert space.

(iii) The orthogonal basis $\{ e_i, i \in \mathbb{N} \}$ is called the canonical basis of $E_\omega$, where for all $i \in \mathbb{N}, \| e_i \| = |\omega_i|^{\frac{1}{2}}$.

Definition 1.3. [7] Let $X, Y, Z$ be three non-archimedean Banach spaces over $K$, let $A \in B(X, Y)$ and $B \in B(X, Z)$. Then $A$ majorizes $B$, if there exists $M > 0$ such that

$$\text{for all } x \in X, \| Bx \| \leq M \| Ax \|. \quad (1.1)$$

Theorem 1.4. [7] Assume that, either field $K$ is spherically complete or both $Y$ and $Z$ are countable type Banach spaces over $K$. Let $A \in B(X, Y)$ and $B \in B(X, Z)$. Then the statements are equivalent:

(1) $R(A^*) \subset R(B^*)$;

(2) $B$ majorizes $A$;

(3) there exists a continuous linear operator $D : R(B) \longrightarrow Y$, such that $A = DB$.

In this paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$Ax = \lambda Bx$$

for $\lambda \in K$, $x \in X$, and $A, B \in \mathcal{L}(X)$.

Fore more basic concepts of non-archimedean operators theory, we refer to [4]. In [2], the authors extended the notion of pseudospectrum of linear operator $A$ on non-archimedean Banach space $X$ as follows.

Definition 1.5. [2] Let $X$ be a non-archimedean Banach space over $K$ and $\varepsilon > 0$. The pseudospectrum of a linear operator $A$ on $X$ is defined by

$$\sigma_\varepsilon(A) = \sigma(A) \cup \{ \lambda \in K : \| (\lambda - A)^{-1} \| > \varepsilon^{-1} \},$$

by convention $\| (\lambda - A)^{-1} \| = \infty$ if, and only if, $\lambda \in \sigma(A)$. 

2. Main results

We introduce the following definitions.

**Definition 2.1.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \). Let \( A, B \in \mathcal{L}(X) \), the spectrum \( \sigma(A, B) \) of a pencil of linear operator \((A, B)\) is defined by

\[
\sigma(A, B) = \{ \lambda \in \mathbb{K} : A - \lambda B \text{ is not invertible in } \mathcal{L}(X) \},
\]

\[
= \{ \lambda \in \mathbb{K} : 0 \in \sigma(A - \lambda B) \}.
\]

The resolvent set \( \rho(A, B) \) of a pencil of bounded linear operator \((A, B)\) is

\[
\rho(A, B) = \{ \lambda \in \mathbb{K} : R(\lambda, A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(X) \}.
\]

\( R(\lambda, A, B) \) is called the resolvent of pencil of bounded linear operator \((A, B)\).

**Definition 2.2.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \). Let \( A, B \in \mathcal{L}(X) \), the couple \((A, B)\) is said to be regular, if \( \rho(A, B) \neq \emptyset \).

For a regular couple \((A, B)\), we have the following definitions.

**Definition 2.3.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( A, B \in \mathcal{L}(X) \) and \( \varepsilon > 0 \). The pseudospectrum \( \sigma_\varepsilon(A, B) \) of a pencil of bounded linear operator \((A, B)\) on \( X \) is defined by

\[
\sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{ \lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| > \varepsilon^{-1} \}.
\]

The pseudoresolvent \( \rho_\varepsilon(A, B) \) of a pencil of bounded linear operator \((A, B)\) is defined by

\[
\rho_\varepsilon(A, B) = \rho(A, B) \cap \{ \lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}\| \leq \varepsilon^{-1} \},
\]

by convention \( \|(A - \lambda B)^{-1}\| = \infty \) if, and only if, \( \lambda \in \sigma(A, B) \).

**Definition 2.4.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( A, B \in \mathcal{L}(X) \) and \( \varepsilon > 0 \). The generalized pseudospectrum of a pencil of bounded linear operator \((A, B)\) on \( X \) is defined by

\[
\Sigma_\varepsilon(A, B) = \sigma(A, B) \cup \{ \lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1} \}.
\]

By convention \( \|(A - \lambda B)^{-1}B\| = \infty \) if, and only if, \( \lambda \in \sigma(A, B) \).

**Remark 2.5.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( A, B \in \mathcal{L}(X) \). Then

(i) If \( B = I \), then, \( \Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A) \) where \( \sigma_\varepsilon(A) \) is the pseudospectrum of \( A \).

(ii) Definition 2.3 is a natural generalization of Definition 1.5.

In the rest of this section, we suppose that \((A, B)\) is regular. The next proposition gives a comparison between \( \sigma_\varepsilon(A, B) \) and \( \Sigma_\varepsilon(A, B) \).

**Proposition 2.6.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( A, B \in \mathcal{L}(X) \). Then for all \( \varepsilon > 0 \),

\[
\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).
\]

**Proof.** Let \( \varepsilon > 0 \) and \( \lambda \in \Sigma_\varepsilon(A, B) \), then \( \lambda \in \sigma(A, B) \) and

\[
\frac{1}{\varepsilon} < \|(A - \lambda B)^{-1}B\| \quad (2.1)
\]

\[
\leq \|(A - \lambda B)^{-1}\| \|B\|. \quad (2.2)
\]

Hence

\[
\frac{1}{\|B\|\varepsilon} < \|(A - \lambda B)^{-1}\|.
\]

Thus \( \lambda \in \sigma_{\varepsilon\|B\|}(A, B) \). Consequently

\[
\Sigma_\varepsilon(A, B) \subset \sigma_{\varepsilon\|B\|}(A, B).
\]

\( \square \)
Lemma 2.7. Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $\|B\| = 1$. Then for all $\varepsilon > 0$, 
\[ \Sigma_\varepsilon(A, B) \subset \sigma_\varepsilon(A, B). \]

Theorem 2.8. Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B, C \in \mathcal{L}(X)$ such that $C^{-1} \in \mathcal{L}(X)$. Then

(i) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \Sigma_\varepsilon(CA, CB)$.

(ii) For all $\varepsilon > 0$, $\Sigma_\varepsilon(A, C) = \sigma_\varepsilon(C^{-1}A)$. In particular $C = I$, $\Sigma_\varepsilon(A, I) = \sigma_\varepsilon(A)$.

Proof. (i) For all $\lambda \in \rho(A, B)$, we have $(A - \lambda B)^{-1}C^{-1} = (CA - \lambda CB)^{-1}$. Then, $\sigma(A, B) = \sigma(CA, CB)$. In addition, it is clear that

\[ \|(CA - \lambda CB)^{-1}CB\| = \|(A - \lambda B)^{-1}B\|. \]

Hence $\lambda \in \Sigma_\varepsilon(A, B)$, if, and only if, $\lambda \in \Sigma_\varepsilon(CA, CB)$.

(ii) Assume that $C$ is invertible, then $(A - \lambda C)^{-1} = (C^{-1}A - \lambda I)^{-1}$. Then $\lambda \in \Sigma_\varepsilon(A, C)$, if and only if $\lambda \in \sigma_\varepsilon(C^{-1}A)$.

\[ \square \]

Proposition 2.9. Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$. For all $\varepsilon > 0$, we have

(i) $\sigma(A, B) = \bigcap_{\varepsilon > 0} \Sigma_\varepsilon(A, B)$.

(ii) If $0 < \varepsilon_1 \leq \varepsilon_2$, then $\sigma(A, B) \subset \Sigma_{\varepsilon_1}(A, B) \subset \Sigma_{\varepsilon_2}(A, B)$.

Proof. (i) By Definition 2.4, we have for all $\varepsilon > 0$, $\sigma(A, B) \subset \Sigma_\varepsilon(A, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon > 0} \Sigma_\varepsilon(A, B)$, then for all $\varepsilon > 0$, $\lambda \in \Sigma_\varepsilon(A, B)$. If $\lambda \not\in \sigma(A, B)$, then $\lambda \in \{\lambda \in \mathbb{K} : \|(A - \lambda B)^{-1}B\| > \varepsilon^{-1}\}$, taking limits as $\varepsilon \to 0^+$, we get $\|(A - \lambda B)^{-1}B\| = \infty$. Thus $\lambda \in \sigma(A, B)$.

(ii) For $0 < \varepsilon_1 \leq \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A, B)$, then $\|(A - \lambda B)^{-1}B\| > \varepsilon_1^2 \geq \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A, B)$.

\[ \square \]

Proposition 2.10. Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $X$ be a free Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$ be analytic operator with compact spectrum $\sigma(A) \neq \emptyset$. Then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B, C)$. In addition we have, for all $\varepsilon > 0$, $\Sigma_\varepsilon(B, C) = \sigma_\varepsilon(A)$.

Proof. Let $\alpha \in \rho(A)$, we set $C = (A - \alpha I)^{-1}$ and $B = A(A - \alpha I)^{-1}$. Then

\[ \lambda \in \rho(A) \iff (A - \lambda I)^{-1} \in \mathcal{L}(X) \]
\[ \iff (A - \lambda I)(A - \alpha I)^{-1} \in \mathcal{L}(X) \]
\[ \iff A(A - \alpha I)^{-1} - \lambda(A - \alpha I)^{-1} \in \mathcal{L}(X) \]
\[ \iff B - \lambda C \in \mathcal{L}(X) \]
\[ \iff (B - \lambda C)^{-1} \in \mathcal{L}(X) \]
\[ \iff \lambda \in \rho(B, C). \]

Thus, $\sigma(A) = \sigma(B, C)$, let $\varepsilon > 0$, $z \in \sigma_\varepsilon(A)$, then $z \in \sigma(A)$ and

\[ \frac{1}{\varepsilon} < \|(A - zI)^{-1}\|, \]
\[ = \|(B - zC)^{-1}C\|. \]

Thus $\sigma_\varepsilon(A) = \Sigma_\varepsilon(B, C)$.

\[ \square \]
Proposition 2.11. Let $X$ be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{C}(X)$ with $\rho(A) \neq \emptyset$, then there exists $B, C \in \mathcal{L}(X)$ such that $\sigma(A) = \sigma(B,C)$.

Proof. Similar to the proof of Proposition 2.10. \hfill $\square$

We have the following examples.

Example 2.12. Let $\mathbb{K} = \mathbb{Q}_p$. Let $2 \times 2$ square matrix $A$ and $B$ over $\mathbb{Q}_p \times \mathbb{Q}_p$ and $a, b, c, d \in \mathbb{Q}_p^*$ such that $a \neq b$ and $c \neq d$. Then:

(i) If

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}. $$

It is easy to see that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = -\lambda a$, then $\sigma(A, B) = \{0\}$. Simple calculation, we get

$$(A - \lambda B)^{-1}B = \begin{pmatrix} \frac{-1}{\lambda} & 0 \\ 0 & 0 \end{pmatrix}, $$

thus, for all $\varepsilon > 0$, $\Sigma_\varepsilon(A, B) = \{0\} \cup \{\lambda \in \mathbb{Q}_p : |\lambda|_p < \varepsilon\}$.

(ii) If

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}. $$

Note that, for all $\lambda \in \mathbb{Q}_p$, $\det(A - \lambda B) = (a - \lambda c)(b - \lambda d)$, then $\sigma(A, B) = \{\frac{a}{c}, \frac{b}{d}\}$ and

$$\|(A - \lambda B)^{-1}B\| = \max \left\{ \frac{|c|_p}{|a - \lambda c|_p}, \frac{|d|_p}{|b - \lambda d|_p} \right\}. $$

Hence, the generalized pseudospectrum of $(A, B)$ is

$$\Sigma_\varepsilon(A, B) = \left\{ \frac{a}{c}, \frac{b}{d} \right\} \cup \left\{ \lambda \in \mathbb{Q}_p : \max \left\{ \frac{1}{|ac^{-1} - \lambda|_p}, \frac{1}{|bd^{-1} - \lambda|_p} \right\} > \frac{1}{\varepsilon} \right\}. $$

Example 2.13. Let $A, B \in \mathcal{L}(\mathbb{E}_\omega)$ be two diagonal operators such that for all $i \in \mathbb{N}$, $Ae_i = a_i e_i$ and $Be_i = b_i e_i$, where $a_i, b_i \in \mathbb{Q}_p$ and $B$ is invertible and $\sup_{i \in \mathbb{N}} |a_i|_p$ and $\sup_{i \in \mathbb{N}} |b_i|_p$ are finite and $0 < \inf_{i \in \mathbb{N}} |b_i|_p \leq \sup_{i \in \mathbb{N}} |b_i|_p \leq 1$. It is easy to see that

$$\sigma(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i - \lambda b_i| = 0\} = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0\}$$

and for all $\lambda \in \rho(A, B)$, we have

$$\|(A - \lambda B)^{-1}B_e_i\| = \sup_{i \in \mathbb{N}} \frac{|(A - \lambda B)^{-1}B e_i|}{\|e_i\|} = \sup_{i \in \mathbb{N}} \left| \frac{b_i}{a_i - \lambda b_i} \right| = \frac{1}{\inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda|}. $$

Hence,

$$\left\{ \lambda \in \mathbb{Q}_p : \|(A - \lambda B)^{-1}B\| > \frac{1}{\varepsilon} \right\} = \left\{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| > \frac{1}{\varepsilon} \right\}. $$

Consequently,

$$\Sigma_\varepsilon(A, B) = \{\lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| = 0\} \cup \left\{ \lambda \in \mathbb{Q}_p : \inf_{i \in \mathbb{N}} |a_i b_i^{-1} - \lambda| < \varepsilon \right\}. $$
We have the following results.

**Theorem 2.14.** Let $X, Y$ be two non-Archimedean Banach spaces over $\mathbb{K}$, let $A \in \mathcal{C}(X, Y)$, $B \in \mathcal{L}(X, Y)$ and $\lambda \in \mathbb{K}$. If $A - \lambda B$ is one-to-one and onto, then $(A - \lambda B)^{-1}$ is a closed linear operator.

**Proof.** Let $\lambda \in \mathbb{K}$ and a sequence $(y_n)_n \subseteq Y$ such that $y_n$ converges to $y$ in $Y$ and $(A - \lambda B)^{-1}y_n$ converges to $x$ in $X$. Setting $x_n = (A - \lambda B)^{-1}y_n$, then $x_n \in D(A)$ and $y_n = (A - \lambda B)x_n \in Y$. Since $(A - \lambda B)x_n \to y$ and $x_n \to x$ and $A - \lambda B$ is a closed linear operator which implies that $x \in D(A)$ and $y = (A - \lambda B)x$, then $y \in Y$ and $x = (A - \lambda B)^{-1}y$. Thus $(A - \lambda B)^{-1}$ is a closed linear operator. \hfill \Box

**Corollary 2.15.** Let $X, Y$ be two non-Archimedean Banach spaces over $\mathbb{K}$. Let $A$ be a linear operator from $X$ into $Y$ and $B$ be a non null bounded linear operator from $X$ into $Y$. If $A$ is a non closed operator, then $\sigma(A, B) = \mathbb{K}$.

**Proof.** Let $A$ be a linear operator which is not closed. We argue by contradiction. Suppose that $\rho(A, B)$ is not empty, then there exists $\lambda \in \mathbb{K}$ such that $\lambda \in \rho(A, B)$, consequently, $(A - \lambda B)^{-1}$ is a bounded operator. Hence, $A - \lambda B$ is a closed operator. In addition, we can write $A = A - \lambda B + \lambda B$. We conclude that $A$ is a closed operator, which is a contradiction. \hfill \Box

**Corollary 2.16.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $\sigma(A, B) = \mathbb{K}$, then $A$ is not invertible.

**Proposition 2.17.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ and $B^{-1} \in \mathcal{L}(X)$, then $\sigma_p(B^{-1}A) = \sigma_p(A, B)$.

**Proof.** Let $\lambda \in \sigma_p(A, B)$, then there exists $x \in X \setminus \{0\}$ such that $Ax = \lambda Bx$. Since $B^{-1} \in \mathcal{L}(X)$, we have $B^{-1}Ax = \lambda x$, thus $\lambda \in \sigma_p(B^{-1}A)$, therefore $\sigma_p(A, B) \subseteq \sigma_p(B^{-1}A)$. Similarly, we obtain $\sigma_p(B^{-1}A) \subseteq \sigma_p(A, B)$. Thus $\sigma_p(A, B) = \sigma_p(B^{-1}A)$.

**Theorem 2.18.** Let $A, B \in \mathcal{L}(\mathbb{K}^n)$. If $A$ is invertible and $A^{-1}B$ or $BA^{-1}$ is nilpotent, then $\sigma(A, B) = \emptyset$.

**Proof.** Assume that $A$ is invertible and $A^{-1}B$ or $BA^{-1}$ is nilpotent, then for all $\lambda \in \mathbb{K}$, $I - \lambda A^{-1}B$ or $I - \lambda BA^{-1}$ is invertible, hence for all $\lambda \in \mathbb{K}$, $(A - \lambda B)^{-1}$ exists in $\mathcal{L}(\mathbb{K}^n)$. Thus, $\sigma(A, B) = \emptyset$. \hfill \Box

**Theorem 2.19.** Let $X$ be a Banach space of countable type over $\mathbb{Q}_p$, let $A, B \in \mathcal{L}(X)$ such that $B$ majorizes $A$ and $B$ is not invertible, then $\sigma(A, B) = \mathbb{Q}_p$.

**Proof.** If $B$ majorizes $A$. From Theorem 1.4, there exists a continuous linear operator $D : R(B) \to X$ such that $A = DB$, then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B = (D - \lambda)B$, since $B$ is not invertible then, for all $\lambda \in \mathbb{Q}_p$, $A - \lambda B$ is not invertible. Thus, $\sigma(A, B) = \mathbb{Q}_p$. \hfill \Box

**Proposition 2.20.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ and $0 \in \rho(A) \cap \rho(B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$.

**Proof.** From $A - \lambda B = -\lambda B(A^{-1} - \lambda^{-1}B^{-1})A$, we obtain that $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda} \in \sigma(A^{-1}, B^{-1})$. \hfill \Box

**Proposition 2.21.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, B \in \mathcal{L}(X)$ such that $AB = BA$ with $\sigma(A, B) \neq \emptyset$. If $\mu \in \rho(A, B)$, then $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\mu - \lambda} \in \sigma((A - \mu B)^{-1}B)$.

**Proof.** Let $A, B \in \mathcal{L}(X)$ and $\mu \in \rho(A, B)$. For $\lambda \in \mathbb{K}$ with $\lambda \neq \mu$, we have

$$A - \lambda B = (A - \mu B)((A - \mu B)^{-1}B - (\lambda - \mu)^{-1})(\mu - \lambda).$$

Hence $\lambda \in \sigma(A, B)$ if and only if $\frac{1}{\lambda - \mu} \in \sigma((A - \mu B)^{-1}B)$. \hfill \Box
3. Non-archimedean generalized spectrum approximation

In [1], the authors extended the following definitions to non-archimedean case.

**Definition 3.1.** [1] Let $X$ be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$.

1. A sequence $(A_n)$ of bounded linear operators on $X$ is said to be norm convergent to $A$, denoted by $A_n \to A$, if $\lim_{n \to \infty} \|A_n - A\| = 0$.
2. A sequence $(A_n)$ of bounded linear operators on $X$ is said to be pointwise convergent to $A$, denoted by $A_n \overset{p}{\to} A$, if for all $x \in X$, $\lim_{n \to \infty} \|A_n x - Ax\| = 0$.

**Definition 3.2.** [1] Let $X$ be a non-archimedean Banach space over $\mathbb{K}$ and let $A \in \mathcal{L}(X)$. A sequence $(A_n)$ of bounded linear operators on $X$ is said to be $\nu$-convergent to $A$, denoted by $A_n \overset{\nu}{\to} A$, if

1. $(\|A_n\|)$ is bounded,
2. $\|(A_n - A)A\| \to 0$ as $n \to \infty$, and
3. $\|(A_n - A)A_n\| \to 0$ as $n \to \infty$.

**Definition 3.3.** [1] Let $X$ be a non-archimedean Banach space over a locally compact field $\mathbb{K}$ and let $A \in \mathcal{L}(X)$. A sequence $(A_n)$ of bounded linear operators on $X$ is said to be convergent to $A$ in the collectively compact convergence, denoted by $A_n \overset{c.c}{\to} A$, if $A_n \overset{P}{\to} A$, and for some positive integer $N$,

$\bigcup_{n \geq N} \{(A_n - A)x : x \in X, \|x\| \leq 1\}$

has compact closure of $X$.

We have the following results.

**Proposition 3.4.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \to A$ or $B_n \to B$, then for any $C \in B(X)$, we have

$$\|(A_n - A)C(B_n - B)\| \to 0.$$  

**Proof.** Since $A_n \to A$ or $B_n \to B$, then for any $C \in B(X)$, we have

$$\|(A_n - A)C(B_n - B)\| \leq \|(A_n - A)\|\|C\|\|(B_n - B)\| \to 0.$$  

$\square$

**Proposition 3.5.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $A, A_n, B, B_n \in \mathcal{L}(X)$. If $A_n \overset{\nu}{\to} A$, $B_n \overset{\nu}{\to} B$ and $0 \in \rho(A) \cap \rho(B)$, then for all $\lambda \in \mathbb{K}$, we have

$$A_n - \lambda B_n \to A - \lambda B.$$  

**Proof.** Suppose that $A_n \overset{\nu}{\to} A$, $B_n \overset{\nu}{\to} B$, $0 \in \rho(A) \cap \rho(B)$, and for all $\lambda \in \mathbb{K}$, we have

$$\|(A_n - \lambda B_n) - (A - \lambda B)\| \leq \max \left\{ \|(A_n - A)\|; |\lambda|\|(B_n - B)\| \right\} \to 0.$$  

Since,

$$\|(A_n - A)\| = \|(A_n - A)AA^{-1}\| \leq \|(A_n - A)A\|\|A^{-1}\| \to 0.$$  

$\square$
Similarly, we obtain \( \| (B_n - B) \| \to 0 \).

**Proposition 3.6.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{Q}_p \) such that \( \| X \| \leq |\mathbb{Q}_p| \), let \( A, A_n, B, B_n \in \mathcal{L}(X) \). If \( A_n \xrightarrow{p} A \) and \( B_n \xrightarrow{c} B \), then for any \( C \in \mathcal{L}(X) \), we have
\[
\|(A_n - A)C(B_n - B)\| \to 0.
\]

**Proof.** Since \( A_n \xrightarrow{p} A \) and \( B_n \xrightarrow{c} B \), and \( \| X \| \leq |\mathbb{Q}_p| \), hence \( A_n \xrightarrow{p} A \) and \( B_n \xrightarrow{p} B \) and
\[
C \left( \bigcup_{n \geq N} \{ (B_n - B)x : x \in X, \| x \| \leq 1 \} \right)
\]
has compact closure of \( X \). Then
\[
\|(A_n - A)C(B_n - B)\| \to 0.
\]

\( \Box \)

The aim of the following results is to discuss the spectrum of a sequence of a pencil of linear operators in a non-archimedean Banach space.

**Theorem 3.7.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( (A_n) \) be a sequence of bounded linear operators on \( X \) and \( A \in \mathcal{L}(X) \). If \( A_n \to A \), then there exists \( N \in \mathbb{N} \), we have
\[
\text{for all } n \geq N, \, \sigma(A_n) \subset \sigma(A).
\]

**Proof.** Let \( \lambda \in \rho(A) \). Then for all \( n \in \mathbb{N} \), we have
\[
\lambda I - A_n = (\lambda I - A) \left( I + (\lambda I - A)^{-1}(A - A_n) \right).
\]

Since \( A_n \to A \) then \( \lim_{n \to \infty} \| A_n - A \| = 0 \), hence for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \| A_n - A \| < \varepsilon \). In particular, for \( \varepsilon = \| (\lambda I - A)^{-1} \|^{-1} \), we have
\[
\text{for all } n \geq N, \, \| A_n - A \| < \| (\lambda I - A)^{-1} \|^{-1}.
\]

Thus, for all \( n \geq N \), we have
\[
\| (\lambda I - A)^{-1} (A - A_n) \| \leq \| (\lambda I - A)^{-1} \| \| (A - A_n) \| < 1.
\]

Then for all \( n \geq N \), \( \left( I + (\lambda I - A)^{-1}(A - A_n) \right)^{-1} \in \mathcal{L}(X) \), hence for all \( n \geq N \), \( (\lambda - A_n)^{-1} \in \mathcal{L}(X) \). Thus, for all \( n \geq N \), \( \lambda \in \rho(A_n) \).

\( \Box \)

We have the following proposition.

**Proposition 3.8.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( A, B, C, D \in \mathcal{L}(X) \) such that \( \rho(A, C) \neq \emptyset \). For all \( z \in \rho(A, C) \) such that \( \| R(z, A, C)[(A - B) - z(C - D)] \| < 1 \), we have \( z \in \rho(B, D) \).

**Proof.** Since, for all \( z \in \rho(A, C) \) such that \( \| R(z, A, C)[(A - B) - z(C - D)] \| < 1 \) and
\[
B - zD = (A - zC) \left[ I - R(z, A, C) \left( (A - B) - z(C - D) \right) \right].
\]

Hence \( (B - zD) \) is invertible and \( (B - zD)^{-1} \in \mathcal{L}(X) \). Thus, \( z \in \rho(B, D) \).

\( \Box \)

**Theorem 3.9.** Let \( X \) be a non-archimedean Banach space over \( \mathbb{K} \), let \( (A_n) \) and \( (B_n) \) be a sequences of bounded linear operators on \( X \) and \( A, B \in \mathcal{L}(X) \). If \( A_n \to A \) and \( B_n \to B \), then there exists \( N \in \mathbb{N} \), we have
\[
\text{for all } n \geq N, \, \sigma(A_n, B_n) \subset \sigma(A, B).
\]
Proof. Let $\lambda \in \rho(A, B)$. Then for all $n \in \mathbb{N}$, we can write

$$A_n - \lambda B_n = \left( I - (E_n - \lambda F_n) \right) (A - \lambda B),$$

where $E_n = (A - A_n)R(\lambda, A, B)$ and $F_n = (B - B_n)R(\lambda, A, B)$. Since $A_n \to A$ and $B_n \to B$, then for all $\lambda \in \rho(A, B)$, $\lim_{n \to \infty} \|E_n - \lambda F_n\| = 0$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|E_n - \lambda F_n\| < 1$.

Then, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$(I - (E_n - \lambda F_n))^{-1} \in \mathcal{L}(X).$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $A_n - \lambda B_n$ is invertible $\mathcal{L}(X)$. Then, for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. \hfill \Box

**Theorem 3.10.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $(A_n)$, $(B_n)$ be sequences of bounded linear operators on $X$ and $A, B \in \mathcal{L}(X)$. If $A_n \to A$ and $B_n \to B$, then for all $\lambda \in \rho(A, B)$, $(A_n - \lambda B_n)^{-1} \to (A - \lambda B)^{-1}$.

Proof. Let $\lambda \in \rho(A, B)$. Since $A_n \to A$ and $B_n \to B$, then by using Theorem 3.9, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\lambda \in \rho(A_n, B_n)$. Then for all $n \geq N$,

$$\| (A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1} \|$$

(3.1)

$$\begin{align*}
(3.1) & = \| (A_n - \lambda B_n)^{-1} (A - \lambda B) - (A_n - \lambda B_n) (A - \lambda B)^{-1} \| \\
& = \| (A_n - \lambda B_n)^{-1} (A - A_n - \lambda (B - B_n)) (A - \lambda B)^{-1} \| \\
& \leq \max \left\{ \| (A_n - A_n) \|, \| \lambda \| \| (B - B_n) \| \right\} \| (A_n - \lambda B_n)^{-1} \| \| (A - \lambda B)^{-1} \| \\
\end{align*}$$

Since $A_n \to A$ and $B_n \to B$, and $\| (A_n - \lambda B_n)^{-1} \| \| (A - \lambda B)^{-1} \| < \infty$. Thus,

$$\lim_{n \to \infty} \| (A_n - \lambda B_n)^{-1} - (A - \lambda B)^{-1} \| = 0.$$ 

\hfill \Box

**Proposition 3.11.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $(A_n)$ be a sequence of bounded linear operators on $X$ and $A \in \mathcal{L}(X)$. If $A_n \to A$, then $A_n \overset{\mathbb{K}}{\to} A$.

Proof. Obvious. \hfill \Box

**Theorem 3.12.** Let $X$ be a non-archimedean Banach space over $\mathbb{K}$, let $(A_n)$, $(B_n)$ be sequences of bounded linear operators on $X$ and $A, B \in \mathcal{L}(X)$. If $A_n \to A$ and $B_n \to B$, then $(A_n - \lambda B_n)^{-1} \overset{\mathbb{K}}{\to} (A - \lambda B)^{-1}$ for all $\lambda \in \rho(A, B)$.

Proof. It suffices to apply Theorem 3.10 and Proposition 3.11. \hfill \Box

**References**


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