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Between Strongly θ-continuous and Weakly Continuous Functions

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ABSTRACT: In this paper, we investigate some properties of θ_g-continuous and weakly g-continuous functions in a topological spaces. Moreover, the relationships with other related functions are investigated.

Key Words: θ-continuous, strongly θ-continuous, weakly g-continuous, θ_g-continuous.

Contents

1. Introduction

Let (X,τ) be a topological space with no separation axioms assumed. If $A \subseteq X$, $Cl(A)$ (or $cl(A)$) and $Int(A)$ (or $int(A)$) will denote the closure and interior of A in (X,τ), respectively.

In 1968, Veličko [14] introduced the class of θ-open sets. A set A is said to be θ-open [14] if every point of A has an open neighborhood whose closure is contained in A. The θ-interior [14] of A in X is the union of all θ-open subsets of A and is denoted by $Int_\theta(A)$. Naturally, the complement of a θ-open set is said to be θ-closed. Equivalently $Cl_\theta(A) = \{x \in X : Cl(U) \cap A \neq \phi, U \in \tau \text{ and } x \in U\}$ and a set A is θ-closed if and only if $A = Cl_\theta(A)$. Note that all θ-open sets form a topology on X, coarser than τ that is $\tau_\theta \subseteq \tau$, denoted by τ_θ and that a space (X,τ) is regular if and only if $\tau = \tau_\theta$. Note also that the θ-closure of a given set need not be a θ-closed set.

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be θ-continuous [6] (resp. strongly θ-continuous [12], weakly continuous [7]) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(Cl(U)) \subseteq Cl(V)$ (resp. $f(Cl(U)) \subseteq V$, $f(U) \subseteq Cl(V)$). Some other locally closed set related continuity has been discussed in [1,2,3,5,9,10]. In the present paper, we investigate some properties of θ_g-continuous and weakly g-continuous functions in a topological spaces. Moreover, the relationships with other related functions are investigated.

Definition 1.1. *[10] Let (X,τ) be a topological space and $A \subseteq X$. Then*

1. *A is generalized closed (briefly g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.*
2. *A is generalized open(briefly g-open) if $X \setminus A$ is g-closed.*
3. *(X,τ) is $T_{\frac{1}{2}}$-space if every g-closed set is closed.*

The intersection of all g-closed sets containing A is called the g-closure of A [4] and denoted by $Cl^*(A)$ or $cl^*(A)$, and the g-interior of A is the union of all g-open sets contained in A and is denoted by $Int^*(A)$ or $int^*(A)$. A is said to be τ^*-closed if $Cl^*(A) = A$. The complement of a τ^*-closed set is called a τ^*-open set.

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2. Preliminaries

Let us start by the following Lemma:

Lemma 2.1. *[4] Let (X, τ) be a topological space and $A \subseteq X$. Then*

1. *$A \subseteq Cl^*(A) \subseteq Cl(A)$.*
2. *Cl^* is a Kuratowski closure operator on X and $\tau^* = \{A : Cl^*(X - A) = X - A\}$ is a topology on X generated by Cl^* in the usual manner.*
3. *$\tau \subseteq \tau^*$ with equality if and only if (X, τ) is $T_{\frac{1}{2}}$-space.*

Let (X, τ) be a topological space and $A \subseteq X$. A point x of X is called a θ_g-cluster point of A if $Cl^*(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ_g-cluster points of A is called the θ_g-closure of A and is denoted by $Cl^*_\theta(A)$. A is said to be θ_g-closed if $Cl^*_\theta(A) = A$. The complement of a θ_g-closed set is called a θ_g-open set. The family of all θ_g-open sets in (X, τ) will be denoted by τ^*_θ.

Definition 2.2. *Let (X, τ) be a topological space. A point x of X is called a θ_g-interior point of A if there exists an open set U containing x such that $Cl^*(U) \subseteq A$. The set of all θ_g-interior points of A is called the θ_g-interior of A and is denoted by $Int^*_\theta(A)$.*

Remark 2.3. *For a set A of X, $Int^*_\theta(X - A) = X - Cl^*_\theta(A)$ so that A is θ_g-open if and only if $A = Int^*_\theta(A)$. In this respect, $Int^*_\theta \sim^X Cl^*_\theta$ [11].*

Lemma 2.4. *Let (X, τ) be a topological space and let $A \subseteq X$. Then*
$Cl^*(A) \subseteq Cl(A) \subseteq Cl^*_\theta(A) \subseteq Cl_\theta(A)$ and $\tau_\theta \subseteq \tau^*_\theta \subseteq \tau \subseteq \tau^*$.

Lemma 2.5. *[14] Let (X, τ) be a topological space. Then*

1. *for each $A \in \tau$, $Cl_\theta(A) = Cl(A)$.*
2. *X is regular if and only if $\tau = \tau_\theta$.*

Theorem 2.6. *Let (X, τ) be a topological space and $A \subseteq X$. Then $A \in \tau^*_\theta$ if and only if for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$.*

Proof. Suppose that $A \in \tau^*_\theta$ and let $x \in A$. Then $X - A$ is θ_g-closed and $x \notin X - A$. Thus, $x \notin Cl^*_\theta(X - A)$ and hence there is $U \in \tau$ such that $x \in U$ and $Cl^*(U) \cap (X - A) = \emptyset$. Therefore, we have $x \in U \subseteq Cl^*(U) \subseteq A$.

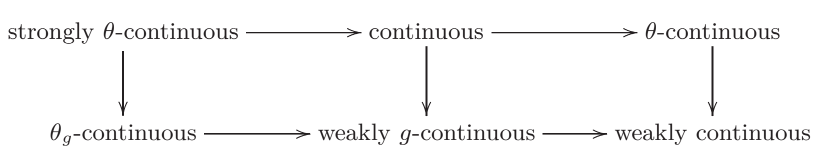
Conversely, suppose for each $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$ and suppose on that contrary that $A \notin \tau^*_\theta$. Then $X - A$ is not θ_g-closed and $Cl^*_\theta(X - A) \neq X - A$. Choose $x \in Cl^*_\theta(X - A) - (X - A)$. Since $x \in A$, there is $U \in \tau$ such that $x \in U \subseteq Cl^*(U) \subseteq A$. Thus we have $x \in U \in \tau$ and hence $Cl^*(U) \cap (X - A) = \emptyset$. Therefore $x \notin Cl^*_\theta(X - A)$, a contradiction. □

Corollary 2.7. *In a topological space (X, τ) every open and τ^*-closed set is θ_g-open.*

Proof. Let (X, τ) be a topological space and let A be open and τ^* -closed set in (X, τ). Let $x \in A$. Since A is τ^*-closed, then $Cl^*(A) = A$. Take $U = A$. Then $U \in \tau$ and $x \in U = Cl^*(U) = A \subseteq A$. Thus by Theorem 2.6, it follows that A is θ_g-open. □

Definition 2.8. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly g-continuous (resp. θ_g-continuous) if for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq cl^*(V)$ (resp. $f(cl^*(U)) \subseteq cl^*(V)$).*

By the above definitions, we have the following diagram and none of these implications is reversible as shown by examples.



The following strict implications are well-known:

$$\text{strongly } \theta\text{-continuous} \longrightarrow \text{continuous} \longrightarrow \theta\text{-continuous} \longrightarrow \text{weakly continuous}$$

Example 2.9. *Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{1\}, \{2,3\}\}$ and $Y = \{a,b,c\}$, $\sigma = \{\emptyset, Y, \{a\}\}$. Then $\sigma^* = \{\emptyset, Y, \{a\}, \{b,c\}, \{a,c\}, \{a,b\}, \{b\}, \{c\}\}$ We define a function $f : (X,\tau) \to (Y,\sigma)$ as $f = \{(1,b),(2,a),(3,c)\}$. Then f is weakly continuous function but not weakly g-continuous.*

1. *Let $1 \in X$ such that $f(1) = b \in V = Y$, then there exists an open set $U = \{1\} \in \tau$ containing 1 such that $f(U) = \{b\} \subseteq cl(V) = Y$.*
2. *Let $2 \in X$ such that $f(2) = a \in V = \{a\}$ or $V = Y$, then there exists an open set $U = \{2,3\}$ containing 2 such that $f(U) = \{a,c\} \subseteq cl(V) = Y$.*
3. *Let $3 \in X$ such that $f(3) = c \in V = Y$, then there exists an open set $U = \{2,3\}$ containing 3 such that $f(U) = \{a,c\} \subseteq cl(V) = Y$.*

By (1), (2) and (3) f is weakly continuous. On the other hand, let $2 \in X$ such that $f(2) = a \in V = \{a\}$. But, for every open set U containing 2 where $U = \{2,3\}$ or $U = X$. Then $\{a,c\} \subseteq f(U) \nsubseteq cl^*(V) = \{a\}$. Therefore, $f : (X,\tau) \to (Y,\sigma)$ is not weakly g-continuous.

Example 2.10. *Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{2\}, \{3\}, \{2,3\}\}$ and $Y = \{a,b,c\}$, $\sigma = \{\emptyset, Y, \{a\}\}$. Then $\tau^* = \{\emptyset, X, \{2\}, \{3\}, \{2,3\}\}$ and $\sigma^* = \{\emptyset, Y, \{a\}, \{b,c\}, \{a,c\}, \{a,b\}, \{b\}, \{c\}\}$. We define a function $f : (X,\tau) \to (Y,\sigma)$ as $f = \{(1,b),(2,a),(3,c)\}$. Then f is weakly g-continuous but not θ_g-continuous. We show that f is weakly g-continuous*

1. *$1 \in X$ and $f(1) = b \in V = Y$. If we take $U = X$, then $f(U) = Y \subseteq cl^*(V)$.*
2. *$2 \in X$ and $f(2) = a \in V = \{a\} \subseteq cl^*(V) = \{a\}$ or $V = Y \subseteq cl^*(V) = Y$. If we take $U = \{2\}$, then $f(U) = \{a\} \subseteq cl^*(V)$ for all V containing a.*
3. *$3 \in X$ and $f(3) = c \in V = Y \subseteq cl^*(V) = Y$. If we take $U = \{3\}$, then $f(U) = \{c\} \subseteq cl^*(V)$ for all V containing c.*

Then for each $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteq cl^*(V)$.

We show that $f : (X,\tau) \to (Y,\sigma)$ is not θ_g-continuous. Let $2 \in X$ and $V = \{a\} \in \sigma$ such that $f(2) = a \in V \in \sigma$. But, for every open set $U \subseteq X$ such that $2 \in U$, where $U = \{2\}$ or $U = \{2,3\}$ or $U = X$, $Cl^*(U) = \{1,2\}$ or $Cl^*(U) = X$. Then, for all open set U containing we have 2, $f(Cl^*(U) \nsubseteq cl^*(V) = \{a\}$. Therefore, $f : (X,\tau) \to (Y,\sigma)$ is not θ_g-continuous.

Example 2.11. *Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{2\}, \{3\}, \{2,3\}\}$ and $Y = \{a,b,c\}$, $\sigma = \{\emptyset, Y, \{a,b\}\}$. Then $\tau^* = \{\emptyset, X, \{2\}, \{3\}, \{2,3\}\}$ and $\sigma^* = \{\emptyset, Y, \{a,b\}, \{b\}, \{a\}\}$ We define a function $f : (X,\tau) \to (Y,\sigma)$ as $f = \{(1,a),(2,b),(3,c)\}$. Then f is θ_g-continuous but not continuous.*

1. *Let $1 \in X$ and $V = \{a,b\}$ or $V = Y$ such that $f(1) = a \in V$, then there exists an open set $U = X \in \tau$ containing 1 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*
2. *Let $2 \in X$ and $V = \{a,b\}$ or $V = Y$ such that $f(2) = b \in V$, then there exists an open set $U = \{2\}$ containing 2 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*
3. *Let $3 \in X$ and $V = Y$ such that $f(3) = c \in V$, then there exists an open set $U = \{2,3\}$ containing 3 such that $f(cl^*(U)) \subseteq cl^*(V) = Y$.*

By (1), (2) and (3) f is θ_g-continuous. On the other hand, let $1 \in X$ and $V = \{a,b\} \in \sigma$ such that $f(1) = a \in V$. But, for every open set $U \subseteq X$ such that $1 \in U = X$. Then $f(U) = Y \nsubseteq V = \{a,b\}$. Therefore, $f : (X,\tau) \to (Y,\sigma)$ is not continuous.

3. Characterizations of θ_g-continuous functions

In this section, we obtain several characterizations of θ_g-continuous functions in topological spaces.

Theorem 3.1. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*

1. *f is θ_g-continuous;*

2. $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$ *for every subset* B *of* Y*;*
3. $f(Cl^*_\theta(A)) \subseteq cl^*_\theta(f(A))$ *for every subset* A *of* X*.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \notin f^{-1}(cl^*_\theta(B))$. Then $f(x) \notin cl^*_\theta(B)$ and there exists an open set V containing $f(x)$ such that $cl^*(V) \cap B = \emptyset$. Since f is θ_g-continuous, there exists an open set U containing x such that $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, we have $f(Cl^*(U)) \cap B = \emptyset$ and $Cl^*(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin Cl^*_\theta(f^{-1}(B))$. Thus, we obtain $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$.
(2) $\Rightarrow$ (1): Let $x \in X$ and V be an open set of Y containing $f(x)$. Then we have $cl^*(V) \cap (Y - cl^*(V)) = \emptyset$ and $f(x) \notin cl^*_\theta(Y - cl^*(V))$. Therefore, $x \notin f^{-1}(cl^*_\theta(Y - cl^*(V)))$ and by (2) we have $x \notin Cl^*_\theta(f^{-1}(Y - cl^*(V)))$. There exists an open set U containing x such that $Cl^*(U) \cap f^{-1}(Y - cl^*(V)) = \emptyset$ and hence $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, f is θ_g-continuous.
(2) $\Rightarrow$ (3): Let A be any subset of X. Then we have $Cl^*_\theta(A) \subseteq Cl^*_\theta(f^{-1}(f(A))) \subseteq f^{-1}(cl^*_\theta(f(A)))$ and hence $f(Cl^*_\theta(A)) \subseteq cl^*_\theta(f(A))$.
(3) $\Rightarrow$ (2): Let B be a subset of Y. We have $f(Cl^*_\theta(f^{-1}(B))) \subseteq cl^*_\theta(f(f^{-1}(B))) \subseteq cl^*_\theta(B)$ and hence $Cl^*_\theta(f^{-1}(B)) \subseteq f^{-1}(cl^*_\theta(B))$. □

Theorem 3.2. *For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following implications: (1)* $\Leftrightarrow$ *(2)* $\Rightarrow$ *(3)* $\Leftrightarrow$ *(4) hold. Moreover, the implication (4)* $\Rightarrow$ *(1) holds if* (Y, σ) *is a* $T_{\frac{1}{2}}$*-space.*

1. f *is* θ_g*-continuous;*
2. $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$ *for every open set* V *of* Y*;*
3. $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ *for every open set* V *of* Y*;*
4. *For each* $x \in X$ *and each open set* V *of* Y *containing* $f(x)$*, there exists an open set* U *of* X *containing* x *such that* $f(Cl^*(U)) \subseteq cl(V)$*.*

Proof. (1) $\Rightarrow$ (2): Let V be any open set in Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists an open set U containing x such that $f(Cl^*(U)) \subseteq cl^*(V)$. Therefore, $x \in U \subseteq Cl^*(U) \subseteq f^{-1}(cl^*(V))$. This shows that $x \in Int^*_\theta(f^{-1}(cl^*(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$.
(2) $\Rightarrow$ (1): Let $x \in X$ and $V \in \sigma$ containing $f(x)$. Then, by (2) $f^{-1}(V) \subseteq Int^*_\theta(f^{-1}(cl^*(V)))$. Since $x \in f^{-1}(V)$, there exists an open set U containing x such that $Cl^*(U) \subseteq f^{-1}(cl^*(V))$. Therefore, $f(Cl^*(U)) \subseteq cl^*(V)$ and hence f is θ_g-continuous.
(2) $\Rightarrow$ (3): Let V be any open set in Y and $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and there exists an open set W containing $f(x)$ such that $W \cap V = \emptyset$; hence $cl^*(W) \cap V \subseteq cl(W) \cap V = \emptyset$. Therefore, we have $f^{-1}(cl^*(W)) \cap f^{-1}(V) = \emptyset$. Since $x \in f^{-1}(W)$, by (2) $x \in Int^*_\theta(f^{-1}(cl^*(W)))$. There exists an open set U containing x such that $Cl^*(U) \subseteq f^{-1}(cl^*(W))$. Thus we have $Cl^*(U) \cap f^{-1}(V) = \emptyset$ and hence $x \notin Cl^*_\theta(f^{-1}(V))$. This shows that $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.
(3) $\Rightarrow$ (4): Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $V \cap (Y - cl(V)) = \emptyset$ and $f(x) \notin cl(Y - cl(V))$. Therefore $x \notin f^{-1}(cl(Y - cl(V)))$ and by (3) $x \notin Cl^*_\theta(f^{-1}(Y - cl(V)))$. There exists an open set U containing x such that $Cl^*(U) \cap f^{-1}(Y - cl(V)) = \emptyset$. Therefore, we obtain $f(Cl^*(U)) \subseteq cl(V)$.
(4) $\Rightarrow$ (3): Let V be any open set of Y. Suppose that $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and there exists an open set W containing $f(x)$ such that $W \cap V = \emptyset$. By (4), there exists an open set U containing x such that $f(Cl^*(U)) \subseteq cl(W)$. Since $V \in \sigma$, $cl(W) \cap V = \emptyset$ and $f(Cl^*(U)) \cap V \subseteq cl(W) \cap V = \emptyset$. Therefore, $Cl^*(U) \cap f^{-1}(V) = \emptyset$ and hence $x \notin Cl^*_\theta(f^{-1}(V))$. This shows that $Cl^*_\theta(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.
(4) $\Rightarrow$ (1): Since (Y, σ) is a $T_{\frac{1}{2}}$-space, $cl(V) = cl^*(V)$ for every open set V of Y and hence f is θ_g-continuous. □

Proposition 3.3. *A function* $f : (X, \tau) \to (Y, \sigma)$ *from a* $T_{\frac{1}{2}}$*-space to a* $T_{\frac{1}{2}}$*-space is* θ_g*-continuous if and only if it is* θ*-continuous.*

Proof. This follows from the Lemma 2.1 (3). □

4. Some properties of θ_g-continuous functions

Definition 4.1. *A topological space* (X, τ) *is said to be* θ_g*-*T_2 *(resp. g-Urysohn) if for each distinct points* $x, y \in X$*, there exist two* θ_g*-open (resp. open) sets* $U, V \in X$ *containing* x *and* y*, respectively, such that* $U \cap V = \emptyset$ *(resp.* $Cl^*(U) \cap Cl^*(V) = \emptyset$*).*

Theorem 4.2. *If* $f, g : (X, \tau) \to (Y, \sigma)$ *are* θ_g*-continuous functions and* (Y, σ) *is g-Urysohn , then* $A = \{x \in X : f(x) = g(x)\}$ *is a* θ_g*-closed set of* (X, τ)*.*

Proof. We prove that $X - A$ is a θ_g-open set. Let $x \in X - A$. Then $f(x) \neq g(x)$. Since Y is g-Urysohn, there exist open sets V_1 and V_2 containing $f(x)$ and $g(x)$, respectively, such that $cl^*(V_1) \cap cl^*(V_2) = \emptyset$. Since f and g are θ_g-continuous, there exists an open set U_1 containing x such that $f(Cl^*(U_1)) \subseteq cl^*(V_1))$ and there exists an open set U_2 containing x such that $g(Cl^*(U)) \subseteq cl^*(V_2))$. Let $U = U_1 \cap U_2$ which is an open set U containing x such that $f(Cl^*(U)) \subseteq cl^*(V_1))$ and $g(Cl^*(U)) \subseteq cl^*(V_2))$. Hence we obtain that $Cl^*(U) \subseteq f^{-1}(cl^*(V_1))$ and $Cl^*(U) \subseteq g^{-1}(cl^*(V_2))$. From here we have $Cl^*(U) \subseteq f^{-1}(cl^*(V_1)) \cap g^{-1}(cl^*(V_2))$. Moreover $f^{-1}(cl^*(V_1)) \cap g^{-1}(cl^*(V_2)) \subseteq X - A$. This shows that $X - A$ is θ_g-open.

□

Definition 4.3. *A topological space (X, τ) is said to be g-regular if for each closed set F and each point $x \notin F$, there exist an open set V and an τ^*-open set $U \in \tau^*$ such that $x \in V$, $F \subseteq U$ and $U \cap V = \emptyset$.*

Example 4.4. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ with $\tau^* = \{\emptyset, X, \{a\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b\}, \{c\}\}$, then (X, τ) is a g-regular space which is not regular.*

Lemma 4.5. *A topological space (X, τ) is g-regular if and only if for each open set U containing x there exists an open set V such that $x \in V \subseteq Cl^*(V) \subseteq U$.*

Lemma 4.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is weakly g-continuous if and only if for each open set V, $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$.*

Proof. Necessity. Let V be any open set of Y and $x \in f^{-1}(V)$. Since f is weakly g-continuous, there exists an open set U such that $x \in U$ and $f(U) \subseteq cl^*(V)$. Hence $x \in U \subseteq f^{-1}(cl^*(V))$ and $x \in Int(f^{-1}(cl^*(V)))$. Therefore, we obtain $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$.

Sufficiency. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Let $U = Int(f^{-1}(cl^*(V)))$. Then $f(U) \subseteq f(Int(f^{-1}(cl^*(V)))) \subseteq f(f^{-1}(cl^*(V))) \subseteq cl^*(V)$. Hence f is weakly g-continuous.

□

Lemma 4.7. *If a space (Y, σ) is a $T_{\frac{1}{2}}$-space and a function $f : (X, \tau) \to (Y, \sigma)$ is weakly g-continuous, then $Cl^*(f^{-1}(G)) \subseteq f^{-1}(cl^*(G))$ for every open set G in Y.*

Proof. Suppose there exists a point $x \in Cl^*(f^{-1}(G)) - f^{-1}(cl^*(G))$. Then $f(x) \notin cl^*(G)$, we have $f(x) \notin G$. Since Y is a $T_{\frac{1}{2}}$-space $f(x) \notin cl(G)$. Hence there exists an open set W containing $f(x)$ such that $W \cap G = \emptyset$. Since G is open, $G \cap cl(W) = \emptyset$ and hence we have $G \cap cl^*(W) = \emptyset$. Since f is weakly g-continuous, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq cl^*(W)$. Thus we obtain $f(U) \cap G = \emptyset$. On the other hand, since $x \in Cl^*(f^{-1}(G))$, we have $x \in Cl(f^{-1}(G))$ and hence $U \cap f^{-1}(G) \neq \emptyset$. Thus $f(U) \cap G \neq \emptyset$, a contradiction. Hence $Cl^*(f^{-1}(G)) \subseteq f^{-1}(cl^*(G))$ for every open set G in Y. □

Theorem 4.8. *Let (Y, σ) be a $T_{\frac{1}{2}}$-space. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is weakly g-continuous;*
2. *$Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set V of Y;*
3. *f is weakly continuous.*

Proof. (1) ⇒ (2): Let V be any open set of Y. Suppose that $x \notin f^{-1}(cl^*(V))$. Then $f(x) \notin cl^*(V)$. Since (Y, σ) is a $T_{\frac{1}{2}}$-space, $f(x) \notin cl(V)$ and there exists $W \in \sigma$ containing $f(x)$ such that $W \cap V = \emptyset$, hence $cl^*(W) \cap V = cl(W) \cap V = \emptyset$. Since f is weakly g-continuous, there exists $U \in \tau$ containing x such that $f(U) \subseteq cl^*(W)$. Therefore, we have $f(U) \cap V = \emptyset$ and $U \cap f^{-1}(V) = \emptyset$. Since $U \in \tau$, $U \cap Cl(f^{-1}(V)) = \emptyset$ and hence $x \notin Cl(f^{-1}(V))$. Therefore, we obtain $Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$.
(2) ⇒ (3): Let V be any open set of Y. Since (Y, σ) is a $T_{\frac{1}{2}}$-space, by (2) we have $Cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$. It follows from Theorem 7 of [13] that f is weakly continuous.
(3) ⇒ (1): Let f be weakly continuous. By Theorem 1 of [7] $f^{-1}(V) \subseteq Int(f^{-1}(cl(V)))$ for every open set V of Y. Since (Y, σ) is a $T_{\frac{1}{2}}$-space, $cl(V) = cl^*(V)$ and we have $f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Therefore, by Lemma 4.6 f is weakly g-continuous. □

Definition 4.9. *An topological space (X, τ) is said to be g-extremally disconnected if the g-closure of every open subset of X is open.*

Proposition 4.10. *Let (X, τ) be an g-regular space. Then $f : (X, \tau) \to (Y, \sigma)$ is θ_g-continuous if and only if it is weakly g-continuous.*

Proof. Every θ_g-continuous function is weakly g-continuous. Suppose that f is weakly g-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then, there exists an open set U containing x such that $f(U) \subseteq cl^*(V)$. Since X is g-regular, by Lemma 4.5 there exists an open set W such that $x \in W \subseteq Cl^*(W) \subseteq U$. Therefore, we obtain $f(Cl^*(W)) \subseteq cl^*(V)$. This shows that f is θ_g-continuous. □

Theorem 4.11. *Let a topological space (Y, σ) be a $T_{\frac{1}{2}}$-space and g-extremally disconnected. Then $f : (X, \tau) \to (Y, \sigma)$ is θ_g-continuous if and only if it is weakly g-continuous.*

Proof. It is clear that every θ_g-continuous function is weakly g-continuous. Conversely, suppose that f is weakly g-continuous. Let $x \in X$ and V be an open set of Y containing $f(x)$. Then by Lemma 4.6, $x \in f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V)))$. Let $U = Int(f^{-1}(cl^*(V)))$. Since (Y, σ) is an $T_{\frac{1}{2}}$-space and g-extremally disconnected, by using Lemma 4.7,

$$\begin{aligned} f(Cl^*(U)) &= f(Cl^*(Int(f^{-1}(cl^*(V)))) \\ &\subseteq f(Cl^*(f^{-1}(cl^*(V))) \\ &\subseteq f(f^{-1}(cl^*(cl^*(V))) \subseteq cl^*(V). \end{aligned}$$

Hence f is θ_g-continuous. □

Corollary 4.12. *Let a topological space (Y, σ) be a $T_{\frac{1}{2}}$-space and g-extremally disconnected. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is θ_g-continuous;*
2. *f is weakly g-continuous;*
3. *$f^{-1}(V) \subseteq Int(f^{-1}(cl^*(V))$ for every open set V in Y;*
4. *$f^{-1}(V) \subseteq Int(f^{-1}(cl(V)))$ for every open set V of Y;*
5. *f is weakly continuous.*

Proof. By Theorem 4.11, we have the equivalence of (1) and (2). The equivalences of (2), (3) and (4) follow from Lemma 4.6 and Lemma 2.1 (1). The equivalence of (4) and (5) is shown in Theorem 1 of [7]. □

A subset A of a topological space (X, τ) is said to be pre-g-open if $A \subseteq Int(Cl^*(A))$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pre-g-continuous if the inverse image of every open set of Y is pre-g-open in X.

Theorem 4.13. *If $f : (X, \tau) \to (Y, \sigma)$ is a pre-g-continuous function and $Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$ for every open set U in Y, then f is θ_g-continuous.*

Proof. Let $x \in X$ and U be an open set in Y containing $f(x)$. By hypothesis, $Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$. Since f is pre-g-continuous, $f^{-1}(U)$ is pre-g-open in X and so $f^{-1}(U) \subseteq Int(Cl^*(f^{-1}(U)))$. Since $x \in f^{-1}(U) \subseteq Int(Cl^*(f^{-1}(U)))$, there exists an open set V containing x such that $x \in V \subseteq Cl^*(V) \subseteq Cl^*(f^{-1}(U)) \subseteq f^{-1}(cl^*(U))$ and so $f(Cl^*(V)) \subseteq cl^*(U)$ which implies that f is θ_g-continuous. □

The following corollary follows from Lemmas 4.6 and 4.7 also Theorems 4.8 and 4.13.

Corollary 4.14. *Let $f : (X, \tau) \to (Y, \sigma)$ be pre-g-continuous and (Y, σ) is a $T_{\frac{1}{2}}$-space. The following properties are equivalent:*

1. *f is θ_g-continuous;*
2. *$Cl^*(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set V in Y;*
3. *$Cl(f^{-1}(V)) \subseteq f^{-1}(cl^*(V))$ for every open set V in Y;*
4. *f is weakly g-continuous.*

5. Preservation theorems

A subset A of a space X is said to be quasi H^*-closed relative to X if for every cover $\{V_\alpha : \alpha \in \Lambda\}$ of A by open sets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteq \cup\{Cl^*(V_\alpha) : \alpha \in \Lambda_0\}$. A space X is said to be quasi H^*-closed if X is quasi H^*-closed relative to X

Theorem 5.1. *If $f : (X, \tau) \to (Y, \sigma)$ is θ_g-continuous and K is quasi H^*-closed relative to X, then $f(K)$ is quasi H^*-closed relative to Y.*

Proof. Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a θ_g-continuous function and K is quasi H^*-closed relative to X. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a cover of $f(K)$ by open sets of Y. For each point $x \in K$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ_g-continuous, there exists an open set U_x containing x such that $f(Cl^*(U_x)) \subseteq cl^*(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of K by open sets of X and hence there exists a finite subset K_* of K such that $K \subseteq \cup_{x \in K_*} Cl^*(U_x)$. Therefore, we obtain $f(K) \subseteq \cup_{x \in K_*} cl^*(V_{\alpha(x)})$. This shows that $f(K)$ is quasi H^*-closed relative to Y. □

Definition 5.2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be θ_g-irresolute if for every θ_g-open set U in Y, $f^{-1}(U)$ is θ_g-open in X.*

Theorem 5.3. *Every θ_g-continuous function $f : (X, \tau) \to (Y, \sigma)$ is θ_g-irresolute.*

Proof. Let f be a θ_g-continuous function and U be a θ_g-open set in Y. Let $x \in f^{-1}(U)$. Then, $f(x) \in U$. Since U is θ_g-open, there exists an open set V in Y such that $f(x) \in V \subseteq cl^*(V) \subseteq U$. By θ_g-continuity of f, there exists an open set W in X containing x such that $f(Cl^*(W)) \subseteq cl^*(V) \subseteq U$. Thus $x \in W \subseteq Cl^*(W) \subseteq f^{-1}(U)$. Hence $f^{-1}(U)$ is θ_g-open and hence f is θ_g-irresolute. □

Definition 5.4. *(1) A topological space (X, τ) is said to be θ_g-compact if every cover of X by θ_g-open sets admits a finite subcover.*
(2) A subset A of a topological space (X, τ) is said to be θ_g-compact relative to X if every cover of A by θ_g-open sets of X admits a finite subcover.

Proposition 5.5. *In a topological space (X, τ) every quasi H^*-closed set is θ_g-compact.*

Proof. More generally, we show that if A is quasi H^*-closed relative to a space X, then A is θ_g-compact relative to X. Let $A \subseteq \cup\{V_\alpha : \alpha \in \Lambda\}$, where each V_α is θ_g-open, and A be quasi H^*-closed relative to X, then for each $x \in A$ there exists an $\alpha(x) \in \Lambda$ with $x \in V_{\alpha(x)}$. Then there exists an open set $U_{\alpha(x)}$ with $x \in U_{\alpha(x)}$ such that $Cl^*(U_{\alpha(x)}) \subseteq V_{\alpha(x)}$. Since $\{U_{\alpha(x)} : x \in A\}$ is a cover of A by open set in X, then there is a finite subset $\{x_1, x_2, ..., x_n\} \subseteq A$ such that $A \subseteq \cup\{Cl^*(U_{\alpha(x_i)}) : i = 1, 2, ..., n\} \subseteq \cup\{V_{\alpha(x_i)} : i = 1, 2, ..., n\}$. Hence A is θ_g-compact relative to X. □

Theorem 5.6. *If $f : (X, \tau) \to (Y, \sigma)$ is a θ_g-irresolute surjection and (X, τ) is θ_g-compact, then Y is θ_g-compact.*

Proof. Let $\mathcal{V}$ be a θ_g-open covering of Y. Then, since f is θ_g-irresolute, the collection $\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{V}\}$ is a θ_g-open covering of X. Since X is θ_g-compact, there exists a finite subcollection $\{f^{-1}(U_i) : i = 1, ..., n\}$ of $\mathcal{U}$ which covers X. Now since f is onto, $\{U_i : i = 1, ..., n\}$ is a finite subcollection of $\mathcal{V}$ which covers Y. Hence Y is a θ_g-compact space. □

Corollary 5.7. *The θ_g-continuous surjective image of a θ_g-compact space is θ_g-compact.*

Definition 5.8. *A topological space (X, τ) is said to be g-Lindelöf if for every open cover $\{U_\alpha : \alpha \in \Lambda\}$ of X there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Lambda$ such that $X = \cup_{n \in \mathbb{N}} Cl^*(U_{\alpha_n})$.*

Theorem 5.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a θ_g-continuous (resp. weakly g-continuous) surjection. If X is g-Lindelöf (resp. Lindelöf), then Y is g-Lindelöf.*

Proof. Suppose that f is θ_g-continuous and X is g-Lindelöf. Let $\{V_\alpha : \alpha \in \Lambda\}$ be an open cover of Y. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $f(x) \in V_{\alpha(x)}$. Since f is θ_g-continuous, there exists an open set $U_{\alpha(x)}$ of X containing x such that $f(Cl^*(U_{\alpha(x)})) \subseteq cl^*(V_{\alpha(x)})$. Now $\{U_{\alpha(x)} : x \in X\}$ is an open cover of the g-Lindelöf space X. So there exists a countable subset $\{U_{\alpha(x_n)} : n \in \mathbb{N}\}$ such that $X = \cup_{n \in \mathbb{N}}(Cl^*(U_{\alpha(x_n)}))$. Thus $Y = f(\cup_{n \in \mathbb{N}}(Cl^*(U_{\alpha(x_n)}))) \subseteq \cup_{n \in \mathbb{N}} f(Cl^*(U_{\alpha(x_n)})) \subseteq \cup_{n \in \mathbb{N}} cl^*(V_{\alpha(x_n)})$. This shows that Y is g-Lindelöf. In case X is Lindelöf the proof is similar. □

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be θ_g-closed if for each θ_g-closed set F in X, $f(F)$ is θ_g-closed in Y. The following characterization of θ_g-closed functions will be used in the sequel.

Theorem 5.10. *A surjective function $f : (X, \tau) \to (Y, \sigma)$ is θ_g-closed if and only if for each set $B \subseteq Y$ and for each θ_g-open set U containing $f^{-1}(B)$, there exists a θ_g-open set V containing B such that $f^{-1}(V) \subseteq U$.*

Proof. Necessity. Suppose that f is θ_g-closed. Since U is θ_g-open in X, $X - U$ is θ_g-closed and so $f(X - U)$ is θ_g-closed in Y. Now, $V = Y - f(X - U)$ is θ_g-open, $B \subseteq V$ and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - f^{-1}(f(X - U)) \subseteq X - (X - U) = U$.

Sufficiency. Let A be a θ_g-closed set in X. To prove that $f(A)$ is θ_g-closed, we shall show that $Y - f(A)$ is θ_g-open. Let $y \in Y - f(A)$. Then $f^{-1}(y) \cap f^{-1}(f(A)) = \emptyset$ and so $f^{-1}(y) \subseteq X - f^{-1}(f(A)) \subseteq X - A$. By hypothesis there exists a θ_g-open set V containing y such that $f^{-1}(V) \subseteq X - A$. So $A \subseteq X - f^{-1}(V)$ and hence $f(A) \subseteq f(X - f^{-1}(V)) = Y - V$. Thus $V \subseteq Y - f(A)$ and so the set $Y - f(A)$ being the union of θ_g-open sets is θ_g-open. □

Theorem 5.11. *Let $f : (X, \tau) \to (Y, \sigma)$ be a θ_g-closed surjection such that for each $y \in Y$, $f^{-1}(y)$ is θ_g-compact relative to X. If Y is θ_g-compact, then X is θ_g-compact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a θ_g-open covering of X. Since for each $y \in Y$, $f^{-1}(y)$ is θ_g-compact relative to X, we can choose a finite subset Λ_y of Λ such that $\{U_\beta : \beta \in \Lambda_y\}$ is a covering of $f^{-1}(y)$. Now, by Theorem 5.10, there exists a θ_g-open set V_y containing y such that $f^{-1}(V_y) \subseteq \cup\{U_\beta : \beta \in \Lambda_y\}$. The collection $\mathcal{V} = \{V_y : y \in Y\}$ is a θ_g-open covering of Y. In view of θ_g-compactness of Y there exists a finite subcollection $\{V_{y_1}, ..., V_{y_n}\}$ of $\mathcal{V}$ which covers Y. Then the finite subcollection $\{U_\beta : \beta \in \Lambda_{y_i}, i = 1, ..., n\}$ of $\mathcal{U}$ covers X. Hence X is a θ_g-compact space. □

Theorem 5.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function, D be a dense subset in the topological space (Y, σ^*) and $f(X) \subseteq D$. Then the following properties are equivalent:*

1. *$f : (X, \tau) \to (Y, \sigma)$ is θ_g-continuous;*
2. *$f : (X, \tau) \to (D, \sigma_D)$ is θ_g-continuous.*

Proof. (1) $\Rightarrow$ (2): Let $x \in X$ and W be any open set of D containing $f(x)$, that is $f(x) \in W \in \sigma_D$ where $\sigma_D = \{U \cap D\}$ and $U \in \sigma$. Then exists $V \in \sigma$ such that $W = D \cap V$. Since $f : (X, \tau) \to (Y, \sigma)$ is θ_g-continuous and $f(x) \in V \in \sigma$, there exists $U \in \tau$ such that $x \in U$ and $f(Cl^*(U)) \subseteq cl^*(V)$. If D is a dense subset in the topological space (Y, σ^*), then D is a dense subset in the topological space (Y, σ) since $cl^*(D) \subseteq cl(D)$. Since $\sigma \subseteq \sigma^*$, $V \in \sigma^*$. So, $cl^*(D \cap V) = cl^*(V)$ since D is dense. Thus $f(Cl^*(U)) \subseteq cl^*(V) \cap f(X) \subseteq cl^*(D \cap V) \cap D \subseteq cl^*(V) \cap D$. Since $W = D \cap V$, $cl^*_D(W) = cl^*(V) \cap D$, $f(Cl^*(U)) \subseteq cl^*_D(W)$. Hence we obtain that $f : (X, \tau) \to (D, \sigma_D)$ is θ_g-continuous.

(2) $\Rightarrow$ (1): Let $x \in X$ and V be any open set Y containing $f(x)$. Since $f(x) \in D \cap V$ and $D \cap V \in \sigma_D$, by (2) there exists $U \in \tau$ containing x such that $f(Cl^*(U)) \subseteq cl^*_D(D \cap V) = cl^*(D \cap V) \cap D \subseteq cl^*(V)$. This shows that f is θ_g-continuous. □

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References

1. A. Al-Omari, T. Noiri, *Weak forms of $\mathcal{G}$-α-open sets and decompositions of continuity via grills*, Bol. Soc. Paran. Mat. 31(2), 19-29, (2013).
2. A. Al-Omari and T. Noiri, *On $\theta_{(I,J)}$ -continuous functions*, Rend. Istit. Mat. Univ. Trieste, 44, 399–411, (2012).
3. A. Al-Omari and T. Noiri, *Weakly Φ-continuous functions in grill topological spaces*, Hacettepe Journal of Mathematics and Statistics, 41 (6), 785-793, (2012).
4. W. Dunham, *A new closure operator for non-T_1 topologies*, Kyungpook Math J, 22, 55-60, (1982).
5. E. Ekici, *On $\mathcal{A}^*_I$-sets, $\mathcal{C}_I$-sets, $\mathcal{C}^*_I$-sets and decompositions of continuity in ideal topological spaces*, An. Stiinţ. Univ. Al. I. Cuza Iasi Mat. (N. S.), 59 , 173-184, (2013).
6. S. Fomin, *Extensions of topological spaces*, Ann. of Math., 44, 471-480, (1943).
7. N. Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, 68, 44-46, (1961).
8. N. Levine, *Generalized closed sets in topology*, Rend Circ Mat Palermo, 19,, 89-96, (1970).
9. S. Modak and T. Noiri, *Remarks on locally closed set*, Acta et Commentationes Universitatis Tartuensis de Mathematica, 22(1), 57-64, (2018).
10. S. Modak and T. Noiri, *Some generalizations of locally closed sets*, Iranian Journal of Mathematical Sciences and Informatics, 14(1), 159-165, (2019).

11. S. Modak and Sk. Selim, *Set operator and associated functions*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 70(1), 456-466, (2021).
12. T. Noiri, *On δ-continuous functions*, J. Korean Math. Soc., 16, 161-166, (1980).
13. D. A. Rose, *Weak continity and almost continuity*, Internat. J. Math. Math. Sci., 7 (2), 311-318, (1984).
14. N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., 78, 103-118, (1968).

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