



Nonlocal Elliptic Problems on Compact Riemannian Manifold

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ABSTRACT: In this paper, we investigate the existence and uniqueness of a non-trivial solution for a class of nonlocal equations involving the fractional p -Laplacian operator defined on compact Riemannian manifold, namely,

$$\begin{cases} (-\Delta_g)_p^s u(x) + |u|^{p-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } M \setminus \Omega, \end{cases} \quad (0.1)$$

and Ω is an open bounded subset of M with a smooth boundary.

Keywords: Non-linear fractional elliptic equation, weak solution, existence and uniqueness, Riemannian manifold.

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1. Introduction

Non-local operator problems have recently received a lot of attention in the literature. A good amount of investigation have focused on the existence and regularity of solutions to such problems governed by the fractional Laplacian in Euclidean space \mathbb{R}^n , see for instance [13,22,23,24,26,28,29,31]. For the prototype form $(-\Delta)^s$, $0 < s < 1$, which is the infinitesimal generator of the s -stable Lévy processes [4], the associate equation was treated by R. Servadei and E. Valdinoci in reference [31] by proving the existence of a solution to the following problem:

$$\begin{cases} (-\Delta)^s u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \quad (1.1)$$

where $(-\Delta)^s$ is a non- local operator defined as follow:

$$(-\Delta)^s(u(x)) = C(N, s)PV \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+ps}} dy,$$

see for instante the reference [31] for more details. For the non-linear involving the p -fractional Laplacian which is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

where $sp < N$ with $s \in (0, 1)$, $p \in (1, \infty)$, Ω is an open bounded subset of \mathbb{R}^N with smooth boundary and $B_\epsilon(x)$ the ball of \mathbb{R}^N of center x and radius ϵ , we refer to [13,22,23,24,26,28,29]. This type of problem arises in many applications such as continuum mechanics, phase phenomena [15], population dynamics, game theory , crystal dislocation [18], optimization, finance [14,19], stratified materials, conversation

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laws and minimal surfaces [9,11,12,30].

For the framework of the above problems on the open set of N -dimensional, real Euclidean space \mathbb{R}^N , we recommend [10,17].

Here, in this paper, we are interested in the non-Euclidian case, i.e., the case of Riemannian manifold, and the treatment of non-linear fractional operators defined on Riemannian manifold defined as follow:

$$(-\Delta_g)_p^s u(x) = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{M \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{(d_g(x, y))^{N+ps}} d\mu_g(y),$$

for $x \in M$, where M is a compact Riemannian N -manifold, $d\mu_g(y) = \sqrt{\det(g_{ij})} dy$ is the Riemannian volume element on (M, g) , $B_\varepsilon(x)$ denotes the geodesic ball of centre x and radius ε , dy is the Lebesgue volume element of \mathbb{R}^N , $d_g(x, y)$ defines a distance on M , and g a C^∞ Riemannian metric, see the reference [5] and section 2 for more information. Consider the following non-linear problem with lower order:

$$\begin{cases} (-\Delta_g)_p^s u(x) + |u|^{p-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } M \setminus \Omega, \end{cases} \quad (1.2)$$

where $N > ps$ with $s \in (0, 1)$, $p \in (1, \infty)$, and Ω is an open bounded subset of M with smooth boundary $\partial\Omega$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(f1) There exist $\beta > 0$ and $1 < q < p_s^* = \frac{Np}{N-ps}$ such that

$$|f(x, t)| \leq \beta(1 + |t|^{q-1}),$$

for a.e. $x \in \Omega$, $t \in \mathbb{R}$.

(f2) For $1 < q < p_s^*$, we have

$$\lim_{t \rightarrow \infty} \frac{f(x, t)}{|t|^{q-1}} = 0 \text{ uniformly for a.e } x \in \Omega.$$

(f3)

$$\lim_{\zeta \rightarrow 0} \frac{f(x, \zeta)}{|\zeta|^{p-1}} = 0 \text{ uniformly for a.e } x \in \Omega;$$

(f4) (AR condition) There exists $\mu > p$ such that

$$0 < \mu F(x, t) \leq t f(x, t) \text{ for a.e } x \in \Omega \text{ and } t > 0,$$

$$\text{where } F(x, t) = \int_0^t f(x, s) ds.$$

(f5) The function $h : \Omega \times (0, \infty) \rightarrow \mathbb{R}^+$ as

$$h(x, t) = \frac{f(x, t)}{t^{p-1}}$$

is decreasing in $(0, \infty)$ for a.e $x \in \Omega$.

Example 1.1 For $c > 0$, the function $f(x, t) = c|t|^p \exp(-t)$ satisfying the above conditions.

Remark 1.1 The conditions (f1)–(f4) are used to prove the solution's existence, with the auxiliary condition (f5) proving the solution's uniqueness.

Our goal in this paper is to prove the existence and uniqueness of a non-trivial solution, via variational methods in the framework of fractional Sobolev space on Riemannian manifold, in which, we extend the results proved in references [5,3,13,24,16,25,27,32] in the non-Euclidean case, this generates some complications due to the non locality character of the operator, that is having a well defined Dirichlet

problem in the non-local framework. So it is not enough to prescribe the boundary condition $\partial\Omega$, since to compute the value of $(-\Delta_g)_p^s u(x)$ at $x \in \Omega$, we need to know the value of $u(x)$ in the whole M . Other complications are due to the non-Euclidean framework of our equation. For that, checking for example the density of Shwartz space $\mathcal{D}(M)$ in $W^{s,p}(M)$, it's not useful to consider a function C^∞ on \mathbb{R} , as in the proof of Euclidean case, because for Riemannian manifold $[d(P, Q)]^2$ is only Lipschitz function in M and in $Q \in M$, P being fixed point of M . For more functional properties of Sobolev space to compact Riemannian manifold, we refer to [5,7,21]. In addition, another challenge is to verify that the chosen test functions are admissible.

The rest of this paper is structured as follows: In section 2 we recall some Definitions, Lemmas, and Theorems, that will help us in our analysis. In section 3 we will prove the existence of a non-trivial solution using the Mountain Pass Theorem, and in section 4, we will show the uniqueness of a non-trivial solution of our problem.

2. Background Material

First of all, we recall the most important and relevant properties and notations, by referring to [1,2,5,8,21] for more details.

Definition 2.1 Let (M, g) be an N -dimensional Riemannian manifold and let ∇ be the Levi-Civita connection. For $u \in C^\infty(M)$, then $\nabla^k u$ denotes the k -th covariant derivative of u . In local coordinates, the pointwise norm of $\nabla^k u$ is given by

$$|\nabla^k u| = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^{k u})_{i_1 i_2 \dots i_k} (\nabla^k u)_{j_1 j_2 \dots j_k}.$$

When $k = 1$, the components of ∇u in local coordinates are given by

$$(\nabla u)_i = \nabla^i u$$

. By definition, one has that

$$|\nabla u| = \sum_{i,j=1}^{\infty} g^{ij} \nabla^i u \nabla^j u.$$

Definition 2.2 Let (M, g) be an N -dimensional Riemannian manifold and $\gamma : [a, b] \subset \mathbb{R} \rightarrow M$ a curve of class C^1 , the length of c is

$$l(\gamma) = \int_a^b (g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt.$$

Definition 2.3 Let (M, g) be an N -dimensional Riemannian manifold, and let $C_{x,y}^1$ be the space of piecewise C^1 curves $\gamma : [a, b] \subset \mathbb{R} \rightarrow M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. The distance between x and y is defined by

$$d(x, y) = \inf\{l(\gamma), \gamma \text{ is a differentiable curve connecting } x \text{ and } y\}.$$

Remark 2.1 Let (M, g) be an N -dimensional Riemannian manifold, then (M, g) is a metric space.

Definition 2.4 Consider X to be a topological space, and $U = \{U_i, i \in I\}$ to be an open covering of X . A subordinate to U is a family of finite functions that satisfy two conditions, $\sum_{i \in I} \eta_i = 1$ and $\text{Supp } \eta_i \subset U_i$.

Theorem 2.1 [5]. Let X be a paracompact differential manifold and let U_i be an open cover of X , then there exists a locally finite partition of class $C^\infty(X)$ in X , subordinate to U_i .

Definition 2.5 Let (M, g) be an N -dimensional Riemannian manifold, $(\Omega_i, \varphi_i)_{i \in I}$ an atlas of M and $(\Omega_i, \varphi_i, \eta_i)$ a partition of a subordinate to $(\Omega_i, \varphi_i)_{i \in I}$. We can define the Riemannian measure as follows $u : M \rightarrow \mathbb{R}$ with compact support by

$$\int_M u(x) \mu_g(x) = \sum_{k \in J} \int_{\varphi_k(\Omega_k)} \left(\sqrt{\det(g_{ij})} \eta_k u \right) \circ \varphi_k^{-1}(x) dx,$$

where $g_{ij}, i, j \in I$ are the components of the Riemannian metric g in the chart (Ω_i, φ_i) and dx is the Lebesgue volume element of \mathbb{R}^N .

Let $0 < s < 1$, and $1 < p < \infty$ be real numbers. The fractional Sobolev space $W^{s,p}(M)$ is defined as follows: It is endowed with the natural norm

$$\|u\|_{W^{s,p}(M)} = \left(\int_M |u(x)|^p d\mu_g(x) + [u]_{W^{s,p}(M)}^p \right)^{1/p},$$

with

$$[u]_{W^{s,p}(M)} = \left(\iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \right)^{1/p},$$

where $[u]_{W^{s,p}}$ is the Gagliardo semi-norm.

Let us denote by $W_0^{s,p}(M)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(M)$. Notice that $W_0^{s,p}(M)$ and $W^{s,p}(M)$ are reflexive and separable Banach spaces, for all $0 < s < 1 < p < \infty$. We refer to [24] for more details.

Lemma 2.1 [24] *Let (M, g) be an N -dimensional Riemannian manifold. Then,*

- (1) *There exists a positive constant $C_1 = C_1(N, p, q, s)$ such that for any $u \in W_0^{s,p}(M)$ and $1 \leq q \leq p_s^*$,*

$$\|u\|_{L^q(M)}^p \leq C_1 \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y).$$

- (2) *There exists a constant $\tilde{C} = \tilde{C}(N, p, q, s)$ such that for any $u \in W_0^{s,p}(M)$,*

$$\begin{aligned} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) &\leq \|u\|_{W^{s,p}(M)}^p \\ &\leq \tilde{C} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y). \end{aligned}$$

Consequently, the space $W_0^{s,p}(M)$ is continuously embedded in $L^q(M)$ for any $q \in [p, p_s^*]$, where p_s^* is a critical exponent defined by:

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ \infty & \text{if } sp \geq N. \end{cases}$$

Definition 2.6 *We say that a functional ψ satisfies the Palais-Smale condition in $W_0^{s,p}(M)$, if for any sequence $u_n \subset W_0^{s,p}(M)$ such that $\psi(u_n) \rightarrow c$ and $\psi'(u_n) \rightarrow 0$ in $W_0^{s,p}(M)^*$ as $n \rightarrow \infty$, then u_n admits a convergent subsequence.*

Lemma 2.2 [6] *(Fractional Picone inequality). Let $u, v \in W_0^{s,p}(M)$ with $u > 0$. Assume that $(-\Delta_g)_p^s u$ is a positive bounded Radon measure in Ω . Then*

$$\int_\Omega \frac{v^p}{u^{p-1}} (-\Delta_g)_p^s u d\mu_g(x) \leq \|v\|_{W^{s,p}(M)}^p.$$

Remark 2.2 *We will use Picone's Lemma to show that the functions of type $\frac{v^p}{u^{p-1}}$ and $\frac{u^p}{v^{p-1}}$ are admissible test functions, which play a crucial role in the uniqueness part.*

3. Main Results

In this section, we study the existence of a non-trivial weak solution of problem (1.2).

Definition 3.1 A function $u \in W_0^{s,p}(\Omega)$ is said to be a weak solution of problem (1.2), if

$$\begin{aligned} & \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ & + \int_M |u(x)|^{p-2} u(x)v(x) d\mu_g(x) = \int_\Omega f(x, u(x))v(x) d\mu_g(x), \end{aligned}$$

for any $v \in W_0^{s,p}(M)$.

In the following, we will prove the existence of a non-trivial solution for the case where $q \in (1, p)$.

Theorem 3.1 Under assumptions (f1)–(f3). If $1 < q < p$, then the problem (1.2) has a non-trivial weak solution in $W_0^{s,p}(M)$.

The energy functional in $W_0^{s,p}(M)$ is defined by

$$\begin{aligned} \psi(u) &= \frac{1}{p} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) + \frac{1}{p} \int_M |u(x)|^p d\mu_g(x) \\ & - \int_\Omega F(x, u(x)) d\mu_g(x) \\ & := I_1(u) + I_2(u) - K(u), \end{aligned}$$

where $F(x, t) = \int_0^t f(x, s) ds$.

The energy functional is $C^1(W_0^{s,p}(M), \mathbb{R})$, as we will show in the following two lemmas.

Lemma 3.1 Suppose that (f1) is true, then the functional $K \in C^1(W_0^{s,p}(M), \mathbb{R})$ and

$$\langle K'(u), v \rangle = \int_\Omega f(x, u)v d\mu_g(x), \quad \text{for all } u, v \in W_0^{s,p}(\Omega).$$

Proof: Let $u, v \in W_0^{s,p}(M)$, $x \in M$ and $0 < t < 1$, we have

$$\begin{aligned} \frac{1}{t}(F(x, u + tv) - F(x, u)) &= \frac{1}{t} \int_0^{u+tv} f(m, s) ds - \frac{1}{t} \int_0^u f(m, s) ds \\ &= \frac{1}{t} \int_u^{u+tv} f(m, s) ds. \end{aligned}$$

By the mean value theorem, there exists a $0 < z < 1$ such that

$$\frac{1}{t}(F(x, u + tv) - F(x, u)) = f(x, u + ztv)v.$$

We use the (f1) and Young's inequality, we obtain

$$\begin{aligned} |f(x, u + ztv)v| &\leq \beta |1 + |u + ztv|^{q-1}| |v| \\ &\leq \beta \left| \frac{1}{q} |v|^q + \frac{1}{q'} + \frac{1}{q'} |u + ztv|^q + \frac{1}{q} |v|^q \right| \\ &\leq \beta (2|v|^q + 1 + 2^2(|v|^q + |u|^q)) \\ &\leq \beta 2^{q+1} (1 + |u|^q + |v|^q), \end{aligned}$$

where q' is the conjugate of q . Lebesgue's dominated convergence Theorem implies

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} (K(u + tv) - K(u)) &= \lim_{t \rightarrow 0} \int_{\Omega} f(x, u + ztv) v d\mu_g(x) \\ &= \int_{\Omega} \lim_{t \rightarrow 0} f(x, u + ztv) v d\mu_g(x) \\ &= \int_{\Omega} f(x, u) v d\mu_g(x). \end{aligned}$$

Let $u_n, u \in W_0^{s,p}(M)$ be such that $u_n \rightarrow u$ strongly in $W_0^{s,p}(M)$ as $n \rightarrow \infty$. According to Lemma 2.1, there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n \rightarrow u$ a.e. in Ω . Since $1 < q < p_s^*$, we use the Lemma 2.1, Hölder's inequality and (f1), to get

$$\begin{aligned} \int_{\Omega} |f(x, u_n)|^{q'} d\mu_g(x) &\leq 2^{\frac{q+1}{q-1}} \beta^{\frac{q+1}{q-1}} \left(\|u_n\|^q \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(\Omega)} + \mu(\Omega) \right) \\ &\leq C(\mu(\Omega))^{\frac{p_s^*-q}{p_s^*}} + C\mu(\Omega) \leq C(q, \beta, \Omega), \end{aligned} \quad (3.1)$$

where $\mu(\Omega)$ denotes the volume of set Ω and q' is the conjugate of q . It follows from inequality 4 that, the sequence $|f(x, u_n) - f(x, u)|^{q'}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. The Vitali convergence Theorem implies

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f(x, u_n) - f(x, u)|^{q'} d\mu_g(x) = 0.$$

Thus, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} \|K'(u_n) - K'(u)\|_{W_0^{s,p}(M)^*} &= \sup_{v \in W_0^{s,p}(\Omega), \|v\|_{W_0^{s,p}(\Omega)} \leq 1} \|\langle K'(u_n) - K'(u), v \rangle\| \\ &\leq \|f(x, u_n) - f(x, u)\|_{L^{q'}(\Omega)} \|v\|_{L^q(\Omega)} \\ &\leq \|f(x, u_n) - f(x, u)\|_{L^{q'}(\Omega)} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, where $W_0^{s,p}(M)^*$ denotes the dual space of $W_0^{s,p}(M)$. □

Lemma 3.2 *The functional $I_1 + I_2 \in C^1(W_0^{s,p}(M), \mathbb{R})$ and*

$$\begin{aligned} \langle (I_1 + I_2)'(u), v \rangle &= \int_M |u(x)|^{p-2} u(x) v(x) d\mu_g(x) \\ &\quad + \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y), \end{aligned}$$

for all $u, v \in W_0^{s,p}(M)$.

Proof: Let $u, v \in W_0^{s,p}(M)$ we have

$$\begin{aligned} \langle (I_2)'(u), v \rangle &= \lim_{t \rightarrow 0} \frac{1}{t} (I_2(u + tv) - I_2(u)) \\ &= \frac{1}{p} \lim_{t \rightarrow 0} \frac{1}{t} \int_M (|u(x) + tv(x)|^p - |u(x)|^p) d\mu_g(x). \end{aligned}$$

We consider the function defined by $K : [0, 1] \rightarrow \mathbb{R}$ as

$$K(y) = |u(x) + tyv(x)|^p.$$

According to the mean value Theorem, there exists a $0 < z < 1$ such that

$$\frac{1}{p} \frac{|u(x) + tv(x)|^p - |u(x)|^p}{t} = |u(x) + ztv(x)|^{p-2} (u(x) + ztv(x))v(x).$$

We apply the mean value Theorem dominate, we get

$$\langle (I_2)'(u), v \rangle = \int_M |u(x)|^{p-2} u(x) v(x) d\mu_g(x).$$

Similarly, we have

$$\langle (I_1)'(u), v \rangle = \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y).$$

Let $u, v \in W_0^{s,p}(M)$, by Hölder's inequality, we have

$$\begin{aligned} \langle (I_1 + I_2)'(u), v \rangle &= \int_M |u(x)|^{p-2} u(x) v(x) d\mu_g(x) \\ &\quad + \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\leq \int_M |u(x)|^{p-1} v(x) d\mu_g(x) \\ &\quad + \iint_{M \times M} \frac{|u(x) - u(y)|^{p-1} (v(x) - v(y))}{(d_g(x, y))^{(N+ps)(\frac{p-1}{p})} (d_g(x, y))^{(N+ps)\frac{1}{p}}} d\mu_g(x) d\mu_g(y) \\ &\leq \left(\iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \right)^{\frac{p-1}{p}} \\ &\quad \times \left(\iint_{M \times M} \frac{|v(x) - v(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \right)^{\frac{1}{p}} \\ &\quad + \left(\int_M |u(x)|^p d\mu_g(x) \right)^{\frac{p-1}{p}} \left(\int_M |v(x)|^p d\mu_g(x) \right)^{\frac{1}{p}}. \end{aligned}$$

Finally, we obtain $\psi \in C^1(W_0^{s,p}(M), \mathbb{R})$ and

$$\begin{aligned} \langle (\psi)'(u), v \rangle &= \int_M |u(x)|^{p-2} u(x) v(x) d\mu_g(x) \\ &\quad + \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y). \end{aligned}$$

□

Now, we will show that the energy functional ψ is weakly lower semi-continuous, and coercive.

Lemma 3.3 *Assume (f1) holds. Then the functional ψ is weakly lower semi-continuous.*

Proof: Let $\{u_n\} \subset W_0^{s,p}(M)$, such that $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(M)$ as $n \rightarrow \infty$. Because $I_1 + I_2$ is convex functional, we concluded that the following inequality holds

$$(I_1 + I_2)(u_n) > (I_1 + I_2)(u) + \langle (I_1 + I_2)'(u), u_n - u \rangle.$$

Then we get that $((I_1 + I_2)(u) \leq \liminf_{n \rightarrow \infty} (I_1 + I_2)(u_n)$.

Since $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(M)$, we get that $u_n \rightarrow u$ strongly in $L^q(\Omega)$. Without loss of generality, we assume that $u_n \rightarrow u$ a.e in M . Similar to the proof of the Lemma 3.1, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} F(x, u_n) d\mu_g(x) = \int_{\Omega} F(x, u) d\mu_g(x).$$

As a result, ψ is weakly lower semi-continuous in $W_0^{s,p}(M)$. □

Finally, by applying condition (f1), we can get $|F(x, z)| < 2\beta(1 + |z|^q)$, and by applying Lemma 2.1, we can get

$$\begin{aligned} \psi(u) &= \frac{1}{p} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) + \frac{1}{p} \int_M |u(x)|^p d\mu_g(x) \\ &\quad - \int_{\Omega} F(x, u(x)) d\mu_g(x) \\ &\geq \frac{1}{p} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) - \int_{\Omega} F(x, u(x)) d\mu_g(x) \\ &\geq \frac{1}{p} \|u\|_{W_0^{s,p}(M)}^p - 2\beta C_1^{\frac{q}{p}} \|u\|_{W_0^{s,p}(M)}^q - 2\beta \mu(\Omega). \end{aligned}$$

Since $q < p$, we have $\psi(u) \rightarrow \infty$ as $\|u\|_{W_0^{s,p}(M)} \rightarrow \infty$. Since ψ is weakly lower semi-continuous, it has a minimum point u_0 in $W_0^{s,p}(M)$, and u_0 is a weak solution of problem (1.2). This completes the proof of Theorem 3.1.

Now, we will prove our second result, the existence of a weak solution in the case $q \in (p, p_s^*)$. We will use the geometric Mountain Pass Theorem.

Theorem 3.2 *Let f be a function satisfying conditions (f1)–(f4) then the problem (1.2) has a weak solution for $p < q < p_s^*$.*

We must prove the following Lemmas in order to prove Theorem 3.

Lemma 3.4 *The functional ψ satisfies the Palais-Smale condition.*

Proof: Let $u_n \subset W_0^{s,p}(M)$ be such that $\psi(u_n) \rightarrow c$ and $\psi'(u_n) \rightarrow 0$ in $(W_0^{s,p}(M))^*$ as $n \rightarrow \infty$, so for n large we have $c + 1 + \|u\|_{W_0^{s,p}(M)} \geq \psi(u_n) - \frac{1}{\mu} |\langle \psi'(u_n), u_n \rangle|$ with $\mu > 0$. By assumption (f1) yields

$$\begin{aligned} c + 1 + \|u_n\|_{W_0^{s,p}(M)} &\geq \psi(u_n) - \frac{1}{\mu} |\langle \psi'(u_n), u_n \rangle| \\ &= \frac{1}{p} \|u_n\|_{W_0^{s,p}(M)}^p + \frac{1}{p} \|u_n\|_{L^p(M)}^p - \int_{\Omega} F(x, u_n(x)) d\mu_g(x) \\ &\quad - \frac{1}{\mu} (\|u_n\|_{W_0^{s,p}(M)}^p + \|u_n\|_{L^p(M)}^p) + \frac{1}{\mu} \int_{\Omega} f(x, u_n(x)) u_n(x) d\mu_g(x) \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \|u_n\|_{W_0^{s,p}(M)}^p. \end{aligned}$$

Since $W_0^{s,p}(M)$ is uniformly convex space, then there is a subsequence that will be noted as (u_n) such that $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(M)$. We use the Hölder's inequality, we have

$$\begin{aligned} &([u_n]_{s,p}^{p-1} - [u]_{s,p}^{p-1})([u_n]_{s,p} - [u]_{s,p}) = [u_n]_{s,p}^p + [u]_{s,p}^p - [u_n]_{s,p}^{p-1}[u]_{s,p} - [u]_{s,p}^{p-1}[u_n]_{s,p} \\ &\leq \iint_{M \times M} \frac{|u_n(x) - u_n(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\quad + \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\quad - \iint_{M \times M} \frac{|u_n(x) - u_n(y)|^p (u_n(x) - u_n(y))(u(x) - u(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\quad - \iint_{M \times M} \frac{|u(x) - u(y)|^p (u(x) - u(y))(u_n(x) - u_n(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &= B_n. \end{aligned}$$

Now, we will show that $B_n \rightarrow 0$ as $n \rightarrow \infty$. We have,

$$\begin{aligned} \langle (\psi)'(u_n) - (\psi)'(u), u_n - u \rangle &= B_n + \int_M (u_n - u)(|u_n(x)|^{p-2}u_n - |u(x)|^{p-2}u) d\mu_g(x) \\ &\quad + \int_M (u_n - u)(f(x, u_n) - f(x, u)) d\mu_g(x). \end{aligned}$$

Since $u_n \rightarrow u$ as $n \rightarrow \infty$ in $W_0^{s,p}(M)$. According to Lemma 1, we have $u_n \rightarrow u$ strongly in $L^q(M)$, for all $q \in [p, p_s^*]$, and $u_n \rightarrow u$ a.e in M . By Theorem 3.32 [5], there exists a sub-sequence noted by $u_n, h_1 \in L^q(M)$ and $h_2 \in L^q(M)$ such that

$$|u_n(x)| \leq h_1(x), |u(x)| \leq h_2(x) \text{ a.e in } M,$$

for all $q \in [p, p_s^*]$. From Hölder's inequality, we get

$$\begin{aligned} &\int_M (u_n - u)(|u_n(x)|^{p-2}u_n - |u(x)|^{p-2}u) d\mu_g(x) \\ &\leq \int_M (u_n - u)|u_n(x)|^{p-1} d\mu_g(x) + \int_M (u_n - u)|u(x)|^{p-1} d\mu_g(x) \\ &\leq \int_M |u_n - u||h_1|^{p-1} d\mu_g(x) + \int_M |u_n - u||h_2|^{p-1} d\mu_g(x) \\ &\leq \|u_n - u\|_{L^p(M)} (\|h_1\|_{L^p(M)}^{p-1} + \|h_2\|_{L^p(M)}^{p-1}). \end{aligned}$$

We also apply Hölder's inequality and the assumption (f1) we obtained

$$\begin{aligned} &|\int_M (f(x, u_n) - f(x, u))(u_n - u) d\mu_g(x)| \\ &\leq \int_M |u_n - u||f(x, u_n)| d\mu_g(x) + \int_M |u_n - u||f(x, u)| d\mu_g(x) \\ &\leq \int_M \beta|u_n - u| d\mu_g(x) + \int_M |h_1|^{q-1}\beta|u_n - u| d\mu_g(x) \\ &\quad + \int_M |h_2|^{q-1}\beta|u_n - u| d\mu_g(x) + \int_M \beta|u_n - u| d\mu_g(x) \\ &\leq \beta(\|h_1\|_{L^q(M)}^{q-1} + \|h_2\|_{L^q(M)}^{q-1})\|u_n - u\|_{L^q(M)} \\ &\quad + 2\beta\|1\|_{L^{\frac{p}{p-1}}(M)}\|u_n - u\|_{L^p(M)}. \end{aligned}$$

Since $u_n \rightarrow u$ as $n \rightarrow \infty$, we have $B_n \rightarrow 0$ as $n \rightarrow \infty$. Finally, we find that $u_n \rightarrow u$ strongly in $W_0^{s,p}(M)$. \square

Lemma 3.5 *Let f be a function satisfying the conditions (f1) and (f3). Then there are two positive real numbers, a and b , such that $\psi(u) \leq a$ and $\|u\|_{W_0^{s,p}(M)} = b$, for all $u \in W_0^{s,p}(M)$ and $p < q < p_s^*$.*

Proof: Let's combine the two conditions (f1) and (f3). There exists a $C > 0$ such that

$$F(x, t) \leq \varepsilon|t|^p + C|t|^q, \text{ for all } \varepsilon > 0.$$

As a result, we have

$$\begin{aligned} \psi(u) &= \frac{1}{p} \iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) + \frac{1}{p} \int_{\Omega} \frac{|u(x)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) \\ &\quad - \int_{\Omega} F(x, u(x)) d\mu_g(x) \\ &\geq \frac{1}{p} \|u\|_{W_0^{s,p}(M)}^p - \varepsilon C_1(p, q) \|u\|_{W_0^{s,p}(M)}^p - C C_1(p, q) \|u\|_{W_0^{s,p}(M)}^q. \end{aligned}$$

If we take $\varepsilon = \frac{1}{2pC_1}$, we have

$$\begin{aligned}\psi(u) &\geq \frac{1}{2p} \|u\|_{W_0^{s,p}(M)}^p - CC_1(q,p) \|u\|_{W_0^{s,p}(M)}^q \\ &= \|u\|_{W_0^{s,p}(M)}^p \left(\frac{1}{2p} - CC_1(q,p) \|u\|_{W_0^{s,p}(M)}^{q-p} \right) \\ &= b^p \left(\frac{1}{2p} - CC_1(q,p) b^{q-p} \right) =: a.\end{aligned}$$

□

Lemma 3.6 *If f satisfying conditions (f1) and (f4), if $p < q < p_s^*$, then there exists a $v \in W_0^{s,p}(M)$ such that $\|v\|_{W_0^{s,p}(M)} \geq b$ and $\psi(v) < 0$.*

Proof: We apply the condition (f4), we have

$$F(x, tv) \geq t^\mu F(x, v), \text{ for } t \geq 1.$$

Consequently, we have

$$\begin{aligned}\psi(tv) &= \frac{t^p}{p} \|u\|_{W_0^{s,p}(M)}^p + \frac{t^p}{p} \|u\|_{L^p(M)}^p - \int_{\Omega} F(x, tv) d\mu_g(x) \\ &\leq \frac{t^p}{p} (\|u\|_{W_0^{s,p}(M)}^p + C \|u\|_{W_0^{s,p}(M)}^p) - \int_{\Omega} F(x, tv) d\mu_g(x) \\ &\leq C' \frac{t^p}{p} \|u\|_{W_0^{s,p}(M)}^p - t^\mu \int_{\Omega} f(x, v) d\mu_g(x).\end{aligned}$$

As $\mu \geq p \geq 1$, we have $\psi(tv) \rightarrow -\infty$ as $t \rightarrow +\infty$. So there exists $t_0 > 0$ large enough such that $\|u\|_{W_0^{s,p}(M)}^p > b$ and $\psi(t_0 u) < 0$. We take $v = ut_0$ with large enough we get the result. Conclusion: From lemmas (3.4)-(3.6) and ψ satisfies the Mountain Pass Theorem, ψ admits a critical value u ; however, this u is a weak solution to the problem (1.2). As a result, the proof is finished. □

4. Uniqueness of weak solution

Now, we will study the following problem:

$$\begin{cases} (-\Delta_g)_p^s u(x) + |u|^{p-2} u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } M \setminus \Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \quad (4.1)$$

Where Ω is an open bounded smooth-boundary subset of M .

Remark 4.1 *The problem 4.1 is well defined.*

Proof: Let u is a weak solution of our problem. Then

$$\begin{aligned}\iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y))}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ + \int_M |u(x)|^{p-2} u(x) v(x) d\mu_g(x) = \int_{\Omega} f(x, u(x)) v(x) d\mu_g(x)\end{aligned} \quad (4.2)$$

for any $v \in W_0^{s,p}(M)$. Using $v = u$ in 4.2, we have

$$\begin{aligned}\iint_{M \times M} \frac{|u(x) - u(y)|^p}{(d_g(x, y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ + \int_M |u(x)|^p d\mu_g(x) = \int_{\Omega} f(x, u(x)) u(x) d\mu_g(x).\end{aligned} \quad (4.3)$$

By condition (f5), we have $u > 0$. □

Lemma 4.1 *Let $v \in W_0^{s,p}(\Omega)$ and g be a function satisfying a Lipschitz condition in \mathbb{R} . Then $g(v) \in W_0^{s,p}(\Omega)$.*

Proof: Let $v \in W_0^{s,p}(\Omega)$. Using the g is a Lipschitz function we get,

$$\begin{aligned} \|g(v)\|_{W_0^{s,p}(\Omega)}^p &= \iint_{\Omega \times \Omega} \frac{|g(v(x)) - g(v(y))|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\leq l(p) \|v\|_{W_0^{s,p}(\Omega)}^p, \end{aligned}$$

where l is Lipschitz constant. \square

Theorem 4.1 *Let f be a function satisfying conditions (f1)–(f5), then the problem (4.1) has a unique non-trivial solution.*

Proof: Let $u, v \in W_0^{s,p}(M)$ two solutions of problem (4.1), we get

$$\begin{aligned} &\iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &+ \int_M |u(x)|^{p-2} u(x) \phi(x) d\mu_g(x) = \int_\Omega f(x, u(x)) \phi(x) d\mu_g(x), \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} &\iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &+ \int_M |u(x)|^{p-2} u(x) \psi(x) d\mu_g(x) = \int_\Omega f(x, u(x)) \varphi(x) d\mu_g(x), \end{aligned} \quad (4.5)$$

for any $\phi, \varphi \in W_0^{s,p}(M)$.

We have $\frac{v^p}{u^{p-1}} \in W_0^{s,p}(M)$, thanks to Lemma 4.1. Since $W_0^{s,p}(M)$ is a space vector, it yields $u - \frac{v^p}{u^{p-1}}$ and $\frac{u^p}{v^{p-1}} - v \in W_0^{s,p}(M)$. Using $\phi = u - \frac{v^p}{u^{p-1}}$ and $\varphi = \frac{u^p}{v^{p-1}} - v$ in (4.4) and (4.5) respectively, we have

$$\langle (I_2)'(u), \phi \rangle - \langle (I_2)'(v), \varphi \rangle = \int_\Omega (u^p - v^p) \left(\frac{f(x, u)}{u^{p-1}} - \frac{f(x, v)}{v^{p-1}} \right).$$

Thanks to (f5) we obtain

$$\int_\Omega (u^p - v^p) \left(\frac{f(x, u)}{u^{p-1}} - \frac{f(x, v)}{v^{p-1}} \right) \leq 0. \quad (4.6)$$

On the other hand

$$\begin{aligned} &\langle (I_2)'(u), \phi \rangle - \langle (I_2)'(v), \varphi \rangle \\ &= \iint_{M \times M} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &+ \iint_{M \times M} \frac{|v(x) - v(y)|^{p-2} (v(x) - v(y)) (\varphi(x) - \varphi(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &= \|u\|_{W_0^{s,p}(M)}^p - \int_\Omega \frac{v^p}{u^{p-1}} (-\Delta_g)_p^s u + \|v\|_{W_0^{s,p}(M)}^p - \int_\Omega \frac{u^p}{v^{p-1}} (-\Delta_g)_p^s v. \end{aligned}$$

Through fractional Picone inequality, we achieve

$$\langle (I_2)'(u), \phi \rangle - \langle (I_2)'(v), \varphi \rangle \geq 0. \quad (4.7)$$

Let us collect with (4.6) and (4.7) let us get

$$\int_{\Omega} (u^p - v^p) \left(\frac{f(x, u)}{u^{p-1}} - \frac{f(x, v)}{v^{p-1}} \right) = 0.$$

The consequence of (f5) is that $u = v$ a.e in Ω . □

References

1. Aberqi, A., Bennouna, J., Benslimane, O., Ragusa, M.A., Existence results for double phase problem in Sobolev Orlicz spaces with variable exponents in complete manifold, *Mediterr. J. Math.* 19, 158, (2022).
2. Aberqi, A., Benslimane, O., Ouaziz, A., Repovs, D. D., On a new fractional Sobolev space with variable exponent oncomplete manifolds, *Bound. Value Probl.* 7, (2022).
3. Aberqi, A., Bennouna, J., Hammouni, M. *Non-uniformaly degenerated parabolic equations with L^1 -data*, AIP Conference Proceedings. 2019 DOI: 10.1063/1.5090619.
4. Applebaum, D. *Lévy processes—from probability to finance quantum groups*, Notices Amer. Math. Soc, 51, 1336-1347, (2004).
5. Aubin, T., *Nonlinear Analysis on Manifolds*, Monge Ampere Equations, vol. 252. Springer, Berlin (1982).
6. Barrios, B., Peral, I., Vita, S. *Some remarks about the summability of nonlocal nonlinear problems*, Adv. Nonlinear Anal. 4(2), 33-107, (2015).
7. Benslimane, O., Aberqi, A., Bennoua, J. , *Existence and uniqueness of weak solution of $p(x)$ - laplacian in Sobolev spaces with variable exponents in completes manifolds*, FILOMAT. 35, 1453-1463, (2021).
8. Benslimane, O., Aberqi, A., Bennoua, J., *Existence and uniqueness of entropy solution of a nonlinear elliptic equation in anisotropic Sobolev-Orlicz space*, Rendiconti del Circolo Matematico di Palermo Series 2, 1-30, (2020).
9. Biler, P., Karch, G., Woyczyński, W. A., *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (5), 613-637, (2001).
10. Bisci, G. M., Radulescu, V. D., Servadei, R., *Variational methods for nonlocal fractional problems*, volume 162 of Encyclopedia of Mathematics and Applications. Cambridge University Press, Cambridge (2016).
11. Caffarelli, L., Roquejoffre, J. M., Savin, O., *Non-local minimal surfaces*, Comm. Pure Appl. Math. 63, 1111-1144, (2010).
12. Caffarelli, L., Valdinoci, E., *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations 41 (1–2), 203-240, (2011).
13. Chang, S. Y. A., González, M. d. M., *Fractional Laplacian in conformal geometry*, Adv. Math, 226, 1410-1432, (2010).
14. Cont, R., Tankov, P., *Financial Modelling with Jump Processes*, Chapman Hall/CRC Financ. Math. Ser., Chapman Hall/CRC, Boca Raton, FL, 2004.
15. Danielli, D., Salsa, S., *Obstacle problems involving the fractional Laplacian*, In Recent developments in nonlocal theory, pages 81-164. De Gruyter, Berlin, (2018).
16. Druet, O., Hebey, E., *Blow-up examples for second order elliptic PDEs of critical Sobolev growth*, Trans. Amer. Math. Soc. 357, 1915-1929, (2004).
17. Di Nezza, E., Palatucci, G., Valdinoci, E., *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math, 136, 521-573, (2012).
18. Dipierro, S. Figalli, A., E. Valdinoci, E., *Strongly nonlocal dislocation dynamics in crystals*, Comm. Partial Differential Equations, 39(12), 2351-2387, (2014).
19. Duvaut, G., Lions, J. L., *Inequalities in Mechanics and Physics*, Grundlehren Math. Wiss., vol. 219, Springer-Verlag, Berlin, 1976. Translated from French by C.W. John.
20. Fiscella, A., Servadei, R., Valdinoci, E., *Density properties for fractional Sobolev spaces*, Ann. Acad. Sci. Fenn. Math, 40, 235-253, (2015).
21. Gadea, P. M., Masqué, J. M., Mykytyuk, I. V., *Analysis and algebra on differentiable manifolds: a workbook for students and teachers*, Springer Science & Business Media, (2012).
22. González, M. d. M., *Recent progress on the fractional Laplacian in conformal geometry*, arXiv:1609.08988.
23. González, M. d. M., Qing, J., *Fractional conformal Laplacians and fractional Yamabe problems*, Analysis & PDE, 6.7, 1535-1576, (2013).

24. Guo, L., *Fractional p -Laplacian equations on Riemannian manifolds*, Electronic Journal of Differential Equations, 156, 1-17, (2018).
25. Hebey, E., *Nonlinear Analysis on Manifolds: Sobolev spaces and inequalities*, Courant Lecture Notes, vol. 5, American Mathematical Society, (2000).
26. Iannizzotto, A., Squassina, M., *Weyl-type laws for fractional p -eigenvalue problems*, Asymptotic Anal, 88, 233-245, (2014).
27. Ilias, S., *Constantes explicites pour les inégalités de Sobolev sur les variétés riemanniennes compactes*, Ann. Inst. Fourier 33, 151—165, (1983).
28. Molica Bisci, G., *Sequences of weak solutions for fractional equations*, Math. Res. Lett, 21, 1-13, (2014).
29. Molica Bisci, G., *Fractional equations with bounded primitive*, Appl. Math. Lett., 27, 53—58, (2014).
30. Savin, O., Valdinoci, E., *Elliptic PDEs with fibered nonlinearities*, J. Geom. Anal. 19 (2), 420-432, (2009).
31. Servadei, R., Valdinoci, E. *Lewy-Stampacchia type estimates for variational inequalities driven by (non)local operators*, Rev. Mat. Iberoam, 29, 1091-1126, (2013).
32. Trudinger, N. S., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22, 265-274, (1968).

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