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A Study on Spectrum Classification of the Operator $D(p,0,0,q)$ over Hahn Sequence Space h

Avinoy Paul

ABSTRACT: The Hahn sequence space is defined as

$$h = \left\{ y = (y_n) \in w : \sum_{k=1}^{\infty} k|\triangle y_k| < \infty \text{ and } \lim_{k\to\infty} y_k = 0 \right\},$$

where $\triangle y_k = y_k - y_{k+1}$ for all $k \in \mathbb{N}$. In this paper we study the spectrum and fine spectrum of the difference operator $D(p,0,0,q)$ over the Hahn sequence space h. Further, we subdivide the spectrum into the approximate point spectrum, the defect spectrum and the compression spectrum.

Key Words: Approximate point spectrum, compression spectrum, defect spectrum, fine Spectrum.

Contents

1. Introduction

Throughout the article, $w, c, c_0, \ell_p, \ell_\infty$, and bv_p denote the sets of all convergent, null, p-absolutely summable, bounded, and p-bounded variation sequences, respectively.
Spectral theory plays an important role in mathematics due to its applications in different branches of science. Many authors have studied the spectrum and fine spectrum of different linear operators on various sequence spaces. In particular, several authors have studied the fine spectrum of certain difference-type operators on different sequence spaces.

Akhmedov and Başar [1,3] studied the fine spectrum of the Cesàro operator over c_0 and bv_p $(1 \leq p < \infty)$, respectively. Further, Akhmedov and Başar [2,4] examined the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p $(1 \leq p < \infty)$ and bv_p $(1 \leq p < \infty)$, respectively.

Altay and Başar [5,7] determined the fine spectrum of the difference operator Δ on the sequence spaces c_0, c, and ℓ_p $(0 < p < 1)$, respectively. Moreover, Altay and Başar [6] investigated the fine spectrum of the generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c.

Bilgiç and Furkan [10] studied the fine spectrum of the operator $B(r,s,t)$ over the sequence spaces ℓ_1 and bv. Dundar and Başar [12] determined the fine spectrum of the upper triangular double-band matrix Δ^+ on the sequence space c_0.

Tripathy and Paul [21] investigated the spectrum and the fine spectrum of the difference operator $D(r,0,0,s)$ on the sequence spaces c_0 and c. Further, Tripathy and Paul [17,22] studied the fine spectrum of the operators $D(r,0,0,s)$ and $D(r,0,s,0,t)$ over the sequence spaces ℓ_p and bv_p. The spectrum of the operator $D(r,0,0,s)$ over the sequence space $bv_0(= bv \cap c_0)$ was studied by Paul and Tripathy [18].

Das [11] investigated the fine spectrum of the operator $B(r,s)$ over the Hahn sequence space. Tripathy and Saikia [23] studied the spectrum of the Cesàro operator C_1 on $\overline{bv_0} \cap \ell_\infty$. The fine spectrum of the operator defined by a lambda matrix over the sequence spaces of null and convergent sequences was

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studied by Yeşilkayagil and Başar [24]. Further, Yeşilkayagil and Başar [25] presented a survey on the spectrum of triangles over sequence spaces.

Başar [9] studied sequence spaces, including topological properties, matrix transformations, matrix domains of triangles, and spectra. Mursaleen and Başar [16] studied sequence spaces and matrix transformations in the context of summability and spaces of summable sequences.

The Hahn [14] sequence space is defined by

$$h = \left\{ y = (y_n) \in w : \sum_{k=1}^{\infty} k|\Delta y_k| < \infty \text{ and } \lim_{k\to\infty} y_k = 0 \right\},$$

where $\Delta y_k = y_k - y_{k+1}$, for all $k \in \mathbb{N}$. The norm

$$\|y\|_h = \sum_{k=1}^{\infty} k|\Delta y_k| + \sup_k |y_k|$$

on the space h was defined by Hahn [14]. Rao [19] defined a new norm on h given by

$$\|y\|_h = \sum_{k=1}^{\infty} k|\Delta y_k|.$$

Moreover, different authors have investigated various properties of the Hahn sequence space.

2. Preliminaries and Definitions

Let Y be a vector space over the complex field $\mathbb{C}$. The collection of all bounded linear operators from Y into itself is denoted by $B(Y)$. Let $F \in B(Y)$, where Y is a Banach space. We associate a complex number β with the operator $(F - \beta I)$, denoted by F_β, which is defined on the same domain $D(F)$, where I is the identity operator. The inverse of F_β, denoted by F_β^{-1}, is called the resolvent operator of F_β.

A complex number β is called a regular value of F if it satisfies the following conditions: (R_1) F_β^{-1} exists,
(R_2) F_β^{-1} is bounded, and
(R_3) the domain of F_β^{-1} is dense in Y.
The collection of all such regular values β of F is called the resolvent set and is denoted by $\rho(F, Y)$. The complement of $\rho(F, Y)$ in $\mathbb{C}$ is called the spectrum of F and is denoted by $\sigma_s(F, Y)$.

Spectrum Classification: We can partition the spectrum $\sigma_s(F, Y)$ into three disjoint sets as follows:
(i) Point spectrum, denoted by $\sigma_{ps}(F, Y)$: A complex number β belongs to $\sigma_{ps}(F, Y)$ if F_β^{-1} does not exist. A member of $\sigma_{ps}(F, Y)$ is called an eigenvalue of F.

(ii) Continuous spectrum, denoted by $\sigma_{cs}(F, Y)$: A complex number β belongs to $\sigma_{cs}(F, Y)$ if F_β^{-1} exists and satisfies (R_3) but not (R_2).

(iii) Residual spectrum, denoted by $\sigma_{rs}(F, Y)$: A complex number β belongs to $\sigma_{rs}(F, Y)$ if F_β^{-1} exists (whether bounded or not) but does not satisfy (R_3).

It is to be noted that in the finite-dimensional case, $\sigma_{cs}(F, Y) = \sigma_{rs}(F, Y) = \varnothing$, and the spectrum consists only of the eigenvalues of F.

Again, we can give Goldberg's classification [13] of the spectrum as follows. There are three possibilities for the range of F (denoted by $R(F)$), namely: (I) $R(F) = Y$,
(II) $R(F) \neq \overline{R(F)} = Y$,
(III) $\overline{R(F)} \neq Y$.

There are also three possibilities for the inverse of F: (1) F^{-1} exists and is continuous,
(2) F^{-1} exists but is discontinuous,
(3) F^{-1} does not exist.

Combining all the possible cases we get nine different states, labelled by $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3$. For example, an operator is in the state III_2 if $\overline{R(F)} \neq Y$ and F^{-1} exists but is discontinuous.

Further, we can classify the spectrum according to Appell et al. [8] in the following way:

A sequence (y_k) in Y is called a Weyl sequence for F if $||y_k|| = 1$ and $||Fy_k|| \to 0$ as $k \to \infty$, where F is a bounded linear operator on the Banach space Y.

(a) Approximate point spectrum, denoted by $\sigma_{aps}(F, Y)$: A complex number β belongs to $\sigma_{aps}(F, Y)$ if there exists a Weyl sequence for F_β.

(b) Defect spectrum, denoted by $\sigma_{\delta s}(F, Y)$: A complex number β belongs to $\sigma_{\delta s}(F, Y)$ if F_β is not surjective.

(c) Compression spectrum, denoted by $\sigma_{cos}(F, Y)$: A complex number β belongs to $\sigma_{cos}(F, Y)$ if $\overline{R(F_\beta)} \neq Y$.

The approximate point spectrum and the defect spectrum (which may not be disjoint) form subdivisions of the spectrum such that

$$\sigma_s(F, Y) = \sigma_{aps}(F, Y) \cup \sigma_{\delta s}(F, Y).$$

Another subdivision of the spectrum can be obtained by using the compression spectrum and the approximate point spectrum (which may not be disjoint):

$$\sigma_s(F, Y) = \sigma_{aps}(F, Y) \cup \sigma_{cos}(F, Y).$$

Clearly,

$$\sigma_{ps}(F, Y) \subseteq \sigma_{aps}(F, Y) \quad \text{and} \quad \sigma_{cos}(F, Y) \subseteq \sigma_{\delta s}(F, Y).$$

Further, notice that from $\sigma_s(F, Y) = \sigma_{ps}(F, Y) \cup \sigma_{cs}(F, Y) \cup \sigma_{rs}(F, Y)$ we have

$$\sigma_{rs}(F, Y) = \sigma_{cos}(F, Y) \setminus \sigma_{ps}(F, Y), \quad \sigma_{cs}(F, Y) = \sigma_s(F, Y) \setminus \big(\sigma_{ps}(F, Y) \cup \sigma_{cos}(F, Y)\big).$$

From the above results we have the following table.

Table 1: Subdivisions of the spectrum of a linear operator

		1	2	3
		F_β^{-1} exists and is bounded	F_β^{-1} exists and is unbounded	F_β^{-1} does not exist
I	$R(F - \beta I) = Y$	$\beta \in \rho(F, Y)$	-	$\beta \in \sigma_{ps}(F, Y)$ $\beta \in \sigma_{aps}(F, Y)$
II	$\overline{R(F - \beta I)} = Y$	$\beta \in \rho(F, Y)$	$\beta \in \sigma_{cs}(F, Y)$ $\beta \in \sigma_{aps}(F, Y)$ $\beta \in \sigma_{\delta s}(F, Y)$	$\beta \in \sigma_{ps}(F, Y)$ $\beta \in \sigma_{aps}(F, Y)$ $\beta \in \sigma_{\delta s}(F, Y)$
III	$\overline{R(F - \beta I)} \neq Y$	$\beta \in \sigma_{rs}(F, Y)$ $\beta \in \sigma_{\delta s}(F, Y)$ $\beta \in \sigma_{cos}(F, Y)$	$\beta \in \sigma_{rs}(F, Y)$ $\beta \in \sigma_{aps}(F, Y)$ $\beta \in \sigma_{\delta s}(F, Y)$ $\beta \in \sigma_{cos}(F, Y)$	$\beta \in \sigma_{ps}(F, Y)$ $\beta \in \sigma_{aps}(F, Y)$ $\beta \in \sigma_{\delta s}(F, Y)$ $\beta \in \sigma_{cos}(F, Y)$

Proposition 2.1 *Appell et al. [[8], Proposition 1.3, p.28] Spectra and subspectra of an operator* $F \in B(Y)$ *and its adjoint* $F^* \in B(Y^*)$ *are related by the following relations:*
(i) $\sigma(F^*, Y^*) = \sigma(F, Y)$.
(ii) $\sigma_{cs}(F^*, Y^*) \subseteq \sigma_{aps}(F, Y)$.
(iii) $\sigma_{aps}(F^*, Y^*) = \sigma_{\delta s}(F, Y)$.
(iv) $\sigma_{\delta s}(F^*, Y^*) = \sigma_{aps}(F, Y)$.
(v) $\sigma_p(F^*, Y^*) = \sigma_{cos}(F, Y)$.
(vi) $\sigma_{cos}(F^*, Y^*) \supseteq \sigma_{ps}(F, Y)$.
(vii) $\sigma(F, Y) = \sigma_{aps}(F, Y) \cup \sigma_{ps}(F^*, Y^*) = \sigma_{ps}(F, Y) \cup \sigma_{aps}(F^*, Y^*)$.

Let G and H be two sequence spaces. An infinite matrix $B = (b_{nk})$ of real or complex numbers, where $n, k \in \mathbb{N} = \{0, 1, 2, \dots\}$, defines a matrix mapping $B : G \to H$ if for every sequence $y = (y_n) \in G$, the sequence $By = \big((By)_n\big)$ is in H, where

$$(By)_n = \sum_{k=0}^{\infty} b_{nk} y_k \quad (n \in \mathbb{N},\ y \in G),$$

provided the right-hand side converges for every $n \in \mathbb{N}$ and $y \in G$. The class of all such matrices is denoted by $(G : H)$.

Let $n \in \mathbb{N}$ be a fixed natural number. Then Tripathy and Esi [20] introduced the following type of difference sequence spaces:

$$Z(\Delta_n) = \{y = (y_k) : (\Delta_n y_k) \in Z\},$$

for $Z = \ell_\infty, c$, and c_0, where $\Delta_n y = (\Delta_n y_k) = (y_k - y_{k+n})$.

In this paper we deal with an infinite lower triangular matrix $D(p, 0, 0, q)$ of the form

$$D(p,0,0,q) = \begin{bmatrix} p & 0 & 0 & 0 & . & . & . \\ 0 & p & 0 & 0 & . & . & . \\ 0 & 0 & p & 0 & . & . & . \\ q & 0 & 0 & p & . & . & . \\ 0 & q & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

where $q \neq 0$.

It is to be noted that if we consider $p = -1$ and $q = 1$, then $D(-1, 0, 0, 1) = \Delta_3$.

To establish our results in this article we use the following results.

Lemma 2.1 *(Kirişci [15], Theorem 3.5) The matrix* $C = (c_{nk})$ *gives rise to a bounded linear operator* $F \in B(h)$ *from* h *to itself if and only if*
(i) $\sum\limits_{n=1}^{\infty} n|c_{nk} - c_{n+1,k}|$ *converges, for each* $k \in \mathbb{N}$,
(ii) $\sup_k \frac{1}{k} \sum\limits_{n=1}^{\infty} n \Big| \sum\limits_{v=1}^{k} (c_{nv} - c_{n+1,v}) \Big| < \infty$,
(iii) $\lim_{n\to\infty} c_{nk} = 0$, *for each* $k \in \mathbb{N}$.

Lemma 2.2 (Goldberg [13], p.59) F^* *is one-to-one if and only if* F *has a dense range.*

Lemma 2.3 (Goldberg [13], p.59) F^* *is onto if and only if* F *has a bounded inverse.*

3. Main Results

Theorem 3.1 *$D(p,0,0,q) : h \to h$ is a bounded linear operator and*

$$\|D(p,0,0,q)\|_{(k,k)} = |p| + 3|q|.$$

Proof: For each k, one can easily prove that $\lim_{n\to\infty} c_{nk} = 0$ and $\sum_{n=1}^{\infty} n|c_{nk} - c_{n+1,k}|$ is convergent.

Also, it is clear that, for each k,

$$\frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(c_{nv} - c_{n+1,v})\right| \leq |p| + \frac{3}{k}|q|,$$

and hence

$$\sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(c_{nv} - c_{n+1,v})\right| \leq |p| < \infty.$$

Now,

$$\begin{aligned}\|D(p,0,0,q)y\|_h &= |py_2 - py_1| + 2|py_3 - py_2| + 3|(qy_1 + py_4) - py_3| \\ &\quad + 4|(qy_2 + py_5) - (qy_1 + py_4)| + 5|(qy_3 + py_6) - (qy_2 + py_5)| + \cdots \\ &\leq |p|\sum_{k=1}^{\infty} k|y_{k+1} - y_k| + |q|\sum_{k=1}^{\infty} k|y_k - y_{k-1}| + 2|q|\sum_{k=1}^{\infty} k|y_k - y_{k-1}| \\ &= (|p| + 3|q|)\|y\|_h,\end{aligned}$$

where we have set $y_0 = 0$.

This implies that

$$\|D(p,0,0,q)\|_{(k,k)} = |p| + 3|q|.$$

Theorem 3.2

$$\sigma_s(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\}.$$

Proof: Suppose $|p - \beta| > |q|$. We prove that $(D(p,0,0,q) - \beta I)^{-1}$ exists and belongs to $(h : h)$, and then show that $(D(p,0,0,q) - \beta I)$ is not invertible for $|p - \beta| \leq |q|$.

Let $\beta \notin \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\}$. Then $\beta \neq p$ (since $q \neq 0$), and hence $(D(p,0,0,q) - \beta I)^{-1}$ exists, because $(D(p,0,0,q) - \beta I)$ is triangular.

Consider $y = (y_n) \in h$ and solve the equation

$$(D(p,0,0,q) - \beta I)y = z$$

for y in terms of z. We obtain:

$$y_1 = \frac{1}{p-\beta}z_1, \quad y_2 = \frac{1}{p-\beta}z_2, \quad y_3 = \frac{1}{p-\beta}z_3,$$

$$y_4 = -\frac{q}{(p-\beta)^2}z_1 + \frac{1}{p-\beta}z_4, \quad y_5 = -\frac{q}{(p-\beta)^2}z_2 + \frac{1}{p-\beta}z_5, \quad y_6 = -\frac{q}{(p-\beta)^2}z_3 + \frac{1}{p-\beta}z_6,$$

$$y_7 = \frac{q^2}{(p-\beta)^3}z_1 - \frac{q}{(p-\beta)^2}z_4 + \frac{1}{p-\beta}z_7,$$

$$y_8 = \frac{q^2}{(p-\beta)^3}z_2 - \frac{q}{(p-\beta)^2}z_5 + \frac{1}{p-\beta}z_8,$$

$$y_9 = \frac{q^2}{(p-\beta)^3}z_3 - \frac{q}{(p-\beta)^2}z_6 + \frac{1}{p-\beta}z_9, \quad \cdots$$

From this pattern, the nth term is

$$y_n = \sum_{i=1}^{n} \frac{(-q)^{\frac{n-i}{3}}}{(p-\beta)^{\frac{n-i}{3}+1}} z_i,$$

where the summand is taken as 0 *whenever* $\frac{n-i}{3} \notin \mathbb{N}$.

Thus,

$$(D(p,0,0,q)-\beta I)^{-1} = (d_{nk}) = \begin{bmatrix} \frac{1}{p-\beta} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{p-\beta} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{p-\beta} & 0 & \cdots \\ \frac{q}{(p-\beta)^2} & 0 & 0 & \frac{1}{p-\beta} & \cdots \\ 0 & \frac{q}{(p-\beta)^2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For each k, $\lim_{n\to\infty} d_{nk} = 0$, *since* $|p-\beta| > |q|$. *Also, for each* k,

$$\sum_{n=1}^{\infty} n|d_{nk} - d_{n+1,k}| \le \frac{1}{|p-\beta|} + \frac{(k+6)|q|}{|p-\beta|^2} + \frac{(k+12)|q|^2}{|p-\beta|^3} + \cdots .$$

This reduces to two convergent series, because $|p-\beta| > |q|$, *namely*

$$1 + \frac{|q|}{|p-\beta|} + \frac{|q|^2}{|p-\beta|^2} + \cdots = \frac{1}{1-\frac{|q|}{|p-\beta|}},$$

and

$$\frac{|q|}{|p-\beta|} + \frac{2|q|^2}{|p-\beta|^2} + \frac{3|q|^3}{|p-\beta|^3} + \cdots = \frac{\frac{|q|}{|p-\beta|}}{\left(1-\frac{|q|}{|p-\beta|}\right)^2}.$$

Hence $\sum_{n=1}^{\infty} n|d_{nk} - d_{n+1,k}|$ *converges for each* k. *Moreover,*

$$\frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(d_{nv} - d_{n+1,v})\right| \le \frac{1}{|p-\beta|}\left(1 + \frac{3|q|}{k|p-\beta|}\right),$$

so

$$\sup_k \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(d_{nv} - d_{n+1,v})\right| < \infty.$$

Therefore, by Lemma 2.1, $(D(p,0,0,q)-\beta I)^{-1} \in (h:h)$, *which implies* $\beta \notin \sigma_s(D(p,0,0,q),h)$ *whenever* $|p-\beta| > |q|$. *Thus,*

$$\sigma_s(D(p,0,0,q),h) \subseteq \{\gamma \in \mathbb{C} : |\gamma - p| \le |q|\}.$$

Conversely, let $\beta \in \{\gamma \in \mathbb{C} : |\gamma - p| \le |q|\}$.

- *If* $\beta \ne p$: *taking* $z = (1,0,0,0,\dots) \in h$ *and solving* $(D(p,0,0,q)-\beta I)y = z$ *gives* $y_{3n} = \frac{(-q)^n}{(p-\beta)^{n+1}}$ *and* $y_{3n+k} = 0$ *for* $k = 1,2$. *Then* $y = (y_n) \notin h$, *so the inverse is not bounded.*
- *If* $\beta = p$: *then*

$$D(p,0,0,q)-\beta I = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ q & 0 & 0 & 0 & \cdots \\ 0 & q & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = D(0,0,0,q),$$

and since $\overline{R(D(0,0,0,q))} \neq h$*, it is not invertible.*

Thus in both cases,

$$\{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\} \subseteq \sigma_s(D(p,0,0,q),h).$$

Hence,

$$\sigma_s(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\}.$$

Theorem 3.3 $\sigma_{ps}(D(p,0,0,q),h) = \varnothing$.
Proof: Let $\beta \in \sigma_{ps}(D(p,0,0,q),h)$*. Then, by definition there exists* $y \neq \theta = (0,0,0,\dots)$ *in* h *such that* $D(p,0,0,q)y = \beta y$.

On solving the system of linear equations we get

$$py_1 = \beta y_1, \quad py_2 = \beta y_2, \quad py_3 = \beta y_3, \quad qy_1 + py_4 = \beta y_4, \quad qy_2 + py_5 = \beta y_5, \quad \cdots$$

in general,

$$qy_k + py_{k+3} = \beta y_{k+3}, \quad (k \geq 1).$$

Suppose $y_{n_0} \neq 0$ *is the first non-zero element of the sequence* $y = (y_n)$*. Then* $\beta = p$*. Again, from the equation* $qy_{n_0} + py_{n_0+3} = \beta y_{n_0+3}$ *we have* $qy_{n_0} = 0$*. Since* $q \neq 0$ *this implies* $y_{n_0} = 0$*, a contradiction. Hence* $\sigma_{ps}(D(p,0,0,q),h) = \varnothing$.

Theorem 3.4 $\sigma_{ps}(D(p,0,0,q)^*,h^*) = \{\gamma \in \mathbb{C} : |\gamma - p| < |q|\}$.
Proof: Let $\beta \in \sigma_{ps}(D(p,0,0,q)^*,h^*)$*. Then by definition there exists* $y \neq \theta = (0,0,0,\dots)$ *in* h^* *such that* $D(p,0,0,q)^*y = \beta y$.

On solving the system of equations we obtain

$$py_1 + qy_4 = \beta y_1, \quad py_2 + qy_5 = \beta y_2, \quad py_3 + qy_6 = \beta y_3, \quad \cdots$$

in general,

$$py_k + qy_{k+3} = \beta y_k, \quad (k \geq 1).$$

From this we deduce

$$y_{3n+k} = \left(\frac{\beta - p}{q}\right)^n y_k, \quad n \geq 1, \ k = 1,2,3.$$

One can verify that

$$\sup_n \frac{1}{n}\Big|\sum_{k=1}^{n} y_k\Big| < \infty$$

if and only if $|\beta - p| < |q|$*. This completes the proof.*

Theorem 3.5 $\sigma_{rs}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| < |q|\}$.
Proof: We know that

$$\sigma_{rs}(F,Y) = \sigma_{cos}(F,Y) \setminus \sigma_{ps}(F,Y).$$

Moreover, from Proposition 2.1, we have $\sigma_{ps}(F^*,Y^*) = \sigma_{cos}(F,Y)$*. Therefore, using this fact together with Theorems 3.3 and 3.4, we obtain the required result.*

Theorem 3.6 $\sigma_{cs}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| = |q|\}$.
Proof: The spectrum of an operator is the disjoint union of the point spectrum, residual spectrum, and continuous spectrum. The result follows immediately by combining this fact with Theorems 3.2, 3.3, and 3.5.

Theorem 3.7 *If* $\beta = p$*, then* $\beta \in III_1\sigma(D(p,0,0,q),h)$.
Proof: When $\beta = p$*, the operator* $(D(p,0,0,q) - \beta I)$ *is equivalent to the operator* $D(0,0,0,q)$*, and the range of* $D(0,0,0,q)$ *is not dense in* h*. Moreover, when* $\beta = p$*, we have* $\beta \in \sigma_{rs}(D(p,0,0,q),h)$.

From Table 1, we obtain

$$\beta \in III_1\sigma_s(D(p,0,0,q),h) \cup III_2\sigma_s(D(p,0,0,q),h).$$

It is easy to check that $D^*(0,0,0,q) : \sigma_\infty \to \sigma_\infty$ *is onto. Hence, we conclude by using Lemma 2.3 that* $D(0,0,0,q) : h \to h$ *has a bounded inverse.*

Theorem 3.8

$$\sigma_{aps}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\} \setminus \{p\}.$$

Proof: Since

$$\sigma_{aps}(D(p,0,0,q),h) = \sigma_s(D(p,0,0,q),h) \setminus III_1\sigma_s(D(p,0,0,q),h),$$

the result

$$\sigma_{aps}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\} \setminus \{p\}$$

follows by using Theorems 3.2 and 3.7.

Theorem 3.9

$$\sigma_{\delta s}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\}.$$

Proof: From Table 1, we have

$$\sigma_{\delta s}(D(p,0,0,q),h) = \sigma_s(D(p,0,0,q),h) \setminus I_3\sigma_s(D(p,0,0,q),h).$$

Now,

$$I_3\sigma_s(D(p,0,0,q),h) \setminus \big(II_3\sigma_s(D(p,0,0,q),h) \cup III_3\sigma_s(D(p,0,0,q),h)\big) = \sigma_{ps}(D(p,0,0,q),h) = \varnothing,$$

by Theorem 3.3. Hence,

$$\sigma_{\delta s}(D(p,0,0,q),h) = \sigma_s(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| \leq |q|\}.$$

Theorem 3.10

$$\sigma_{cos}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| < |q|\}.$$

Proof: From Table 1, we have

$$\sigma_{cos}(D(p,0,0,q),h) = III_1\sigma_s(D(p,0,0,q),h) \cup III_2\sigma_s(D(p,0,0,q),h) \cup III_3\sigma_s(D(p,0,0,q),h).$$

Now,

$$III_1\sigma_s(D(p,0,0,q),h) \cup III_2\sigma_s(D(p,0,0,q),h) = \sigma_{rs}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| < |q|\},$$

by Theorem 3.5. Also,

$$III_3\sigma_s(D(p,0,0,q),h) = \sigma_{ps}(D(p,0,0,q),h) = \varnothing,$$

by Theorem 3.3.

Hence,

$$\sigma_{cos}(D(p,0,0,q),h) = \{\gamma \in \mathbb{C} : |\gamma - p| < |q|\}.$$

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Avinoy Paul,
Department of Mathematics,
Cachar COllege, Silchar-788001, Assam,
India.
E-mail address: avinoypaul@gmail.com