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Stability result for a system of nonlinear $\mathcal{K}$-wave equations with damping and source terms

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ABSTRACT: In this paper, we consider a system of nonlinear $\mathcal{K}$ −wave equations ($\mathcal{K} \geq 2$) with damping acting in all equations and source terms. We will prove that the solution of the problem is stable for some conditions with a small positive initial energy, by using the integral inequality due to Komornik.

Key Words: Wave equation, source term, global existence, stability solution.

Contents

1. Introduction

In this work, we consider the following system of nonlinear $\mathcal{K}$ wave equations:

$$\begin{cases} u_{1,tt} - \Delta u_1 + |u_{1,t}|^{m-2} u_{1,t} = f_1(u_1, ..., u_{\mathcal{K}}), \\ \quad\quad ... \\ \quad\quad ... \\ u_{\mathcal{K},tt} - \Delta u_{\mathcal{K}} + |u_{\mathcal{K},t}|^{m-2} u_{\mathcal{K},t} = f_{\mathcal{K}}(u_1, ..., u_{\mathcal{K}}), \end{cases} \tag{1.1}$$

where $m > 2$, $(x,\ t) \in \Omega \times (0,\ T)$ and Ω is a bounded domain with smooth boundary $\partial\Omega$ in $R^n\ (n \geq 1)$, and $\mathcal{K}$ functions $f_j(u_1, ..., u_{\mathcal{K}})$, for $j = 1$ to $\mathcal{K}$ are given by:

$$\begin{aligned} &f_1(u_1, ..., u_{\mathcal{K}}) = a|u_1 + u_2|^{2(\rho+1)}(u_1 + u_2) + b|u_1|^{\rho} u_1 |u_2|^{\rho+2}. \\ &f_{\mathcal{K}}(u_1, ..., u_{\mathcal{K}}) = a|u_{\mathcal{K}-1} + u_{\mathcal{K}}|^{2(\rho+1)}(u_{\mathcal{K}-1} + u_{\mathcal{K}}) + b|u_{\mathcal{K}}|^{\rho} u_{\mathcal{K}} |u_{\mathcal{K}-1}|^{\rho+2} \\ &f_j(u_1, ..., u_{\mathcal{K}}) = a|u_{j-1} + u_j|^{2(\rho+1)}(u_{j-1} + u_j) + b|u_j|^{\rho} u_j |u_{j-1}|^{\rho+2} \\ &+a|u_j + u_{j+1}|^{2(\rho+1)}(u_j + u_{j+1}) + b|u_j|^{\rho} u_j |u_{j+1}|^{\rho+2}, \text{ for } j = 2, ..., _{\mathcal{K}-1}. \end{aligned}$$

The system (1.1) is supplemented with the following initial conditions:

$$(u_1(0), ..., u_{\mathcal{K}}(0)) = (u_{1,0}, ..., u_{\mathcal{K},0}),\ \ (u_{1,t}(0), ..., u_{\mathcal{K},t}(0)) = (u_{1,1}, ..., u_{\mathcal{K},1}),\ x \in \Omega \tag{1.2}$$

and boundary conditions

$$u_1(x) = u_2(x) = ... = u_{\mathcal{K}}(x) = 0,\ x \in \partial\Omega. \tag{1.3}$$

Some special case of the single wave equation with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{p-1} u_t = b|u|^{q-1} u, \tag{1.4}$$

with the presence of different mechanisms of dissipation, damping and for more general forms of non-linearities has been extensively studied and results concerning existence, nonexistence and asymptotic

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behavior of solutions have been established by several authors and many results appeared in the literature over the past decades, see ([1], [5]- [8], [10], [12], [14], [16], [19]). Said-Houari in [18] considered the following nonlinear system

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1} u_t = f_1(u, v), \\ v_{tt} - \Delta v + |v_t|^{r-1} v_t = f_2(u, v). \end{cases} \quad (1.5)$$

He proved that the solution of system (1.5) blows up in finite time with the initial data are large enough. Ouaoua and Maouni in [13] considered the following coupled nonlinear Klein-Gordon equations with degenerate damping and source terms

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + \left(|u|^k + |v|^l\right) |u_t|^{p-1} u_t = f_1(u, v), \\ v_{tt} - \Delta v + m_2^2 v + \left(|v|^\theta + |u|^\varrho\right) |v_t|^{q-1} v_t = f_2(u, v), \end{cases} \quad (1.6)$$

and they proved that the positive initial-energy solution grows exponentially. The absence of the terms $m_1^2 u$ and $m_2^2 u$, equations (1.6) take the form

$$\begin{cases} u_{tt} - \Delta u + \left(|u|^k + |v|^l\right) |u_t|^{p-1} u_t = f_1(u, v), \\ v_{tt} - \Delta v + \left(|v|^\theta + |u|^\varrho\right) |v_t|^{q-1} v_t = f_2(u, v). \end{cases} \quad (1.7)$$

In [17] Rammaha and Sakuntasathien focus on the global well-posedness of the system of nonlinear wave equation (1.7). Wu in [20] studied blow up of solutions of the system (1.7) for $n = 3$ and $k = l = \theta = \varrho = 0$. Agre and Rammaha [3] studied the global existence and the blow up of the solution of problem (1.7) when $k = l = \theta = \varrho$, and also Alves et al [4], investigated the existence, uniform decay rates and blow up of the solution. In [15] Erhen Pişkin proved the blow up of solutions of (1.7) in finite time with negative initial energy and nondegenerate damping terms.

In the work [11], Messaoudi and Said-Houari considered the following nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(x,s)\, ds + |u_t|^{p-1} u_t = f_1(u, v), \\ v_{tt} - \Delta v + \int_0^t h(t-s) \Delta v(x,s)\, ds + |v_t|^{q-1} v_t = f_2(u, v), \end{cases} \quad (1.8)$$

and they proved a global nonexistence for certain solutions with positive initial energy, the main tool proof is a method used by vitillaro [19] and developed in [18].

Our objective in this paper is to study: In section 2, some notations, assumptions and preliminaries are introduced, section 3, the global existence of solution and the stability results of this article are proved.

2. Preliminaries

In this section, we shall give some Lemmas which will be used throughout in this work.

Lemma 2.1 (*Young's inequality*) *Let $a, b \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ for $0 < p, q < +\infty$, then one has the inequatity*

$$ab \leq \delta a^p + c(\delta) b^q,$$

where $\delta > 0$ is an constant, and $c(\delta)$ is a positive constant depending on δ.

Lemma 2.2 (*Sobolev-Poincare inequality [2]*). *Let p be a number with $2 \leq p < \infty$ $(n = 1, 2)$ or $2 \leq p \leq \frac{2n}{n-2}$ $(n \geq 3)$, then there is a constant $C_* = C_*(p, \Omega)$ such that*

$$\|u\|_p \leq C_* \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\Omega).$$

Lemma 2.3 *[9]Let $G:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be a non-increasing function and assume that there are two constants $\alpha>0$ and $C>0$ such that*

$$\int_t^\infty G^{\alpha+1}(s)\,ds\leq CG^\alpha(0)\,G(s)\,,\quad \forall t\in\mathbb{R}_+.$$

Then we have

$$G(t)\leq G(0)\left(\frac{C+\alpha t}{C+\alpha C}\right)^{\frac{-1}{\alpha}},\quad \forall\ t\geq C.$$

It is not hard to prove the following corollary by recurrence with the fact

$$\left(\sum_{j=1}^{\mathcal{K}}a_j\right)^\gamma\leq 2^{\gamma-1}\sum_{j=1}^{\mathcal{K}}a_j^\gamma,$$

and using the embedding theorem of $L^m(\Omega)\hookrightarrow L^2(\Omega)$.

Corollary 2.1 *For any m real number such that $m>2$. Then, we have*

$$\sum_{j=1}^{\mathcal{K}}\|u_{j,t}\|_2^2\leq c\left(\sum_{j=1}^{\mathcal{K}}\|u_{j,t}\|_m^m\right)^{\frac{2}{m}}.$$

Definition 2.1 A $\mathcal{K}$ of functions $(u_1,...,u_{\mathcal{K}})$ is said to be a weak solution of (1.1) on $[0,\ T]$ if $u_1,...,u_{\mathcal{K}}\in C_w\left([0,\ T],\ H_0^1(\Omega)\right),\ u_{1,t},...,u_{\mathcal{K},t}\in C_w\left([0,\ T],\ L^2(\Omega)\right),\ u_{i,t}\in L^m(\Omega\times(0,\ T))$ for $i=1$ to $\mathcal{K}$, $(u_1(0),...,u_{\mathcal{K}}(0))=(u_{1,0},...,u_{\mathcal{K},0})\in\left(H_0^1(\Omega)\right)^{\mathcal{K}},(u_{1,t}(0),...,u_{\mathcal{K},t}(0))=(u_{1,1},...,u_{\mathcal{K},1})\in\left(L^2(\Omega)\right)^{\mathcal{K}}$ and $(u_1,...,u_{\mathcal{K}})$ satisfies

$$\int_\Omega u_{i,t}(t)\,\phi_1dx-\int_\Omega u_{i,1}(t)\,\phi_idx+\int_\Omega\nabla u_i\nabla\phi_idx\ +\int_0^t\int_\Omega|u_{i,t}(t)|^m\,u_{i,t}(t)\,\phi_idxds$$
$$=\int_0^t\int_\Omega f_i(u_1,...,u_{\mathcal{K}})\,\phi_idxds\ \text{ for } i=1\ to\ \mathcal{K},$$

for all fonctions $\phi_1,...,\ \phi_{\mathcal{K}}\in H_0^1(\Omega)\times L^m(\Omega)$ and for almost all $t\in[0,\ T]$.

We assume that

$$\begin{cases}\rho>-1 \text{ if } n=1,2\\ -1<\rho\leq\frac{4-n}{n-2} \text{ if } n\geq3.\end{cases}\tag{2.1}$$

We can easily verify that

$$u_1f_1(u_1,...,u_{\mathcal{K}})+...+u_{\mathcal{K}}f_{\mathcal{K}}(u_1,...,u_{\mathcal{K}})=2(\rho+2)\,F(u_1,...,u_{\mathcal{K}})\,,\ \forall(u_1,...,u_{\mathcal{K}})\in\mathbb{R}^{\mathcal{K}},\tag{2.2}$$

where

$$F(u_1,...,u_{\mathcal{K}})=\frac{1}{2(\rho+2)}\left(a\sum_{j=1}^{\mathcal{K}\text{-}1}|u_j+u_{j+1}|^{2(\rho+2)}+2b\sum_{j=1}^{\mathcal{K}\text{-}1}|u_ju_{j+1}|^{\rho+2}\right).$$

Lemma 2.4 *There exist two positive constants c_1 and c_2 such that*

$$c_1\sum_{j=1}^{\mathcal{K}}|u_j|^{2(\rho+2)}\leq 2(r+2)\,F(u_1,...,u_{\mathcal{K}})\leq c_2\sum_{j=1}^{\mathcal{K}}|u_j|^{2(\rho+2)},\tag{2.3}$$

is satisfied.

Remark 2.1 The previous Lemma is a generalization of the Lemma 2.1 in [11].

3. Global existence and stability of solution

In the order to state and prove our result, we define the following energy function associated with a solution $(u_1, ..., u_{\mathcal{K}})$ of problem (1.1)-(1.3)

$$E(t) = \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - \int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx, \tag{3.1}$$

and

$$I(t) = \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - 2(\rho+2)\int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx. \tag{3.2}$$

Lemma 3.1 *Suppose that (2.1), let $(u_1, ..., u_{\mathcal{K}})$ be the solution of the system (1.1)-(1.3), then the energy functional is a decreasing function, that is*

$$E'(t) = -\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_m^m \leq 0, \quad \forall t \geq 0,$$

and

$$E(t) \leq E(0).$$

Proof: *We multiply the equations of (1.1) by $u_{j,t}$ for $j = 1$ to $\mathcal{K}$ respectively, and integrating over the domain Ω, using integration by parts, and summing, we get*

$$\frac{d}{dt}\left(\frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - \int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx\right)$$
$$= -\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_m^m,$$

then

$$\frac{d}{dt}E(t) = -\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_m^m \leq 0. \tag{3.3}$$

Integrating (3.3) over $(0,\ t)$, we obtain

$$E(t) \leq E(0).$$

□

Lemma 3.2 *Let $(u_1, ..., u_{\mathcal{K}})$ be a solution of (1.1)-(1.3), assume that $E(0) > 0,\ \ I(0) > 0$ and*

$$c_2 c_*^{2(\rho+2)}\left(\frac{\rho+2}{\rho+1}E(0)\right)^{\rho+1} = \theta < 1, \tag{3.4}$$

with c_* is the best embedding constant of $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$. Then $I(t) > 0$, for all $t \in [0,\ T]$.

Proof: By continuity, there exists T_*, such that

$$I(t) \geq 0, \quad \text{for all } t \in [0,\ T_*]. \tag{3.5}$$

Now, we have for all $t \in [0,\ T_*]$:

$$\begin{aligned} E(t) &= \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - \int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx \\ &\geq \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - \frac{1}{2(\rho+2)}\left(\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - I(t)\right) \\ &\geq \frac{\rho+1}{\rho+2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 + \frac{1}{2(\rho+2)} I(t) \end{aligned}$$

using (3.5), we obtain

$$\frac{\rho+1}{\rho+2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 \leq E(t), \qquad \text{for all } t \in [0,\ T_*]. \tag{3.6}$$

Then

$$\frac{\rho+1}{\rho+2} \|\nabla u_j(t)\|_2^2 \leq E(t), \qquad \text{for } j = 1, ..., \mathcal{K}, \text{ and for all } t \in [0,\ T_*]. \tag{3.7}$$

By the definition of E, we get

$$\|\nabla u_j(t)\|_2^2 \leq \frac{\rho+2}{\rho+1} E(t) \leq \frac{\rho+2}{\rho+1} E(0), \text{ for } j = 1, ..., \mathcal{K}, \text{ and for all } t \in [0,\ T_*]. \tag{3.8}$$

Thank the Lemma 2.4 and the embedding of $H_0^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$, we have

$$\begin{aligned} 2(\rho+2)\int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx &\leq c_2 c_*^{2(\rho+2)} \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^{2(\rho+2)} \\ &\leq c_2 c_*^{2(\rho+2)} \sum_{j=1}^{\mathcal{K}} \left(\|\nabla u_j(t)\|_2^2\right)^{\rho+1} \|\nabla u_j(t)\|_2^2 \\ &\leq c_2 c_*^{2(\rho+2)} \sum_{j=1}^{\mathcal{K}} \left(\frac{\rho+2}{\rho+1} E(0)\right)^{\rho+1} \|\nabla u_j(t)\|_2^2 \\ &\leq \theta \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2. \end{aligned}$$

Then, we get

$$2(\rho+2)\int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx \leq \theta \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2, \qquad \text{for all } t \in [0,\ T_*]. \tag{3.9}$$

Since $\theta < 1$, then

$$2(\rho+2)\int_\Omega F(u_1, ..., u_{\mathcal{K}})\, dx \leq \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2, \qquad \text{for all } t \in [0,\ T_*]. \tag{3.10}$$

This implies that

$$I(t) > 0, \qquad \text{for all } t \in [0,\ T_*].$$

By repeating the above procedure, we can extend T_* to T. □

Now, we state our main result:

Theorem 3.1 *Under the assumptions of Lemma 3.2, the local solution of (1.1)-(1.3) is global.*

Proof: We have by (3.10)

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 - \int_\Omega F(u_1,...,u_{\mathcal{K}})\,dx \\ &\geq \frac{1}{2}\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \frac{\rho+1}{\rho+2}\sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 . \end{aligned}$$

So that

$$\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 \leq C\ E(t). \tag{3.11}$$

Using the Lemma 3.1, we obtain

$$\sum_{j=1}^{\mathcal{K}} \|u_{j,t}(t)\|_2^2 + \sum_{j=1}^{\mathcal{K}} \|\nabla u_j(t)\|_2^2 \leq C\ E(0), \tag{3.12}$$

where C is a constant depending only of ρ.
This implies that the local solution is global in time. □

Lemma 3.3 *Suppose that the assumptions of Lemma 3.2 hold, then there exists a positive constant c such that*

$$\int_\Omega |u_j(t)|^m\,dx \leq cE(t), \quad \textit{for } j=1,...,\mathcal{K}.$$

Proof: We have

$$\begin{aligned} \int_\Omega |u_j(t)|^m\,dx &\leq c_*^m\ \|\nabla u_j(t)\|_2^m \\ &\leq c_*^m\,\|\nabla u_j(t)\|_2^{m-2} \times \|\nabla u_j(t)\|_2^2 . \end{aligned}$$

By using (3.8), we obtain

$$\int_\Omega |u_j(t)|^m\,dx \leq cE(t), \text{ for } j=1,...,\mathcal{K}.$$

□

Theorem 3.2 *Let the assumptions of Theorem 3.1, then there exists the positive constant $C>0$, such that*

$$E(t) \leq C(1+t)^{-\frac{2}{m-2}}, \quad \textit{for all } t\geq 0.$$

Proof: *Multiplying each equation of (1.1) by $u_j(t)\,E^{\frac{m-2}{2}}(t)$, for $j=1,...,\mathcal{K}$ respectively, integrating over $\Omega\times(S,\ T)$ $(S<T)$, and summing with respect to j, we obtain*

$$\begin{aligned} &\int_S^T\int_\Omega \sum_{j=1}^{\mathcal{K}} u_j(t)\,E^{\frac{m-2}{2}}(t)\left[u_{j,tt}(t) - \Delta u_j(t) + |u_{k,t}(t)|^{m-2}\,u_{k,t}(t)\right]dxdt \\ &= \int_S^T\int_\Omega E^{\frac{m-2}{2}}(t)\sum_{j=1}^{\mathcal{K}} u_j(t)\,f_j(u_1,...,u_{\mathcal{K}})\,dxdt. \end{aligned}$$

So that

$$\int_S^T\int_\Omega\sum_{j=1}^{\mathcal{K}}E^{\frac{m-2}{2}}(t)\left[u_ju_{j,tt}(t)+|\nabla u_j(t)|^2+|u_{k,t}(t)|^m\right]dxdt$$
$$=\int_S^T\int_\Omega E^{\frac{m-2}{2}}(t)\sum_{j=1}^{\mathcal{K}}u_j(t)f_j(u_1,...,u_{\mathcal{K}})\,dxdt.$$

We add and substract the terms

$$\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\left(\theta\sum_{j=1}^{\mathcal{K}}|\nabla u_j(t)|^2+(1+\theta)\sum_{j=1}^{\mathcal{K}}|u_{k,t}(t)|^2\right)dxdt$$

and use (3.9), *to get*

$$(1-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\left(\sum_{j=1}^{\mathcal{K}}|\nabla u_j(t)|^2+\sum_{j=1}^{\mathcal{K}}|u_{k,t}(t)|^2\right)dxdt$$
$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}(u_ju_{j,t})_t\,dxdt-(2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}|u_{k,t}|^2\,dxdt$$
$$+\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}u_j(t)|u_{j,t}|^{m-2}u_{j,t}(t)\,dxdt$$
$$=-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\left(\theta\sum_{j=1}^{\mathcal{K}}|\nabla u_j(t)|^2-2(\rho+2)F(u_1,...,u_{\mathcal{K}})\right)dt\leq 0. \tag{3.13}$$

Then

$$(1-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\left(\frac{1}{2}\sum_{j=1}^{\mathcal{K}}|\nabla u_j|^2+\frac{1}{2}\sum_{j=1}^{\mathcal{K}}|u_{k,t}|^2-F(u_1,...,u_{\mathcal{K}})\right)dxdt$$
$$\leq-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}(u_ju_{j,t})_t\,dxdt-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}u_j|u_{j,t}|^{m-2}u_{j,t}dxdt$$
$$+(2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}|u_{k,t}|^2\,dxdt. \tag{3.14}$$

Using the definition of $E(t)$ *and the following relation*

$$\frac{d}{dt}\left(E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}u_j(t)u_{j,t}(t)\,dx\right)=E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}(u_j(t)u_{j,t}(t))_t\,dx$$
$$+\frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}u_j(t)u_{j,t}(t)\,dxdt.$$

Inequality (3.14) *becomes*

$$\begin{aligned}(1-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\,dt \leq & -\int_S^T \frac{d}{dt}\left(E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(t)\,u_{j,t}(t)\,dx\right)dt \\ & -\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j\left|u_{j,t}\right|^{m-2}u_{j,t}dxdt \\ & -\frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\,\frac{d}{dt}E(t)\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(t)\,u_{j,t}(t)\,dxdt \\ & +(2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}\left|u_{k,t}\right|^2 dxdt. \qquad (3.15)\end{aligned}$$

We estimate the terms on the right-hand side of (3.15) *as follows:*
For the first term, we have

$$\begin{aligned}& -\int_S^T \frac{d}{dt}\left(E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(t)\,u_{j,t}(t)\,dx\right)dt \\ & \leq \left|E^{\frac{m-2}{2}}(t)\int_\Omega\left(\sum_{j=1}^{\mathcal{K}} u_j(S)\,u_{j,t}(S)\,dx - E^{\frac{m-2}{2}}(t)\sum_{j=1}^{\mathcal{K}} u_j(T)\,u_{j,t}(T)\right)dx\right| \qquad (3.16)\end{aligned}$$

By the Young, Poincaré inequalities,(3.11) *and Lemma 3.1, we obtain*

$$\begin{aligned}& -\int_S^T \frac{d}{dt}\left(E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(t)\,u_{j,t}(t)\,dx\right)dt \\ & \leq E^{\frac{m-2}{2}}(t)\left|\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(S)\,u_{j,t}(S)\,dx\right| + E^{\frac{m-2}{2}}(t)\left|\int_\Omega\sum_{j=1}^{\mathcal{K}} u_j(T)\,u_{j,t}(T)\,dx\right| \\ & \leq \frac{1}{2}E^{\frac{m-2}{2}}(t)\int_\Omega\sum_{j=1}^{\mathcal{K}}\left(\left|\nabla u_j(S)\right|^2+\left|u_{j,t}(S)\right|^2\right)dx + \int_\Omega\sum_{j=1}^{\mathcal{K}}\left(\left|\nabla u_j(T)\right|^2+\left|u_{j,t}(T)\right|^2\right)dx \\ & \leq cE^{\frac{m}{2}}(S)+cE^{\frac{m}{2}}(T)\leq cE^{\frac{m-2}{2}}(0)\,E(S) \\ & \leq cE(S)\,. \qquad (3.17)\end{aligned}$$

For the second term of (3.15)*, we use the following Young inequality:*

$$XY \leq \frac{\varepsilon}{\lambda_1}X^{\lambda_1}+\frac{1}{\lambda_2\varepsilon^{\frac{\lambda_2}{\lambda_1}}}Y^{\lambda_2},\ X,\ Y\geq 0,\ \varepsilon>0 \text{ and } \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1.$$

with $\lambda_1 = m,\ \lambda_2 = \frac{m}{m-1}$.

By Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}
&-\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \sum_{j=1}^{\mathcal{K}} u_j(t)\left|u_{j,t}(t)\right|^{m-2} u_{j,t}(t)\,dxdt \\
&\le \int_S^T E^{\frac{m-2}{2}}(t)\left(\frac{\varepsilon}{\lambda_1}\sum_{j=1}^{\mathcal{K}}\|u_j(t)\|_m^m + \frac{1}{\lambda_2 \varepsilon^{\frac{\lambda_2}{\lambda_1}}}\sum_{j=1}^{\mathcal{K}}\|u_{j,t}(t)\|_m^m\right) dt \\
&\le \varepsilon c\int_S^T E^{\frac{m-2}{2}}(t)\int_\Omega \sum_{j=1}^{\mathcal{K}}|u_j(t)|^m\,dxdt + c_\varepsilon \int_S^T E^{\frac{m-2}{2}}(t)\left(-E'(t)\right)dt \\
&\le \varepsilon c\int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon E(S).
\end{aligned} \tag{3.18}$$

For the next term, by Young's, Poincare's inequalities, (3.11) and Lemma 3.1, we obtain

$$\begin{aligned}
&-\frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\frac{d}{dt}E(t)\int_\Omega \sum_{j=1}^{\mathcal{K}} u_j(t)\,u_{j,t}(t)\,dxdt \\
&\le \frac{m-2}{2}\int_S^T E^{\frac{m-2}{2}-1}(t)\left(-E'(t)\right)\int_\Omega\left(\frac{1}{2}\sum_{j=1}^{\mathcal{K}}|u_j(t)|^2 + \frac{1}{2}\sum_{j=1}^{\mathcal{K}}|u_{j,t}(t)|^2\right)dxdt \\
&\le c\int_S^T E^{\frac{m-2}{2}}(t)\left(-E'(t)\right)dt \\
&\le cE^{\frac{m}{2}}(S) - E^{\frac{m}{2}}(T) \\
&\le cE^{\frac{m}{2}-1}(0)\,E(S) \le cE(S).
\end{aligned} \tag{3.19}$$

For the last term of (3.15), by the Corollary 2.1, we have

$$\begin{aligned}
(2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\sum_{j=1}^{\mathcal{K}}\|u_{j,t}(t)\|_2^2\,dt &\le (2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\left(\sum_{j=1}^{\mathcal{K}}\|u_{j,t}(t)\|_m^m\right)^{\frac{2}{m}} dt \\
&\le (2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m}} dt.
\end{aligned}$$

We use the Young's inequality, with $\lambda_1 = \frac{m}{m-2}$, $\lambda_2 = \frac{m}{2}$, we obtain

$$\int_S^T E^{\frac{m-2}{2}}(t)\left(-E'(t)\right)^{\frac{2}{m}} dt \le \varepsilon c\int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon\int_S^T\left(-E'(t)\right)dt.$$

This implies

$$(2-\theta)\int_S^T E^{\frac{m-2}{2}}(t)\sum_{j=1}^{\mathcal{K}}\|u_{j,t}(t)\|_2^2\,dt \le \varepsilon c\int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon E(S). \tag{3.20}$$

By insert (3.17), (3.18), (3.19) and (3.20) in (3.15), we arrive at

$$\gamma\int_S^T E^{\frac{m}{2}}(t)\,dt \le \varepsilon c\int_S^T E^{\frac{m}{2}}(t)\,dt + c_\varepsilon E(S).$$

Choosing ε small enough for that

$$\int_S^T E^{\frac{m}{2}}(t)\,dt \leq cE(S).$$

By taking T goes to ∞, we get

$$\int_S^\infty E^{\frac{m}{2}}(t)\,dt \leq cE(S).$$

By Komornik's integral inequality 2.3, we obtain the result. □

Remark 3.1 The results that we reached in this work in the case when $\mathcal{K}=2$ are identical to the work of Agre and Rammaha in [3]. The condition $4^{\frac{p}{2}}c_0E(0)^{\frac{p-1}{2}}<1$ which was used as a main to prove the global existence of the solution.

4. Numerical example

In this section, we present an application to illustrate numerically the stability result of Theorem 3.2. For this purpose, we numerically solve problem (1.1), for $\mathcal{K}=2$, $m=6$, $\rho=2$ and $n=2$ where the domain is taken to be $\Omega=[-1,\ 1]^2$. We chosen $u_0(x_1,x_2)=3(x_1+1)(x_1-1)+2(x_2+1)(x_2-1)$, $u_1(x_1,x_2)=0$ $v_0(x_1,x_2)=-2(x_1+1)(x_1-1)+5(x_2+1)(x_2-1)$ and $v_1(x_1,x_2)=0$, where will be chosen such that $E(0)>0$ and small enough for that $c_2c_*^8\left(\frac{4}{3}E(0)\right)^3<1$.

4.1. Numerical method

We first introduce a suitable numerical scheme to discretize (1.1) using finite differences for the time variable $t\in[0,T]$ and the space variable $x=(x_1,x_2)\in\Omega$. We subdivide the time interval $[0,\ T]$ into N equal subintervals $[t_{n-1},\ t_n]$, $t_n=n\ \delta t$, $n=1,2,...,N+1$, where δt is the time step.

Let $U^n(x_1,x_2)=u(x_1,x_2,\ t_n)$ and $V^n(x_1,x_2)=v(x_1,x_2,\ t_n)$, and use the finite-difference formulas: the first-order backward difference for

$$\partial_t W^n(x_1,x_2)=\frac{W^n(x_1,x_2)-W^{n-1}(x_1,x_2)}{\delta t}.$$

and the second-order center difference for

$$\partial_{tt} W^n(x_1,x_2)=\frac{W^{n+1}(x_1,x_2)-2W^n(x_1,x_2)+W^{n-1}(x_1,x_2)}{(\delta t)^2}.$$

Then the time discrete problem of (1.1) reads: Given $(u_0,\ v_0)$ and $(u_1,\ v_1)$, find

$$\left\{(U^2,\ V^2),\ (U^3,\ V^3),...,(U^{n+1},\ V^{n+1})\right\}$$

such that

$$\begin{cases} \frac{U^{n+1}}{(\delta t)^2}-\Delta U^{n+1}=\frac{2U^n-U^{n-1}}{(\delta t)^2}-|\partial_t U^n|^{m-2}\partial_t U^n & \\ a\,|U^n+V^n|^{2(\rho+1)}(U^n+V^n)+|U^n|^\rho U^n|V^n|^{\rho+2}, & \text{in } \Omega\\ \frac{V^{n+1}}{(\delta t)^2}-\Delta V^{n+1}=\frac{2V^n-V^{n-1}}{(\delta t)^2}-|\partial_t V^n|^{m-2}\partial_t V^n & \\ a\,|U^n+V^n|^{2(\rho+1)}(U^n+V^n)+b\,|V^n|^\rho V^n|U^n|^{\rho+2} & \text{in } \Omega\\ U^{n+1}=V^{n+1}=0 & \text{on } \partial\Omega\\ U^0=u_0(x_1,x_2),\quad U^1=U^0+(\delta t)\,u_1(x_1,x_2), & \text{in } \Omega\\ V^0=v_0(x_1,x_2),\quad V^1=V^0+(\delta t)\,v_1(x_1,x_2), & \text{in } \Omega \end{cases} \tag{4.1}$$

Note that the above problem is linear in U^{n+1} and also linear in V^{n+1}, which is achieved by using the history data U^n, V^n, U^{n-1} and V^{n-1} in the second side of the equations. Problem(4.1) is solved iteratively as for given regular $(U^n,\ V^n)$, the solution $\left(U^{n+1},\ V^{n+1}\right)$ satisfies the boundary-value problem:

$$\begin{cases} \frac{U^{n+1}}{(\delta t)^2} - \Delta U^{n+1} = F_1\left(U^n,\ V^n,\ U^{n-1}\right), & \text{in } \Omega_h \\ \frac{V^{n+1}}{(\delta t)^2} - \Delta V^{n+1} = F_2\left(U^n,\ V^n,\ V^{n-1}\right), & \text{in } \Omega_h \\ U^{n+1} = V^{n+1} = 0, & \text{on } \partial\Omega_h \end{cases} \tag{4.2}$$

where

$$F_1\left(U^n,\ V^n,\ U^{n-1}\right) = \frac{2U^n - U^{n-1}}{(\delta t)^2} - \left|\partial_t U^n\right|^{m-2} \partial_t U^n + a\left|U^n + V^n\right|^{2(\rho+1)}\left(U^n + V^n\right) + b\left|V^n\right|^{\rho} V^n \left|U^n\right|^{\rho+2},$$

and

$$F_2\left(U^n,\ V^n,\ V^{n-1}\right) = \frac{2V^n - V^{n-1}}{(\delta t)^2} - \left|\partial_t V^n\right|^{m-2} \partial_t V^n + a\left|U^n + V^n\right|^{2(\rho+1)}\left(U^n + V^n\right) + b\left|U^n\right|^{\rho} U^n \left|V^n\right|^{\rho+2}.$$

4.2. Numerical results

In this subsection, we present and discuss the stability results of the numerical scheme(4.1). The numerical results are obtained using the Matlab codes.

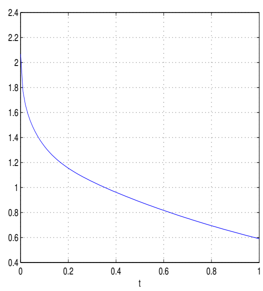

Figure 1: E(t)

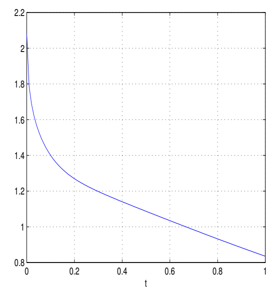

Figure 2: E(t)(1+t)$^{0.5}$.

The parameters that have been set up for numerical expirements are:

- Number of discretisation points is: 100;
- Time step is: $\delta t = 0.01$;

Figures. 1 and 2 presents the energy $E(t)$ and $E(t)(1+t)^{0.5}$ respectively for the times $t_n \in \{1, 2, ..., 100\}$. The numerical solutions of problem (1.1) make the energy function $E(t)$ satisfy

$$E(t)(1+t)^{0.5} \leq 22 \times 10^{-1}.$$

In conclusion, the above numerical application verifies and agrees with the stability results of Theorem 3.2.

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