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On a New Variant of $\mathcal{I}$-Convergence in Topological Spaces

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ABSTRACT: In this write-up, we mainly introduce b-$\mathcal{I}$-convergence of sequences, b-convergence and b-$\mathcal{I}$-convergence of nets in topological spaces, and put forward some important topological investigations. Existence of b-ω-accumulation point is presented via admissible ideal and b-$\mathcal{I}$-cluster point of sequence. It is shown that a map $f : Z \to W$ is quasi-b-irresolute if and only if for every net $(s_d)_{d\in D}$ converging to z_o, the image net $(f(s_d)_{d\in D})$ b-converges to $f(z_o)$. Notion of b-$\mathcal{I}$-cluster point of net is disclosed along with its a nice characterization as: 'Corresponding to a given net $s : D \to Z$, there exists a filter $\mathcal{G}$ on Z such that $z_o \in Z$ is a b-$\mathcal{I}$-cluster point of the net $(s_d)_{d\in D}$ if and only if z_o is a b-cluster point of the filter $\mathcal{G}$'. Another characterization of b-$\mathcal{I}$-cluster point of net with respect to a certain type of class of subsets is demonstrated. Further, we show that b-$\mathcal{I}$-cluster point of a net in a b-compact space always exist.

Key Words: $\mathcal{I}$-convergence, admissible ideal, b-$\mathcal{I}$-convergence, b-open set, b-compact space.

Contents

1. Introduction

We start with the definition of statistical convergence which is an extension of the concept of ordinary convergence of a sequence of real numbers (see [14], [29]) as follows: Let $\mathbb{N}$ denotes the set of all positive integers. For $A \subseteq \mathbb{N}$, the asymptotic or natural density (see [16], [24]) of A is defined by $\delta(A) = \lim_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n}$, provided the limit exists, where $|K|$ denotes the cardinality of the set K. A sequence $(z_n)_{n\in\mathbb{N}}$ of real numbers is called statistically convergent to $z_o \in \mathbb{R}$ (set of all real numbers) if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |z_k - z_o| \geq \varepsilon\}) = 0$. For applications of statistical convergence, interested readers can see references [8,9,21]. In 2002, Baláž *et. al.* (see [7]) gave a new extension, called $\mathcal{I}$-convergence, of statistical convergence of real sequences using ideal of subsets of $\mathbb{N}$. Recall that an ideal (see [18]) $\mathcal{I}$ on a non-empty set X is a non-empty family of subsets of X that satisfies the conditions: (i) $\varnothing \in \mathcal{I}$, (ii) $A \subseteq B \in \mathcal{I}$ implies $A \in \mathcal{I}$ and (iii) A, $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \{\varnothing\}$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ on X is called admissible if $\mathcal{I}$ contains each singleton subsets of X. For example, $\mathcal{I}_{fin} := \{A \subseteq \mathbb{N} : A \text{ is finite}\}$ and $\mathcal{I}_\delta := \{A \subseteq \mathbb{N} : \delta(A) = 0\}$ are admissible ideals on $\mathbb{N}$. On the other hand, a filter (see [18]) $\mathcal{F}$ on a non-empty set X is a non-empty family of subsets of X which obeys the conditions: (i) $\varnothing \notin \mathcal{F}$, (ii) $A \supseteq B \in \mathcal{F}$ implies $A \in \mathcal{F}$, and (iii) A, $B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$. Notice that $\mathcal{I}$ is a non-trivial ideal on X if and only if $\mathcal{F}_\mathcal{I} = \{A \subseteq X : X \setminus A \in \mathcal{I}\}$ is a filter on X. The filter $\mathcal{F}_\mathcal{I}$ is called the associated filter of $\mathcal{I}$. For some new results related to associated filter presented by Modak *et. al.*, interested readers can see [22]. Recently, Lahiri and Das (see [19]) (resp., Di Maio and Kočinac (see [11])) settled the notion of $\mathcal{I}$-convergence (resp., statistical convergence) in topological spaces. On the other hand, in [6], the concept of open set in topological spaces has been extended to b-open set by Andrijević. For more information, readers are referred to [2,3,4,5]. In a very recent, utilizing b-open set, Granados (see [15]) has set up an interesting generalization of the concept of $\mathcal{I}$-convergence in topological spaces by the name of b-$\mathcal{I}$-convergence.

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Since the class of all b-open sets does not form a topology again, it is reasonable to consider b-$\mathcal{I}$-convergence in topological space and to investigate its effect to the basic properties. We organize this write-up by dividing into 5 sections. In section 3, various topological aspects regarding b-$\mathcal{I}$-convergence of sequences and b-$\mathcal{I}$-cluster point of sequences are studied. In section 4, we introduced b-convergence of nets in topological spaces and studied its some properties. Here, we have shown that a map $f : Z \to W$ is quasi-b-irresolute if and only if for every net $(s_d)_{d\in D}$ converging to z_o, the net $(f(s_d)_{d\in D})$ b-converges to $f(z_o)$. In section 5, b-$\mathcal{I}$-convergence and b-$\mathcal{I}$-cluster point of nets has been disclosed and some important topological observations are demonstrated carefully.

2. Known Facts

Throughout this paper, (Z, σ) (or Z) and (W, ρ) (or W) will stand for a topological space on which no separation axioms are permissible unless explicitly recalled, and $\mathcal{I}$ for a non-trivial ideal on $\mathbb{N}$ otherwise mentioned clearly. Now, we recall $\mathcal{I}$-convergence and statistical convergence in topological spaces from literature as follows:

Definition 2.1. *[19] A sequence $(z_n)_{n\in\mathbb{N}}$ in Z is addressed as $\mathcal{I}$-convergent to $z_o \in Z$ if for every open set Q containing z_o, $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathcal{I}$, and is expressed by $z_n \xrightarrow{\mathcal{I}} z_o$.*

Definition 2.2. *[11] A sequence $(z_n)_{n\in\mathbb{N}}$ in Z is said to be statistically convergent to $z_o \in Z$ if for every open set Q containing z_o, $\delta(\{n \in \mathbb{N} : z_n \notin Q\}) = 0$, and is expressed by $z_n \xrightarrow{stat} z_o$.*

In this paragraph, we now collect some basic notions and terminologies from [6], [10] and [1]. A subset Q of Z is called b-open [6] if $Q \subseteq Cl(Int(Q)) \cup Int(Cl(Q))$, where '$Cl$' (resp., '$Int$') denotes the closure (resp., interior) operator in Z. The family of all b-open sets in Z is denoted as $BO(Z)$. Complement of a b-open set is known as b-closed [6]. For $Q \subseteq Z$, its b-closure (resp., b-interior), denoted by $bcl(Q)$ [6] or $Cl_b(Q)$ [10] (resp., $bint(Q)$ [6] or $Int_b(Q)$ [10]), is defined in an analogous manner of Cl (resp., Int) operator. A subset Q of Z is said to be a b-neighbourhood [10] of a point $z_o \in Z$ if there exists a b-open set U such that $z_o \in U \subseteq Q$. We use the notation $\mathcal{N}_b(z_o)$ for the collection of all b-neighbourhoods of z_o. A point $z_o \in Z$ is called a b-limit point [1] of $Q \subseteq Z$ if for every b-open set U containing z_o, we have $U \cap (Q \setminus \{z_o\}) \neq \varnothing$, and the collection of all b-limit points of Q is denoted by $D_b(Q)$.

Definition 2.3. *A space Z is called*

1. *b-T_0 (see [10]) if for any pair of distinct points x and y of Z, there exists a b-open set U containing x but not y or a b-open set V containing y but not x.*
2. *b-T_2 or b-Hausdorff (see [26]) if for any pair of distinct points x and y of Z, there exist U, $V \in BO(Z)$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

Definition 2.4. *A function $f : Z \to W$ is said to be b-continuous (see [13,28]) (resp., b-irresolute (see [12,28])) at $z_o \in Z$ if for each open (resp., b-open) set V containing $f(z_o)$, there exists a b-open set Q containing z_o such that $f(Q) \subseteq V$.*

3. b-$\mathcal{I}$-convergence of sequence in topological spaces

We begin this section by recalling the definition of b-$\mathcal{I}$-convergence from [15].

Definition 3.1. *[15] A sequence $(z_n)_{n\in\mathbb{N}}$ in a space Z is said to be b-$\mathcal{I}$-convergent to a point $z_o \in Z$ if for every b-open set Q containing z_o, we have $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathcal{I}$. Symbolically, we express it as* b-$\mathcal{I}$-$\lim z_n = z_o$ *or $z_n \xrightarrow{b-\mathcal{I}} z_o$, and call z_o as b-$\mathcal{I}$-limit of the sequence $(z_n)_{n\in\mathbb{N}}$.*

Example 3.2. *Let $Z = \{p, q, r\}$ and $\sigma = \{\varnothing, \{p\}, \{q\}, \{p, q\}, Z\}$. Then $BO(Z) = \{\varnothing, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}, Z\}$. Define a sequence $(z_n)_{n\in\mathbb{N}}$ in Z as follows:*

$$z_n = \begin{cases} q, & \text{if } n \text{ is a prime number} \\ p, & \text{if } n \text{ is a square number} \\ r, & \text{otherwise.} \end{cases}$$

Then for any b-open set Q containing r, $\{n \in \mathbb{N} : z_n \notin Q\}$ is the set P of all prime numbers or the set S of all square numbers or $\varnothing$. Consider the ideal $\mathcal{I} = \mathcal{I}_\delta$ on $\mathbb{N}$. Since $\delta(P) = \delta(S) = \delta(\varnothing) = 0$, we have $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathcal{I}$. Hence, the sequence $(z_n)_{n\in\mathbb{N}}$ is b-$\mathcal{I}$-convergent to r.

Lemma 3.3. *[6] Every open set in Z is a b-open set.*

Lemma 3.4. *If $\mathcal{I} = \mathcal{I}_{fin}$, then b-$\mathcal{I}$-convergence in Z implies usual convergence.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z such that $z_n \xrightarrow{b-\mathcal{I}} z_o \in Z$. To show $z_n \to z_o$, let Q be any open set containing z_o. Then Q is a b-open set, and since $z_n \xrightarrow{b-\mathcal{I}} z_o$, so $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathcal{I} = \mathcal{I}_{fin}$. Take $n_o = \max\{n \in \mathbb{N} : z_n \notin Q\}$. Then for all $n \geq n_o$, $z_n \in Q$, as required. □

Corollary 3.5. *If $\mathcal{I} = \mathcal{I}_{fin}$, then b-$\mathcal{I}$-convergence in Z implies b-convergence (see [28]).*

Proposition 3.6. *If Z be such a space that $\sigma = BO(Z)$, and if $\mathcal{I}$ be an admissible ideal not containing any infinite subset of $\mathbb{N}$, then both the concepts of usual convergence and b-$\mathcal{I}$-convergence coincide.*

Proof. The proof is straightforward, and thus removed. □

Lemma 3.7. *If $\mathcal{I} = \mathcal{I}_\delta$, then b-$\mathcal{I}$-convergence in Z implies statistical convergence.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z such that $z_n \xrightarrow{b-\mathcal{I}} z_o \in Z$. To show $z_n \xrightarrow{stat} z_o$, let Q be any open set containing z_o. Then Q is a b-open set, and since $z_n \xrightarrow{b-\mathcal{I}} z_o$, so $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathcal{I} = \mathcal{I}_\delta$. Thus $\delta(\{n \in \mathbb{N} : z_n \notin Q\}) = 0$, as required. □

Theorem 3.8. *Suppose X be a b-$\mathcal{I}$-space with $|X| \geq 2$.*

1. *If b-$\mathcal{I}$-convergence in Z coincides with usual convergence, then $\mathcal{I} = \mathcal{I}_{fin}$.*
2. *If b-$\mathcal{I}$-convergence in Z coincides with statistical convergence, then $\mathcal{I} = \mathcal{I}_\delta$.*

Proof. We give the proof of 1 only. Let $x, y \in Z$ with $x \neq y$. Since Z is a b-T_0-space, there exists $Q \in BO(Z)$ such that $x \in Q$ but $y \notin Q$. Let $A \in \mathcal{I}_{fin}$, and define a sequence $(z_n)_{n\in\mathbb{N}}$ in Z as:

$$z_n = \begin{cases} y, & \text{if } n \in A \\ x, & \text{if } n \notin A. \end{cases}$$

Then $(z_n)_{n\in\mathbb{N}}$ converges to the point x. By hypothesis, $z_n \xrightarrow{b-\mathcal{I}} x$. Since Q is a b-open set containing x, $\{n \in \mathbb{N} : z_n \notin Q\} = A \in \mathcal{I}$. Thus $\mathcal{I}_{fin} \subseteq \mathcal{I}$. We now claim that $\mathcal{I}$ doesn't contain any infinite subset of $\mathbb{N}$. If possible, let $\mathcal{I}$ contains an infinite subset M of $\mathbb{N}$. Since $\mathcal{I}$ is non-trivial, $\mathbb{N} \setminus M$ is also infinite. Define a sequence $(t_n)_{n\in\mathbb{N}}$ in Z as:

$$t_n = \begin{cases} y, & \text{if } n \in M \\ x, & \text{if } n \in \mathbb{N} \setminus M. \end{cases}$$

Obviously then, $t_n \xrightarrow{b-\mathcal{I}} x$. On the other side, $(t_n)_{n\in\mathbb{N}}$ doesn't converge to x. This contradicts our hypothesis. Therefore $\mathcal{I} \subseteq \mathcal{I}_{fin}$ and consequently, $\mathcal{I} = \mathcal{I}_{fin}$. □

Lemma 3.9. *[15] b-$\mathcal{I}$-convergence implies $\mathcal{I}$-convergence, but not conversely.*

Remark 3.10. *Converse of Lemma 3.9 is considered by Granados in Remark 2 of [15] with an additional condition 'discreteness' of the space. Here, we mention that this condition is just a sufficient condition, not a necessary one because in Sierpiński space, $\mathcal{I}$-convergence implies b-$\mathcal{I}$-convergence though it is not a discrete space. In following lemma, we give a positive response of the open problem set by Granados in Remark 3 of [15].*

Lemma 3.11. *If $\sigma = BO(Z)$, then b-$\mathfrak{I}$-convergence coincides with $\mathfrak{I}$-convergence.*

Lemma 3.12. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be two non-trivial ideals on $\mathbb{N}$ such that $\mathfrak{I} \subseteq \mathfrak{J}$. If $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z such that $z_n \xrightarrow{b-\mathfrak{I}} z_o$, then $z_n \xrightarrow{b-\mathfrak{J}} z_o$.*

Proof. Proof is evident. □

Lemma 3.13. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be two non-trivial ideals on $\mathbb{N}$ and $(z_n)_{n\in\mathbb{N}}$ a sequence in Z. If $z_n \xrightarrow{b-\mathfrak{I}} z_o$ and $z_n \xrightarrow{b-\mathfrak{J}} z_o$, then $z_n \xrightarrow{b-\mathfrak{I}\cap\mathfrak{J}} z_o$.*

Proof. Proof is evident. □

Theorem 3.14. *Suppose Z is a b-Hausdorff space. If $(z_n)_{n\in\mathbb{N}}$ be a b-$\mathfrak{I}$-convergent sequence in Z, then b-$\mathfrak{I}$-limit of $(z_n)_{n\in\mathbb{N}}$ is unique.*

Proof. If possible, suppose that the b-$\mathfrak{I}$-convergent sequence $(z_n)_{n\in\mathbb{N}}$ has two b-$\mathfrak{I}$-limits x and y with $x \neq y$. Since Z is a b-Hausdorff space, there exist P, $Q \in BO(Z)$ such that $x \in P$, $y \in Q$ and $P \cap Q = \varnothing$. On the other side, $\{n \in \mathbb{N} : z_n \notin P\} \in \mathfrak{I}$ and $\{n \in \mathbb{N} : z_n \notin Q\} \in \mathfrak{I}$. Now, $\mathbb{N} = \{n \in \mathbb{N} : z_n \in Z\} = \{n \in \mathbb{N} : z_n \in Z \setminus (P \cap Q)\} \subseteq \{n \in \mathbb{N} : z_n \notin P\} \cup \{n \in \mathbb{N} : x_n \notin Q\} \in \mathfrak{I}$ implies $\mathbb{N} \in \mathfrak{I}$, a contradiction contradicting the fact that $\mathfrak{I}$ is non-trivial. Hence, b-$\mathfrak{I}$-limit of $(z_n)_{n\in\mathbb{N}}$ is unique. □

Corollary 3.15. *Suppose Z is a Hausdorff space. If $(z_n)_{n\in\mathbb{N}}$ be a b-$\mathfrak{I}$-convergent sequence in Z, then b-$\mathfrak{I}$-limit of $(z_n)_{n\in\mathbb{N}}$ is unique.*

Theorem 3.16. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$. If there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements in a set $Q \subseteq Z$ which is b-$\mathfrak{I}$-convergent to $z_o \in Z$, then z_o is a b-limit point of Q.*

Proof. Let G be an arbitrary b-open set containing z_o. Since $z_n \xrightarrow{b-\mathfrak{I}} z_o$, $\{n \in \mathbb{N} : z_n \notin G\} \in \mathfrak{I}$ and consequently, $\{n \in \mathbb{N} : z_n \in G\} \notin \mathfrak{I}$ (because: if $\{n \in \mathbb{N} : z_n \in G\} \in \mathfrak{I}$, then $\mathbb{N} = \{n \in \mathbb{N} : z_n \notin G\} \cup \{n \in \mathbb{N} : z_n \in G\} \in \mathfrak{I}$ which contradicts that $\mathfrak{I}$ is non-trivial). Moreover, $\{n \in \mathbb{N} : z_n \in G\}$ is an infinite set (if not, then $\{n \in \mathbb{N} : z_n \in G\}$ is finite, and since $\mathfrak{I}$ is an admissible ideal, so $\{n \in \mathbb{N} : z_n \in G\} = \bigcup_{z_n \in G} \{n\} \in \mathfrak{I}$ which contradicts that $\{n \in \mathbb{N} : z_n \in G\} \notin \mathfrak{I}$). Pick $n_o \in \{n \in \mathbb{N} : z_n \in G\}$ such that $z_{n_o} \neq z_o$. Then $z_{n_o} \in Q \cap (G \setminus \{z_o\})$ proving that $Q \cap (G \setminus \{z_o\}) \neq \varnothing$, as targeted. □

Corollary 3.17. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$. If there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements in a set $Q \subseteq Z$ which is b-$\mathfrak{I}$-convergent to $z_o \in Z$, then z_o is a limit point of Q.*

Corollary 3.18. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$. If there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of distinct elements in a set $Q \subseteq Z$ which is b-$\mathfrak{I}$-convergent to $z_o \in Z$, then $z_o \in Cl_b(Q)$.*

Definition 3.19. *Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in a space Z. A point $z_o \in Z$ is said to be a b-cluster point of $(z_n)_{n\in\mathbb{N}}$ if for every b-open set Q containing z_o, the set $\{n \in \mathbb{N} : z_n \in Q\}$ is infinite.*

Theorem 3.20. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$, and $(z_n)_{n\in\mathbb{N}}$ is a sequence in Z. If $(z_n)_{n\in\mathbb{N}}$ has a b-$\mathfrak{I}$-convergent subsequence, then $(z_n)_{n\in\mathbb{N}}$ has a b-cluster point.*

Proof. Let (z_{n_k}) be a subsequence of $(z_n)_{n\in\mathbb{N}}$ such that $z_{n_k} \xrightarrow{b-\mathfrak{I}} z_o \in Z$. To show z_o is a b-cluster point of $(z_n)_{n\in\mathbb{N}}$, let G be an arbitrary b-open set containing z_o. Then $\{k \in \mathbb{N} : z_{n_k} \notin G\} \in \mathfrak{I}$. Since $\mathfrak{I}$ is an admissible ideal, $\{k \in \mathbb{N} : z_{n_k} \in G\}$ is infinite. Therefore $\{n \in \mathbb{N} : z_n \in G\}$ is an infinite set, and hence z_o is a b-cluster point of $(z_n)_{n\in\mathbb{N}}$. □

Theorem 3.21. *If $(z_n)_{n\in\mathbb{N}}$ be a sequence in a b-closed set $F \subseteq Z$ which is b-$\mathfrak{I}$-convergent to $z_o \in Z$, then $z_o \in F$.*

Proof. Assume on contrary that $z_o \notin F$. Since F is b-closed, $F = Cl_b(F)$. Thus $z_o \notin Cl_b(F)$. Then there exists a b-open set G containing z_o such that $F \cap G = \varnothing$, by Lemma 2.2. of [10]. Since $z_n \xrightarrow{b-\mathfrak{I}} z_o$, $\{n \in \mathbb{N} : z_n \notin G\} \in \mathfrak{I}$ and hence $\{n \in \mathbb{N} : z_n \in G\} \notin \mathfrak{I}$. This gives $\{n \in \mathbb{N} : z_n \in G\} \neq \varnothing$. Pick $n_o \in \{n \in \mathbb{N} : z_n \in G\}$. Then $z_{n_o} \in G$. On the other side, for each n, $z_n \in F$ and this implies $z_{n_o} \in F$. Therefore $F \cap G \neq \varnothing$, a contradiction. Hence $z_o \in F$. $\square$

Corollary 3.22. *If $(z_n)_{n\in\mathbb{N}}$ be a sequence in a closed set $F \subseteq Z$ which is b-$\mathfrak{I}$-convergent to $z_o \in Z$, then $z_o \in F$.*

Theorem 3.23. *Let $g : Z \to W$ be a b-irresolute function. If $(z_n)_{n\in\mathbb{N}}$ be b-$\mathfrak{I}$-convergent to $z_o \in Z$, then $(g(z_n))_{n\in\mathbb{N}}$ is b-$\mathfrak{I}$-convergent to $g(z_o)$.*

Proof. Let Q be any b-open set in W containing $g(z_o)$. Since $g : Z \to W$ is b-irresolute, there exists a b-open set P in Z containing z_o such that $g(P) \subseteq Q$. Since $z_n \xrightarrow{b-\mathfrak{I}} z_o$, $\{n \in \mathbb{N} : z_n \notin P\} \in \mathfrak{I}$. It is obvious that $\{n \in \mathbb{N} : g(z_n) \notin Q\} \subseteq \{n \in \mathbb{N} : z_n \notin P\}$. Consequently, $\{n \in \mathbb{N} : g(z_n) \notin Q\} \in \mathfrak{I}$ which shows that $g(z_n) \xrightarrow{b-\mathfrak{I}} g(z_o)$, and this proves the theorem. $\square$

Theorem 3.24. *Let $f : Z \to W$ be a b-continuous function. If $(z_n)_{n\in\mathbb{N}}$ be b-$\mathfrak{I}$-convergent to $z_o \in Z$, then $(f(z_n))_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to $f(z_o)$.*

Proof. The proof is parallel to that of Theorem 3.23. $\square$

For our next result, we define a new function as follows:

Definition 3.25. *A function $g : Z \to W$ is said to be quasi-b-irresolute if for each $z \in Z$ and for every b-open set Q containing $g(z)$, there exists an open set P containing z such that $g(P) \subseteq Q$.*

Example 3.26. *Consider $Z = \{a, b, c\}$ with $\sigma = \{\varnothing, \{a, b\}, Z\}$ and $W = \{x, y\}$ with $\rho = \{\varnothing, \{x\}, W\}$. Then $BO(W) = \{\varnothing, \{x\}, W\}$. Define $f : Z \to W$ by $f(a) = f(b) = x$ and $f(c) = y$. Then f is a quasi-b-irresolute function.*

Theorem 3.27. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$, and Z is a first countable space. Then $g : Z \to W$ is quasi-b-irresolute if and only if for every sequence $(z_n)_{n\in\mathbb{N}}$ which is $\mathfrak{I}$-convergent to $z_o \in Z$, the sequence $(g(z_n))_{n\in\mathbb{N}}$ is b-$\mathfrak{I}$-convergent to $g(z_o)$.*

Proof. The forward implication is very transparent. For reverse implication, assume that g is not quasi-b-irresolute. Then there is some $z_o \in Z$ at which g is not quasi-b-irresolute. This means that there is a b-open set Q in W containing $g(z_o)$ such that g-image of every open set containing z_o intersects $W \setminus Q$. Since Z is a first countable space, it has a countable local base, say $\{P_1, P_2, \ldots, P_n, \ldots\}$ at z_o. For each $n \in \mathbb{N}$, let $G_n := \bigcap_{k=1}^{n} P_k$. Then $\{G_1, G_2, \ldots, G_n, \ldots\}$ is also a local base at z_o, and $G_1 \supseteq G_2 \supseteq \cdots G_n \supseteq \cdots$. Moreover, for every $n \in \mathbb{N}$, $g(G_n) \cap (W \setminus Q) \neq \varnothing$. So for every $n \in \mathbb{N}$, pick $w_n \in g(G_n) \cap (W \setminus Q)$. Then there exists $z_n \in G_n$ such that $g(z_n) = w_n$ for every n. Since Q is a b-open set containing $g(z_o)$ and $\{n \in \mathbb{N} : g(z_n) = w_n \notin Q\} = \mathbb{N} \notin \mathfrak{I}$ (as $\mathfrak{I}$ is non-trivial), $(g(z_n))_{n\in\mathbb{N}}$ is not b-$\mathfrak{I}$-convergent to $g(z_o)$. Now, we claim that $(z_n)_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to z_o. For this purpose, let U be any open set containing z_o. Since $\{G_1, G_2, \ldots, G_n, \ldots\}$ is a local base at z_o, there is some $n_o \in \mathbb{N}$ such that $G_{n_o} \subseteq U$. Thus for all $n \geq n_o$, $z_n \in G_{n_o}$ and so $z_n \in U$. This yields $\{n \in \mathbb{N} : z_n \notin U\}$ is finite and consequently, $\delta\{n \in \mathbb{N} : z_n \notin U\} = 0$. Since $\mathfrak{I}$ is an admissible ideal, $\{n \in \mathbb{N} : z_n \notin U\} \in \mathfrak{I}$. Thus $z_n \xrightarrow{\mathfrak{I}} z_o$. Therefore by our hypothesis, $g(z_n) \xrightarrow{b-\mathfrak{I}} g(z_o)$. Thus we arrive at a contradiction. Hence g is a quasi-b-irresolute function. $\square$

Corollary 3.28. *Suppose $\mathfrak{I}$ is an admissible ideal on $\mathbb{N}$, and Z is a first countable space. Then $h : Z \to W$ is continuous if and only if for every sequence $(z_n)_{n\in\mathbb{N}}$ which is $\mathfrak{I}$-convergent to $z_o \in Z$, the sequence $(h(z_n))_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to $h(z_o)$.*

Lemma 3.29. *[23] Let $(Z\times W,\tau)$ be the topological product of the spaces (Z,σ) and (W,ρ). If $U\in BO(Z)$ and $V\in BO(W)$, then $U\times V\in BO(Z\times W)$.*

Theorem 3.30. *Let $(\prod_{i=1}^{m} Z_i,\sigma)$ be the topological product of the spaces (Z_i,σ_i) for $i=1,2,\dots,m$, and let $(z_i(n))_{n\in\mathbb{N}}$ be a sequence in Z_i. If $(z_1(n),z_2(n),\dots,z_m(n))_{n\in\mathbb{N}}$ be b-$\mathfrak{I}$-convergent to $(x_1,x_2,\dots,x_m)\in\prod_{i=1}^{m} Z_i$, then $(z_i(n))_{n\in\mathbb{N}}$ is b-$\mathfrak{I}$-convergent to $x_i\in Z_i$ for all $i=1,2,\dots,m$.*

Proof. Pick $i_o\in\{1,2,\dots,m\}$ arbitrarily and then fix it. To show $z_{i_o}(n)\xrightarrow{b-\mathfrak{I}}x_{i_o}$, let Q_{i_o} be any b-open set in Z_{i_o} containing the point x_{i_o}. Define $Q=\prod_{i=1}^{m} U_i$, where

$$U_i=\begin{cases} Z_i, & \text{if } i\neq i_o\\ Q_{i_o}, & \text{if } i=i_o.\end{cases}$$

Then by Lemma 3.29, Q is a b-open set in $\prod_{i=1}^{m} Z_i$. Moreover, $(x_1,x_2,\dots,x_m)\in Q$, by construction of Q. Since $(z_1(n),z_2(n),\dots,z_m(n))_{n\in\mathbb{N}}$ is b-$\mathfrak{I}$-convergent to $(x_1,x_2,\dots,x_m)$, we have $\{n\in\mathbb{N}:(z_1(n),z_2(n),\dots,z_m(n))\notin Q\}\in\mathfrak{I}$. One can easily check that $\{n\in\mathbb{N}:z_{i_o}(n)\notin Q_{i_o}\}\subseteq\{n\in\mathbb{N}:(z_1(n),z_2(n),\dots,z_m(n))\notin Q\}$. Since $\mathfrak{I}$ is an ideal, it follows that $\{n\in\mathbb{N}:z_{i_o}(n)\notin Q_{i_o}\}\in\mathfrak{I}$. Hence $z_{i_o}(n)\xrightarrow{b-\mathfrak{I}}x_{i_o}$. As $i_o\in\{1,2,\dots,m\}$ was arbitrary, the proof completes here. □

Theorem 3.31. *Let $(\prod_{\alpha\in\Delta} Z_\alpha,\sigma)$ be the topological product of a family of topological spaces $\{(Z_\alpha,\sigma_\alpha):\alpha\in\Delta\}$, where Δ is an indexing set, and for each $\alpha\in\Delta$, let $(z_\alpha(n))_{n\in\mathbb{N}}$ be a sequence in Z_α. If $(z_\alpha(n))_{n\in\mathbb{N}}$ be b-$\mathfrak{I}$-convergent to $x_\alpha\in Z_\alpha$ for all $\alpha\in\Delta$, then $((z_\alpha(n))_{\alpha\in\Delta})_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to $(x_\alpha)_{\alpha\in\Delta}\in\prod_{\alpha\in\Delta} Z_\alpha$.*

Proof. To prove $((z_\alpha(n))_{\alpha\in\Delta})_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to $(x_\alpha)_{\alpha\in\Delta}$, let Q be an arbitrary open set in $\prod_{\alpha\in\Delta} Z_\alpha$ containing $(x_\alpha)_{\alpha\in\Delta}$. Then we can find a basic open set $\prod_{\alpha\in\Delta} Q_\alpha$ such that $(x_\alpha)_{\alpha\in\Delta}\in\prod_{\alpha\in\Delta} Q_\alpha\subseteq Q$, where Q_α is open in Z_α for each $\alpha\in\Delta$ and $Q_\alpha=Z_\alpha$ except for finitely many values of α. Let $\Delta_o=\{\alpha\in\Delta:Q_\alpha\neq Z_\alpha\}$. Then Δ_o is a finite subset of Δ. Now, for $k\in\{n\in\mathbb{N}:(z_\alpha(n))_{\alpha\in\Delta}\notin Q\}$, $(z_\alpha(k))_{\alpha\in\Delta}\notin Q$. It implies that $(z_\alpha(k))_{\alpha\in\Delta}\notin\prod_{\alpha\in\Delta} Q_\alpha$ (since $\prod_{\alpha\in\Delta} Q_\alpha\subseteq Q$), and hence there exists at least one $\alpha_o\in\Delta_o$ such that $z_{\alpha_o}(k)\notin Q_{\alpha_o}$. Thus $k\in\{n\in\mathbb{N}:z_{\alpha_o}(n)\notin Q_{\alpha_o}\}\subseteq\bigcup_{\alpha\in\Delta_o}\{n\in\mathbb{N}:z_\alpha(n)\notin Q_\alpha\}$. Therefore $\{n\in\mathbb{N}:(z_\alpha(n))_{\alpha\in\Delta}\notin Q\}\subseteq\bigcup_{\alpha\in\Delta_o}\{n\in\mathbb{N}:z_\alpha(n)\notin Q_\alpha\}$. On the other side, for each $\alpha\in\Delta_o$, Q_α is a b-open subset of Z_α containing x_α, using Lemma 3.3. Since $z_\alpha(n)\xrightarrow{b-\mathfrak{I}}x_\alpha$, we have $\{n\in\mathbb{N}:z_\alpha(n)\notin Q_\alpha\}\in\mathfrak{I}$ for all $\alpha\in\Delta_o$. Since Δ_o is finite, we have $\bigcup_{\alpha\in\Delta_o}\{n\in\mathbb{N}:z_\alpha(n)\notin Q_\alpha\}\in\mathfrak{I}$ and consequently, $\{n\in\mathbb{N}:(z_\alpha(n))_{\alpha\in\Delta}\notin Q\}\in\mathfrak{I}$. Hence $((z_\alpha(n))_{\alpha\in\Delta})_{n\in\mathbb{N}}$ is $\mathfrak{I}$-convergent to $(x_\alpha)_{\alpha\in\Delta}$. □

Definition 3.32. *A point $z_o\in Z$ is said to be b-ω-accumulation (resp., ω-accumulation) point of a subset $Q\subseteq Z$ if for every b-open (resp., open) set U containing z_o, $U\cap Q$ is an infinite set.*

Definition 3.33. *A point z_o of a space Z is said to be a b-$\mathfrak{I}$-cluster (resp., $\mathfrak{I}$-cluster (see [19])) point of a sequence $(z_n)_{n\in\mathbb{N}}$ in Z if for any b-open (resp., open) set Q containing z_o, $\{n\in\mathbb{N}:z_n\in Q\}\notin\mathfrak{I}$.*

Theorem 3.34. *Let $\mathfrak{I}$ be an admissible ideal on $\mathbb{N}$ and $g:Z\to W$ a b-irresolute function. If z_0 be a b-$\mathfrak{I}$-cluster point of a sequence $(z_n)_{n\in\mathbb{N}}$ in Z, then $g(z_0)$ is b-$\mathfrak{I}$-cluster point of $g(z_n)_{n\in\mathbb{N}}$ in W.*

Proof. To show $g(z_0)$ is b-$\mathfrak{I}$-cluster point of $g(z_n)_{n\in\mathbb{N}}$, let T be any b-open set containing $g(z_0)$. By b-irresoluteness of g, there exists a b-open set Q containing z_0 such that $g(Q)\subseteq T$. Since z_0 is a b-$\mathfrak{I}$-cluster point $(z_n)_{n\in\mathbb{N}}$, so $\{n \in \mathbb{N} : z_n \in Q\} \notin \mathcal{I}$. It can be easily verify that $\{n \in \mathbb{N} : z_n \in Q\} \subseteq \{n \in \mathbb{N} : g(z_n) \in T\}$, where $\{n \in \mathbb{N} : z_n \in Q\} \notin \mathcal{I}$. From here we conclude that $\{n \in \mathbb{N} : g(z_n) \in T\}$. Hence $g(z_0)$ is b-$\mathcal{I}$-cluster point of $g(z_n)_{n\in\mathbb{N}}$. □

Theorem 3.35. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If each sequence $(z_n)_{n\in\mathbb{N}}$ in Z has a b-$\mathcal{I}$-cluster point, then every infinite subset of Z possesses a b-ω-accumulation point. Converse is true if $\mathcal{I}$ is an admissible ideal containing no infinite subset of $\mathbb{N}$.*

Proof. Suppose that Q is an infinite subset of Z, and $(z_n)_{n\in\mathbb{N}}$ is a sequence of distinct elements of Q. By hypothesis, $(z_n)_{n\in\mathbb{N}}$ has a b-$\mathcal{I}$-cluster point, say z_o in Z. Then for every b-open set U containing z_o, we have $\{n \in \mathbb{N} : z_n \in U\} \notin \mathcal{I}$. Because of $\mathcal{I}$ is admissible, $\{n \in \mathbb{N} : z_n \in U\}$ is an infinite set. Hence U contains infinitely many points of Q i.e., $U \cap Q$ is infinite. So z_o is a b-ω-accumulation point of Q.

For converse, let $\mathcal{I}$ be an admissible ideal containing no infinite subset of $\mathbb{N}$, and every infinite subset of Z has a b-ω-accumulation point. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in Z, and Q be its range set. Now, if Q be infinite, then by hypothesis, Q has a b-ω-accumulation point $z_o \in Z$. Then for every b-open set U containing z_o, $U \cap Q$ is an infinite set. Consequently, U contains infinitely many points of Q and hence of the sequence $(z_n)_{n\in\mathbb{N}}$. Thus $\{n \in \mathbb{N} : z_n \in U\}$ is infinite and so $\{n \in \mathbb{N} : z_n \in U\} \notin \mathcal{I}$ as $\mathcal{I}$ contains no infinite set. So z_o is a b-$\mathcal{I}$-cluster point of $(z_n)_{n\in\mathbb{N}}$. If Q be finite, then there is a point $y_o \in Z$ such that $z_n = y_o$ for infinitely many n. As a result, for every b-open set U containing y_o, $\{n \in \mathbb{N} : z_n \in U\}$ being infinite is not in $\mathcal{I}$. So y_o is a b-$\mathcal{I}$-cluster point of $(z_n)_{n\in\mathbb{N}}$. □

Corollary 3.36. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If each sequence $(z_n)_{n\in\mathbb{N}}$ in Z has a b-$\mathcal{I}$-cluster point, then every infinite subset of Z possesses an ω-accumulation point.*

For our next result let us recall b-compact and b-Lindelöf spaces. A space Z is a called b-compact or γ-compact (see [13]) (resp., b-Lindelöf (see [25,12]) space if every b-open cover of Z has a finite (resp., countable) subcover.

Theorem 3.37. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If Z be a b-Lindelöf space such that each sequence in Z has a b-$\mathcal{I}$-cluster point, then Z is a b-compact space.*

Proof. Suppose that $\mathcal{Q} = \{Q_\alpha : \alpha \in \Delta\}$ is an arbitrary b-open cover of Z, where Δ is an index set. Since Z is a b-Lindelöf space, $\mathcal{Q}$ has a countable subcover, say $\mathcal{Q}_0 = \{Q_1, Q_2, \ldots, Q_n, \ldots\}$. Inductively, let us define $J_1 = Q_1$ and for $m > 1$, J_m is the first member of the sequence (Q_n) which is not covered by $\bigcup_{i=1}^{m-1} J_i$. We claim that the construction process of J_i's will stop after a finite number of steps. If not, then one can pick a point $z_1 \in J_1$ and for every $m > 1$, $z_m \in J_m$ such that $z_m \notin J_i$ for all $i < m$. Thus $(z_m)_{m\in\mathbb{N}}$ is a sequence in Z. By hypothesis, $(z_m)_{m\in\mathbb{N}}$ has a b-$\mathcal{I}$-cluster point $z_o \in Z$. Then $z_o \in J_{i_o}$ for some i_o because $\{J_m : m \in \mathbb{N}\}$ covers Z. Since J_{i_o} is a b-open set containing z_o, $\{m \in \mathbb{N} : z_m \in J_{i_o}\} \notin \mathcal{I}$. Since $\mathcal{I}$ is an admissible ideal, $M = \{m \in \mathbb{N} : z_m \in J_{i_o}\}$ must be an infinite set. So there exists $m > i_o$ such that $m \in M$ and hence $z_m \in J_{i_o}$. This leads a contradiction. So there exists $m_o \in \mathbb{N}$ such that $\{J_1, J_2, \ldots, J_{m_o}\}$ is a finite subcollection of $\mathcal{Q}$ that covers Z. Hence Z is b-compact. □

Corollary 3.38. *Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$. If Z be a b-Lindelöf space such that every sequence in Z has a b-$\mathcal{I}$-cluster point, then Z is a compact space.*

4. b-convergence of net in topological spaces

Before entering into this section, let us collect following mathematical tools.

Definition 4.1. *[17] A directed set is a pair $(D, \geq)$ where D is a non-empty set and $\geq$ a binary relation on D such that $\geq$ is reflexive, transitive and for every pair of elements m, $n \in D$, there exists $p \in D$ such that $p \geq m$ and $p \geq n$.*

Definition 4.2. *[17] Let X be a non-empty set, and $(D, \geq)$ a directed set. By a net in X, we mean a mapping $s : D \to X$ which will be denoted by $(s_d)_{d\in D}$ or simply by (s_d).*

We define b-convergence of a net in topological space as follows:

Definition 4.3. *A net $s : D \to Z$ in a space Z is said to b-converge to $z_o \in Z$, symbolized as $s \xrightarrow{b} z_o$, if for any b-open set Q containing z_o, there exists $d_o \in D$ such that for all $d \geq d_o$, $s_d \in Q$. In this regard, we call z_o as a b-limit of the net (s_d) and write b-lim $s_d = z_o$.*

Existence of b-convergent of a net in topological space is considered in the following example.

Example 4.4. *Let $Z = \{a, b, c\}$ and $\sigma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, Z\}$. Then $BO(Z) = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, Z\}$. Let $D = \{\{a\}, \{a, b\}, Z\}$, and define $\geq$ on D as: for all $U, V \in D$, $U \geq V$ if and only if $U \subseteq V$. Then $(D, \geq)$ is a directed set. Define a net $s : D \to Z$ by $s_{\{a\}} = s_{\{a,b\}} = c$ and $s_Z = a$. Then $s \xrightarrow{b} c$.*

Remark 4.5. *Since every open set is b-open, it is clear that b-convergence of a net implies ordinary convergence of that net whereas converse is not valid at all. For justification, if we consider the indiscrete topology on $\mathbb{N}$ and a net $s : \mathbb{N} \to \mathbb{N}$ defined by $s_n = n$ for all $n \in \mathbb{N}$, then one can easily check that (s_n) converges to 10 but not b-converges to 10.*

Theorem 4.6. *If Z be a b-T_2 space, then every b-convergent net in Z has unique b-limit.*

Proof. Let $s : D \to Z$ be a net in Z such that $s_d \xrightarrow{b} x$ and $s_d \xrightarrow{b} y$, where $x, y \in Z$ and $x \neq y$. Since Z is b-T_2, there exist $P, Q \in BO(Z)$ such that $x \in P$, $y \in Q$ and $P \cap Q = \varnothing$. Also, there exist $m, n \in D$ such that $s_d \in P$ for every $d \geq m$ and $s_d \in Q$ for every $d \geq n$. Since D is a directed set, there exists $p \in D$ such that $p \geq m$ and $p \geq n$. Thus, for all $d \geq p$, $s_d \in P$ and $s_d \in Q$, showing that $P \cap Q \neq \varnothing$. This is a contradiction. Hence, every b-convergent net in Z has unique b-limit. □

Theorem 4.7. *If every b-convergent net in a B^*-space (see [23]) Z has unique b-limit, then Z is a b-T_2 space.*

Proof. If possible, assume that Z is not b-T_2. Then there exists a pair x, y with $x \neq y$ in Z such that for every $P \in BO(Z, x)$ (the collection of all b-open subsets of Z containing x) and $Q \in BO(Z, y)$, we have $P \cap Q \neq \varnothing$. Consider $D = BO(Z, x) \times BO(Z, y)$ with a binary relation $\geq$ defined by $(P, Q) \geq (U, V)$ if and only if $P \subseteq U$ and $Q \subseteq V$. Since Z is a B^*-space, intersection of two b-open subsets of Z is again a b-open set, and consequently, $(D, \geq)$ is a directed set. Moreover, for every $(P, Q) \in D$, $P \cap Q \neq \varnothing$, and pick $z_{(P,Q)} \in P \cap Q$. Define a net $s : D \to Z$ by $s_{(P,Q)} = z_{(P,Q)}$ for every $(P, Q) \in D$. We now show that the net s b-converges to x. For this, let G be any b-open set containing x. Then $(G, Z) \in D$. Now, for every $(P, Q) \geq (G, Z)$, we have $P \subseteq G$ and $s_{(P,Q)} = z_{(P,Q)} \in P \cap Q \subseteq P$. Thus s b-converges to x. In a similar fashion, we can show that s b-converges to y also. This contradicts our hypothesis. Hence Z is a b-T_2 space. □

Theorem 4.8. *Let z_o be a point of a B^*-space Z, and $Q \subseteq Z$. Then*

1. *$z_o \in D_b(Q)$ if and only if there exists a net $(s_d)_{d \in D}$ in $Q \setminus \{z_o\}$ such that $s_d \xrightarrow{b} z_o$.*
2. *$z_o \in Cl_b(Q)$ if and only if there exists a net $(s_d)_{d \in D}$ in Q such that $s_d \xrightarrow{b} z_o$.*
3. *Q is b-closed if and only if there is no net in Q which b-converges to a point of $Z \setminus Q$.*
4. *Q is b-open if and only if there is no net in $Z \setminus Q$ which b-converges to a point of Q.*

Proof. 1. Let $z_o \in D_b(Q)$. Then for every $A \in BO(Z, z_o)$, $A \cap (Q \setminus \{z_o\}) \neq \varnothing$. Pick $z_A \in A \cap (Q \setminus \{z_o\})$. Now, let $\geq$ be a binary relation on $D = BO(Z, z_o)$ defined by $U \geq V$ if and only if $U \subseteq V$. Since Z is a B^*-space, $BO(Z, z_o)$ is closed under finite intersection. Consequently, $(D, \geq)$ is a directed set. Define a net $s : D \to Q \setminus \{z_o\}$ by $s_U = z_U$ for all $U \in D$. To show $s_U \xrightarrow{b} z_o$, let G be any b-open set containing z_o. Then for every $U \geq G$, we have $U \subseteq G$ and $s_U = z_U \in U \cap (Q \setminus \{z_o\}) \subseteq U \subseteq G$. Thus $s_U \xrightarrow{b} z_o$.

Conversely, suppose that $(s_d)_{d\in D}$ is a net in $Q\setminus\{z_o\}$ and $s_d \xrightarrow{b} z_o$. To show $z_o \in D_b(Q)$, let G be any b-open set containing z_o. Since $s_d \xrightarrow{b} z_o$, there exists $d_o \in D$ such that whenever $d \geq d_o$, $s_d \in G$. On the other hand, $s_d \in Q\setminus\{z_o\}$ for all $d \in D$. Thus for every $d \geq d_o$, $s_d \in G\cap(Q\setminus\{z_o\})$, showing that $G\cap(Q\setminus\{z_o\}) \neq \varnothing$. Hence $z_o \in D_b(Q)$.

2. Proof is similar to that of 1.

3. Let Q be b-closed in Z. If possible, suppose that $(s_d)_{d\in D}$ is a net in Q such that $s_d \xrightarrow{b} z_o \in Z\setminus Q$. Then by 2, $z_o \in Cl_b(Q) = Q$ (since Q is b-closed). Now, $z_o \in Z\setminus Q$ implies $z_o \notin Q$, a contradiction.

Conversely, let there is no net in Q which b-converges to a point of $Z\setminus Q$. Now, let $x \in Cl_b(Q)$. Then by 2, there exists a net $(s_d)_{d\in D}$ in Q such that $s_d \xrightarrow{b} x$. By hypothesis, $x \in Q$. Thus $Cl_b(Q) \subseteq Q$. Since $Q \subseteq Cl_b(Q)$, $Cl_b(Q) = Q$. Hence Q is b-closed.

4. Follows from 3. □

Corollary 4.9. *Let z_o be a point of a space Z, and $Q \subseteq Z$. Then*

1. *if there exists a net $(s_d)_{d\in D}$ in $Q\setminus\{z_o\}$ such that $s_d \xrightarrow{b} z_o$, then $z_o \in D_b(Q)$.*
2. *if there exists a net $(s_d)_{d\in D}$ in Q such that $s_d \xrightarrow{b} z_o$, then $z_o \in Cl_b(Q)$.*
3. *if Q is b-closed, then there is no net in Q which b-converges to a point of $Z\setminus Q$.*
4. *if Q is b-open, then there is no net in $Z\setminus Q$ which b-converges to a point of Q.*

Theorem 4.10. *Let Z and W be two spaces, and $f : Z \to W$ be a function. Then*

1. *f is quasi-b-irresolute if and only if for every net $(s_d)_{d\in D}$ converging to $z_o \in Z$, the net $(f(s_d))_{d\in D}$ b-converges to $f(z_o)$.*
2. *if f is b-irresolute, then whenever a net $(s_d)_{d\in D}$ b-converges to $z_o \in Z$, the net $(f(s_d))_{d\in D}$ b-converges to $f(z_o)$.*
3. *if f is b-continuous, then whenever a net $(s_d)_{d\in D}$ b-converges to $z_o \in Z$, the net $(f(s_d))_{d\in D}$ converges to $f(z_o)$.*

Proof. 1. Firstly, suppose f is quasi-b-irresolute. To show $f(s_d) \xrightarrow{b} f(z_o)$, let Q be any b-open set containing $f(z_o)$. Since f is quasi-b-irresolute, there exists an open set P containing z_o such that $f(P) \subseteq Q$. Since $s_d \to z_o$, there exists $d_o \in D$ such that for all $d \geq d_o$, $s_d \in P$. This implies $f(s_d) \in f(P) \subseteq Q$ for all $d \geq d_o$. Hence $f(s_d) \xrightarrow{b} f(z_o)$.

Conversely, let the condition holds. On contrary, suppose that f is not quasi-b-irresolute. Then there exists a point $z_o \in Z$ and a b-open set $Q \ni f(z_o)$ such that for every $P \in \sigma(z_o) = \{U \subseteq Z : U \in \sigma \text{ and } z_o \in U\}$, $f(P)\cap(W\setminus Q) \neq \varnothing$. Pick $w_P \in f(P)\cap(W\setminus Q)$. Then for every $P \in \sigma(z_o)$, there exists $z_P \in P$ such that $f(z_P) = w_P$. Let $\geq$ be a binary relation on $D = \sigma(z_o)$ defined by $U \geq V$ if and only if $U \subseteq V$. Then clearly, $(D, \geq)$ is a directed set. Consider the net $s : D \to Z$ defined by $s_U = z_U$ for all $U \in D$. It is obvious that $s_U \to z_o$. Then by hypothesis, $f(s_U) \xrightarrow{b} f(z_o)$. But by construction, $f(s_U)$ never b-converges to $f(z_o)$. Thus we reach at a contradiction. Hence f is quasi-b-irresolute.

2, 3. Proofs are omitted for their easiness. □

Recall that a point $z_o \in Z$ is said to be a b-cluster point (see [27]) of a net $s : D \to Z$ if for every b-open set Q containing z_o and for each $d \in D$, there is some $d_o \geq d$ such that $s_{d_o} \in Q$.

Theorem 4.11. *Let $s : D \to Z$ be a net in a space Z, and for each $d_o \in D$, let $Q_{d_o} = \{s_d : d \geq d_o \text{ and } d \in D\}$. Then a point $y \in Z$ is a b-cluster point of $(s_d)_{d\in D}$ if and only if $y \in \bigcap_{d\in D} Cl_b(Q_d)$.*

Proof. Let y is a b-cluster point of the net $(s_d)_{d\in D}$. Then for every b-open set G containing y, the net s_d is frequently in G. That is, for each $d \in D$, there exists $d_o \in D$ such that $d_o \geq d$ and $s_{d_o} \in G$. Moreover, $s_{d_o} \in Q_d$. Thus $Q_d \cap G \neq \varnothing$ for every $d \in D$, and so $y \in Cl_b(Q_d)$, by Lemma 2.2 of [10]. Hence $y \in \bigcap_{d\in D} Cl_b(Q_d)$.

Conversely, if possible, suppose that y is not a b-cluster point of $(s_d)_{d\in D}$. Then there exists a b-open set $G \ni y$ and a $d_o \in D$ such that whenever $d \geq d_o$, $s_d \notin G$, and as a result $Q_{d_o} \cap G = \varnothing$. Thus $y \notin Cl_b(Q_{d_o})$ and hence $y \notin \bigcap_{d\in D} Cl_b(Q_d)$. This is a contradiction. Hence y is a b-cluster point of the net $(s_d)_{d\in D}$. □

Theorem 4.12. *Let $(\prod_{i=1}^{m} Z_i, \sigma)$ be the topological product of the spaces (Z_i, σ_i) for $i = 1, 2, \ldots, m$, and let $(z_i(d))_{d\in D}$ be a net in Z_i. If the net $(z_1(d), z_2(d), \ldots, z_m(d))_{d\in D}$ is b-convergent to $(x_1, x_2, \ldots, x_m) \in \prod_{i=1}^{m} Z_i$, then $(z_i(d))_{d\in D}$ is b-convergent to $x_i \in Z_i$ for all $i = 1, 2, \ldots, m$.*

Proof. Proof is very straightforward. □

5. b-$\mathfrak{I}$-convergence of net in topological spaces

Throughout this section, $\mathfrak{I}$ will stand for a non-trivial ideal on a directed set D. For every $n \in D$, let $D_n = \{m \in D : m \geq n\}$. Then $\mathfrak{F}_o = \{A \subseteq D : A \supseteq D_n$ for some $n\}$ is a filter on D, and $\mathfrak{I}_o = \{A \subseteq D : D \setminus A \in \mathfrak{F}_o\}$ is a non-trivial ideal on D. A non-trivial ideal $\mathfrak{I}$ on D is called D-admissible (see [20]) if $D_n \in \mathfrak{F}_{\mathfrak{I}}$ for all $n \in D$.

Definition 5.1. *Let Z be a space. A net $s : D \to Z$ is said to be b-$\mathfrak{I}$-convergent to $z_o \in Z$, symbolically we write $s_d \xrightarrow{b-\mathfrak{I}} z_o$, if for every b-open set Q containing z_o, we have $\{d \in D : s_d \notin Q\} \in \mathfrak{I}$. We call z_o as b-$\mathfrak{I}$-limit of the net (s_d) and write b-$\mathfrak{I}$-$\lim s_d = z_o$.*

We now give a supporting example in favor of the existence of b-$\mathfrak{I}$-convergence of net in topological spaces.

Example 5.2. *Consider $Z = \{p, q, r\}$ with $\sigma = \{\varnothing, \{p\}, \{q\}, \{p, q\}, Z\}$. Then*

$$BO(Z) = \{\varnothing, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}, Z\}.$$

Consider the directed set $D = \{\{p\}, \{p, q\}, Z\}$ directed by the relation $\geq$ as: for all $U, V \in D$, $U \geq V$ if and only if $U \subseteq V$. Let $\mathfrak{I} = \{\varnothing, \{\{p\}, \{p, q\}\}, \{\{p\}\}, \{\{p, q\}\}\}$. Then $\mathfrak{I}$ is a non-trivial ideal on D. Consider the net $s : D \to Z$ defined by $s_{\{p\}} = s_{\{p,q\}} = r$ and $s_Z = p$. Then for every b-open set Q containing p, $\{d \in D : s_d \notin Q\} = \varnothing$ or $\{\{p\}, \{p, q\}\}$, both of which are members of $\mathfrak{I}$. Thus $s_d \xrightarrow{b-\mathfrak{I}} p$.

Lemma 5.3. *Suppose $(s_d)_{d\in D}$ is a net in a space Z, $z_o \in Z$, and $\mathfrak{I}$ a non-trivial ideal on D. If $\mathfrak{I}$ be D-admissible and $s_d \xrightarrow{b} z_o$, then $s_d \xrightarrow{b-\mathfrak{I}} z_o$. Converse holds if $\mathfrak{I} = \mathfrak{I}_o$.*

Proof. To show $s_d \xrightarrow{b-\mathfrak{I}} z_o$, let Q be any b-open set containing z_o. Since $s_d \xrightarrow{b} z_o$, there exists $n_o \in D$ such that for all $d \geq n_o$, $s_d \in Q$. This implies $D_{n_o} = \{d \in D : d \geq n_o\} \subseteq \{d \in D : s_d \in Q\}$. Since $\mathfrak{I}$ is D-admissible, $D_{n_o} \in \mathfrak{F}_{\mathfrak{I}}$ whence $D \setminus D_{n_o} \in \mathfrak{I}$ and hence $\{d \in D : s_d \notin Q\} = D \setminus \{d \in D : s_d \in Q\} \in \mathfrak{I}$, as required. Conversely, let $s_d \xrightarrow{b-\mathfrak{I}_o} z_o$. To show $s_d \xrightarrow{b} z_o$, let G be any b-open set containing z_o. Then $\{d \in D : s_d \notin G\} \in \mathfrak{I}_o$ implying that $\{d \in D : s_d \in G\} = D \setminus \{d \in D : s_d \notin G\} \in \mathfrak{F}_o$. Thus there exists $d_o \in D$ such that $\{d \in D : s_d \in G\} \supseteq D_{d_o} = \{d \in D : d \geq d_o\}$. This yields that for all $d \geq d_o$, $s_d \in G$. Hence $s_d \xrightarrow{b} z_o$. □

Theorem 5.4. *If Z be a b-T_2 space, and $(s_d)_{d\in D}$ a net in Z such that $s_d \xrightarrow{b-\mathfrak{I}} z \in Z$ and $s_d \xrightarrow{b-\mathfrak{I}} w \in Z$, then $z = w$.*

Proof. Proof is obvious. □

Theorem 5.5. *If every b-$\mathcal{I}$-convergent net in a B^*-space Z has unique b-$\mathcal{I}$-limit for every D-admissible ideal $\mathcal{I}$, then Z is b-T_2.*

Proof. If possible, assume that Z is not b-T_2. Then there exists a pair x, y with $x \neq y$ in Z such that for every $P \in BO(Z,x)$ and $Q \in BO(Z,y)$, we have $P \cap Q \neq \varnothing$. Consider $D = BO(Z,x) \times BO(Z,y)$ with a binary relation $\geq$ defined by $(P,Q) \geq (U,V)$ if and only if $P \subseteq U$ and $Q \subseteq V$. Since Z is a B^*-space, it follows that $(D, \geq)$ is a directed set. Moreover, for every $(P,Q) \in D$, $P \cap Q \neq \varnothing$, and pick $z_{(P,Q)} \in P \cap Q$. Define a net $s : D \to Z$ by $s_{(P,Q)} = z_{(P,Q)}$ for every $(P,Q) \in D$. Then the net s b-converges to x as well as y also. Let $\mathcal{I}$ be any D-admissible ideal on D. Then by Lemma 5.3, the net s b-$\mathcal{I}$-converges to x as well as y. This contradicts our hypothesis. Hence Z is a b-T_2 space. □

Theorem 5.6. *A b-irresolute mapping $f : Z \to W$ preserves b-$\mathcal{I}$-convergence of nets. Conversely, if Z be a B^*-space and $f : Z \to W$ preserves b-$\mathcal{I}$-convergence of nets for every D-admissible ideal $\mathcal{I}$, then f is b-irresolute.*

Proof. Let $(s_d)_{d \in D}$ be a net in Z such that $s_d \xrightarrow{b-\mathcal{I}} z_o \in Z$. To show $f(s_d) \xrightarrow{b-\mathcal{I}} f(z_o)$, let G be any b-open set containing $f(z_o)$. Since f is b-irresolute, there exists a b-open set H in Z containing z_o such that $f(H) \subseteq G$. Because $s_d \xrightarrow{b-\mathcal{I}} z_o$, $\{d \in D : s_d \notin H\} \in \mathcal{I}$. Since $f(H) \subseteq G$, $\{d \in D : f(s_d) \notin G\} \subseteq \{d \in D : s_d \notin H\}$. As $\mathcal{I}$ is an ideal, it follows that $\{d \in D : f(s_d) \notin G\} \in \mathcal{I}$, as desired.

Conversely, if possible, suppose that f is not b-irresolute at some $z_o \in Z$. Then there exists a b-open set G containing $f(z_o)$ such that for every $H \in BO(Z, z_o)$, we have $f(H) \nsubseteq G$. Thus for every $H \in BO(Z, z_o)$, one can pick a point $z_H \in H$ such that $f(z_H) \notin G$. Define a binary relation $\geq$ on $D = BO(Z, z_o)$ such that $U \geq V$ if and only if $U \subseteq V$ for all $U, V \in D$. Then $(D, \geq)$ is a directed set. Let us define a net $s : D \to Z$ by $s_U = z_U$ for all $U \in D$. Then one can easily verify that $s_U \xrightarrow{b} z_o$. Let $\mathcal{I}$ be a D-admissible ideal on D. By Lemma 5.3, it follows that $s_U \xrightarrow{b-\mathcal{I}} z_o$. By hypothesis, $f(s_U) \xrightarrow{b-\mathcal{I}} f(z_o)$. This yields $\{U \in D : f(s_U) \notin G\} \in \mathcal{I}$. But by construction, $\{U \in D : f(s_U) \notin G\} = D$. Hence $D \in \mathcal{I}$, a contradiction as $\mathcal{I}$ is a non-trivial ideal on D. □

We say that a filter $\mathcal{F}$ on a space Z b-converges to $z_o \in Z$ (or z_o is a b-limit of the filter $\mathcal{F}$) if $\mathcal{N}_b(z_o) \subseteq \mathcal{F}$, and z_o is a b-cluster point of the filter $\mathcal{F}$ if every b-neighbourhood of z_o intersects each member of $\mathcal{F}$. These concepts coincide with the Definition 3.7 of [27] where various topological properties regarding these concepts have been presented nicely. Our next result is a new characterization of b-limit (resp., b-cluster point) of a certain type of filter in terms b-$\mathcal{I}$-convergence (resp., b-$\mathcal{I}$-cluster point, which is defined below) of net.

Definition 5.7. *A point $z_o \in Z$ is said to be b-$\mathcal{I}$-cluster point of a net $s : D \to Z$ if for every b-open set Q containing z_o, $\{d \in D : s_d \in Q\} \notin \mathcal{I}$.*

Theorem 5.8. *For every net $s : D \to Z$, there is a filter $\mathcal{G}$ on Z such that $z_o \in Z$ is a b-$\mathcal{I}$-limit of the net $(s_d)_{d \in D}$ if and only if z_o is a b-limit of the filter $\mathcal{G}$. Moreover, z_o is b-$\mathcal{I}$-cluster point of the net $(s_d)_{d \in D}$ if and only if z_o is a b-cluster point of the filter $\mathcal{G}$.*

Proof. Let $s : D \to Z$ be a net, and $\mathcal{I}$ a non-trivial ideal on D. For every $A \in \mathcal{F}_\mathcal{I}$ (associated filter of $\mathcal{I}$), let $A^+ := \{s_d : d \in A\}$. Then each A^+ is a non-empty subset of Z because each $A \in \mathcal{F}_\mathcal{I}$ is non-empty (since $\mathcal{F}_\mathcal{I}$ is filter). We consider the family $\mathcal{B} = \{A^+ : A \in \mathcal{F}_\mathcal{I}\}$ of subsets of Z. It is quite obvious that $\mathcal{B}$ serves as a filter base for some filter on Z. Indeed, for $A^+, B^+ \in \mathcal{B}$, we have $A, B \in \mathcal{F}_\mathcal{I}$. Since $\mathcal{F}_\mathcal{I}$ is a filter, so $A \cap B \in \mathcal{F}_\mathcal{I}$ and hence $(A \cap B)^+ \in \mathcal{B}$. Since $A \cap B \subseteq A$ as well as B, we have $(A \cap B)^+ \subseteq A^+ \cap B^+$, by construction of $(\cdot)^+$. Consider the filter $\mathcal{G}$ generated by the filter base $\mathcal{B}$. We shall now show that $\mathcal{G}$ fulfils our desired properties.

Let $s_d \xrightarrow{b-\mathcal{I}} z_o$. To show z_o is a b-limit of the filter $\mathcal{G}$, let $R \in \mathcal{N}_b(z_o)$. Then there exists $Q \in BO(Z, z_o)$ such that $Q \subseteq R$. Since $s_d \xrightarrow{b-\mathcal{I}} z_o$, so $\{d \in D : s_d \notin Q\} \in \mathcal{I}$ whence $\{d \in D : s_d \in Q\} \in \mathcal{F}_\mathcal{I}$. Name

$\{d \in D : s_d \in Q\} = E$. Then $E^+ \subseteq Q$. Since $E^+ \in \mathcal{B}$, $E^+ \in \mathcal{G}$ and hence $Q \in \mathcal{G}$ which further implies $R \in \mathcal{G}$ (since $\mathcal{G}$ is filter). Thus $\mathcal{N}_b(z_o) \subseteq \mathcal{G}$, as aimed.

Conversely, let z_o be a b-limit point of the filter $\mathcal{G}$. To show $s_d \xrightarrow{b-\mathcal{I}} z_o$, let Q be any b-open set containing z_o. Then $Q \in \mathcal{N}_b(z_o)$. But $\mathcal{N}_b(z_o) \subseteq \mathcal{G}$. Thus $Q \in \mathcal{G}$. Since $\mathcal{B}$ generates $\mathcal{G}$, so there exists $B \in \mathcal{F}_{\mathcal{I}}$ such that $B^+ \subseteq Q$. This implies that $\{d \in D : s_d \notin Q\} \subseteq D \setminus B \in \mathcal{I}$, since $B \in \mathcal{F}_{\mathcal{I}}$. Hence $\{d \in D : s_d \notin Q\} \in \mathcal{I}$. This shows that the net $(s_d)_{d\in D}$ b-$\mathcal{I}$-converges to z_o.

Now, suppose that z_o is a b-$\mathcal{I}$-cluster point of the net $(s_d)_{d\in D}$. To show z_o is a b-cluster point of the filter $\mathcal{G}$, let $U \in \mathcal{N}_b(z_o)$. Then there exists $B \in BO(Z, z_o)$ such that $B \subseteq U$. By hypothesis, we have $\{d \in D : s_d \in B\} \notin \mathcal{I}$. This implies that $\{d \in D : s_d \notin B\} \notin \mathcal{F}_{\mathcal{I}}$. This means that $\{d \in D : s_d \notin B\}$ can't contain any member of $\mathcal{F}_{\mathcal{I}}$. Now. for every $G \in \mathcal{G}$, there exists $A \in \mathcal{F}_{\mathcal{I}}$ such that $A^+ \subseteq G$, since $\mathcal{B}$ is a filter base for $\mathcal{G}$. Since $A \nsubseteq \{d \in D : s_d \notin B\}$, there exists $n \in A$ such that $s_n \in B$. Also $s_n \in A^+$. So $A^+ \cap B \neq \varnothing$. Moreover, $A^+ \cap B \subseteq G \cap U$. Hence $G \cap U \neq \varnothing$. So every b-open set containing z_o intersects every member of $\mathcal{G}$, as aimed.

Conversely, let z_o be a b-cluster point of the filter G, and Q be a b-open set containing z_o. Claim: $\{d \in D : s_d \in Q\} \notin \mathcal{I}$. If possible, suppose that $\{d \in D : s_d \in Q\} \in \mathcal{I}$. Then $\{d \in D : s_d \notin Q\} \in \mathcal{F}_{\mathcal{I}}$. Name $\{d \in D : s_d \notin Q\} = A$. Then $A^+ \in \mathcal{B} \subseteq \mathcal{G}$. By hypothesis, $Q \cap A^+ \neq \varnothing$. Let $y \in Q \cap A^+$. Then $y \in A^+$ implies $y = s_n$ for some $n \in A$ which further yields that $s_n \notin Q$. Thus $y \notin Q$, a contradiction as $y \in Q$. Hence $\{d \in D : s_d \in Q\} \notin \mathcal{I}$, which witnessing that z_o is a b-$\mathcal{I}$-cluster point of the net $(s_d)_{d\in D}$. □

In our following result, existence of b-$\mathcal{I}$-cluster point of net has been investigated carefully. We recall that a space Z is b-compact if and only if every family of b-closed sets having finite intersection property has non-empty intersection (see [27], Proposition 3.3).

Theorem 5.9. *Given a b-compact space Z, every net $s : D \to Z$ has a b-$\mathcal{I}$-cluster point for every non-trivial ideal $\mathcal{I}$ on D. Converse holds if $\mathcal{I}$ is a D-admissible ideal.*

Proof. Let Z be a b-compact space, and $(s_d)_{d\in D}$ a net in Z with a nontrivial ideal $\mathcal{I}$ on D. For every $A \in \mathcal{F}_{\mathcal{I}}$, let $A^+ := \{s_d : d \in A\}$. Then every A^+ is a non-empty subset of Z because each $A \in \mathcal{F}_{\mathcal{I}}$ is non-empty. Evidently, the family $\mathcal{A} = \{A^+ : A \in \mathcal{F}_{\mathcal{I}}\}$ of subsets of Z has the finite intersection property. Indeed, for $A^+, B^+ \in \mathcal{A}$, $A, B \in \mathcal{F}_{\mathcal{I}}$ implies $A \cap B \in \mathcal{F}_{\mathcal{I}}$ yielding that $(A \cap B)^+ \neq \varnothing$. Moreover, $(A \cap B)^+ \subseteq A^+ \cap B^+$. Thus $A^+ \cap B^+ \neq \varnothing$. Hence the family $\mathcal{B} = \{Cl_b(A^+) : A \in \mathcal{F}_{\mathcal{I}}\}$ of b-closed (since every $Cl_b(A^+)$ is b-closed) subsets of Z has the finite intersection property also, since $A^+ \subseteq Cl_b(A^+)$. Since Z is b-compact, so $\cap\{Cl_b(A^+) : A \in \mathcal{F}_{\mathcal{I}}\} \neq \varnothing$. Pick $z_o \in \cap\{Cl_b(A^+) : A \in \mathcal{F}_{\mathcal{I}}\}$. Claim: z_o is a b-$\mathcal{I}$-cluster point of the net $(s_d)_{d\in D}$. For this, let Q be any b-open set containing z_o. If possible, suppose that $\{d \in D : s_d \in Q\} \in \mathcal{I}$. Then $\{d \in D : s_d \notin Q\} \in \mathcal{F}_{\mathcal{I}}$. This implies that $z_o \in Cl_b(\{d \in D : s_d \notin Q\}^+)$. So $Q \cap \{d \in D : s_d \notin Q\}^+ \neq \varnothing$. Pick $x \in Q \cap \{d \in D : s_d \notin Q\}^+$. Then $x = s_n$ for some $n \in \{d \in D : s_d \notin Q\}$. This gives $s_n = x \notin Q$, whereas $x \in Q$ also. Thus we reach at a contradiction. Hence $\{d \in D : s_d \in Q\} \notin \mathcal{I}$, as expected.

Conversely, if possible, suppose that Z is not a b-compact space. Then we have a b-open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ of Z which has no finite subcover, where Δ is an index set. Let $\mathcal{D}$ be the family of all finite subsets of Δ. Then $(\mathcal{D}, \geq)$ is a directed set, where $\geq$ is defined as $J \geq K$ if and only if $K \subseteq J$ for $J, K \in \mathcal{D}$. Since $\mathcal{U}$ has no finite subcover, for every $J \in \mathcal{D}$, we can pick up a point $z_J \in Z \setminus \cup\{U_\alpha : \alpha \in J\}$. Define a net $s : \mathcal{D} \to Z$ by $s_J = z_J$ for all $J \in \mathcal{D}$. Let $\mathcal{I}$ be a $\mathcal{D}$-admissible ideal on $\mathcal{D}$. Then by hypothesis, the net $(s_J)_{J\in\mathcal{D}}$ has a b-$\mathcal{I}$-cluster point, say $z_o \in Z$. So there exists $\alpha_o \in \mathcal{D}$ such that $z_o \in U_{\alpha_o}$. Evidently, $\{J \in \mathcal{D} : s_J \in U_{\alpha_o}\} \notin \mathcal{I}$. This yields that $\{J \in \mathcal{D} : s_J \notin U_{\alpha_o}\} \notin \mathcal{F}_{\mathcal{I}}$. This tells us that $\{J \in \mathcal{D} : s_J \notin U_{\alpha_o}\}$ can't contain any member of $\mathcal{F}_{\mathcal{I}}$. Since $\mathcal{I}$ is a $\mathcal{D}$-admissible ideal, so for every $J \in \mathcal{D}$, $\{K \in \mathcal{D} : K \geq J\} \in \mathcal{F}_{\mathcal{I}}$. In particular, for $\{\alpha_o\} \in \mathcal{D}$, we have $\{K \in \mathcal{D} : K \geq \{\alpha_o\}\} \in \mathcal{F}_{\mathcal{I}}$. Hence $\{K \in \mathcal{D} : K \geq \{\alpha_o\}\} \nsubseteq \{J \in \mathcal{D} : s_J \notin U_{\alpha_o}\}$. Thus there exists $K_o \in \mathcal{D}$ such that $\alpha_o \in K_o$ and $s_{K_o} = z_{K_o} \in U_{\alpha_o}$. But $z_{K_o} \in Z \setminus \cup\{U_\alpha : \alpha \in K_o\}$. This shows that $z_{K_o} \notin U_{\alpha_o}$, a contradiction. Hence Z is a b-compact space. □

We conclude this write-up by stating the following result which characterizes b-$\mathcal{I}$-cluster points of net in terms of a specific subset of Z.

Theorem 5.10. *Let $s : D \to Z$ be a net in a space Z, and $\mathcal{I}$ a non-trivial ideal on D. For every $A \in \mathcal{F}_{\mathcal{I}}$, let $A^+ := \{s_d : d \in A\}$. Then $z_o \in Z$ is a b-$\mathcal{I}$-cluster point of the net $(s_d)_{d\in D}$ if and only if $z_o \in \bigcap_{A\in\mathcal{F}_{\mathcal{I}}} Cl_b(A^+)$.*

References

1. A. Al-Omari and M. Noorani, *On generalized b-closed sets*, Bull. Malays. Math. Sci. Soc., (2) 32(1), 19-30, (2009).
2. A. Al-Omari and M.S.M. Noorani, *Decomposition of continuity via b-open set*, Bol. Soc. Paran. Mat., 26(1-2), 53-64, (2008), doi: 10.5269/bspm.v26i1-2.7402.
3. A. Al-Omari and T. Noiri, *On Ψ^*-operator in ideal m-spaces*, Bol. Soc. Paran. Mat., 30(1), 53-66, (2012), doi: 10.5269/bspm.v30i1.12787.
4. A. Al-Omari and T. Noiri, *Weak forms of G-α-open sets and decompositions of continuity via grills*, Bol. Soc. Paran. Mat., 31(2), 19-29, (2013), doi: 10.5269/bspm.v31i2.13551.
5. A. Al-Omari and T. Noiri, *On quasi compact spaces and some functions*, Bol. Soc. Paran. Mat., 36(4), 121-130, (2018), doi: 10.5269/bspm.v36i4.31125.
6. D. Andrijević, On b-open sets, *Mat. Vesnik*, 48, 59-646, (1996).
7. V. Baláž, J. Červeňanský, P. Kostyrko and T. Šalát, *I-convergence and I-continuity of real functions*, Acta Math. (Nitra), 5, 43-50, (2002).
8. R.C. Buck, *The measure theoretic approach to density*, Amer. J. Math., 68, 560-580, (1946).
9. R.C. Buck, *Generalized asymptotic density*, Amer. J. Math., 75, 335-346, (1953).
10. M. Caldas and S. Jafari, *On some applications of b-open sets in topological spaces*, Kochi J. Math., 2, 11-19, (2007).
11. G. Di Maio and Lj. D. R. Kočinac, *Statistical convergence in topology*, Topology Appl., 156, 28-45, (2008).
12. E. Ekici and M. Caldas, *Slightly γ-continuous functions*, Bol. Soc. Parana. Mat., (3) 22(2), 63-74, (2004).
13. A.A. El-Atik, *A study on some types of mappings on topological spaces*, M.Sc Thesis, Egypt, Tanta University, (1997).
14. H. Fast, *Sur la convergence statistique*, Colloq. Math., 2, 241-244, (1951).
15. C. Granados, *A new notion of convergence on ideal topological spaces*, Sel. Mat., 7(2), 250-256, (2020).
16. H. Halberstem and K.F. Roth, *Sequences*, Springer, New York, (1993).
17. K.D. Joshi, *Introduction to General Topology*, Wiley, (1983).
18. K. Kuratowski, *Topologie I*, PWN, Warszawa, 1961.
19. B.K. Lahiri and P. Das, *I and I*-convergence in topological spaces*, Math. Bohemica, 130 (2), 153-260, (2005).
20. B.K. Lahiri and P. Das, *I and I*-convergence of nets*, Real Anal. Exchange, 33(2), 431-442, (2007/2008).
21. D.S. Mitrinovic, J. Sandor and B. Crstici, *Handbok of Number Theory*, Kluwer Acad. Publ. Dordrecht-Boston-London, (1996).
22. S. Modak, J. Hoque and Sk Selim, *Homeomorphic image of some kernels*, Cankaya Uni. J. Sci. Eng., 17(1), 052-062, (2020).
23. A.A. Nasef, *On b-locally closed sets and related topics*, Choas. Solitons and Fractals, 12, 1909-1915, (2001).
24. I. Niven and H.S. Zuckerman, *An Introduction to the Theory of Numbers*, 4th Ed., John Wiley, New York, 1980.
25. T. Noiri, A. Al-Omari and M. Noorani, *On ωb-open sets and b-Lindelöf spaces*, Eur. J. Pure Appl. Math., 1(3), 3-9, (2008).
26. J.H. Park, *Strongly θ-b-continuous functions*, Acta. Math. Hungar., 110(4), 347-359, (2006).
27. N. Rajesh and V. Vijayabharathi, *Properties of b-compact spaces and b-closed spaces*, *Bol. Soc. Paran. Mat.*, 32(2), 237-247, (2014).
28. T.C.K. Raman and P.K. Biswas, *On interrealtionship of b-Lindelof & second countable spaces as well as countably b-compact & sequentially b-compactness*, Int. J. Eng. Res. Manag. Technol., 2(3), 57-63, (2015).
29. I.J. Schoengberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, 66, 361-375, (1959).

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