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On the Solution of Evolution $p(.)$ −Bilaplace Equation with Variable Exponent

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ABSTRACT: A high-order parabolic $p(.)$ −Bilaplace equation with variable exponent is studied. The well-posedness at each time step of the problem in suitable Lebesgue Sobolev spaces with variable exponent with the help of nonlinear monotone operators theory is investigated. The solvability of the proposed problem as well as some regulrarity results are shown using Roth-Galerkin method.

Key Words: Parabolic $p(.)$ −Bilaplace equation, Galerkin method, time discretization, weak solution, a priori estimates

Contents

1. Introduction

We consider a bounded open domain Ω of $\mathbb{R}^n$, with a Lipschitz-continuous boundary $\partial\Omega$ and $I=[0,T]$, $T\in\mathbb{R}$. Our aim is to prove the existence and uniqueness of weak solution u and some a priori error estimates to the following hight order evolution problem

$$\frac{\partial u}{\partial t}+\triangle\left(|\triangle u|^{p(x)-2}\triangle u\right)=f \qquad \text{in} \quad I\times\Omega \tag{1.1}$$

$$u=0, \quad \nabla u=0 \quad \text{on } \Sigma=I\times \ \partial\Omega \tag{1.2}$$

$$u(0,x)=u_0, \text{ on } \Omega \tag{1.3}$$

where f is contiuous function satisfies

$$|f(\gamma,x)|\leq g(x)+c|\gamma|^{r-1}$$

for some bounded positive function $g:\Omega\longrightarrow\mathbb{R}$ and $r\succ 1$. u_0 is given functions in $W^{2,p(x)}(\Omega)$. Here $p(.):\Omega\longrightarrow\mathbb{R}$ denotes the variable exponent which is assumed to be in $L_+^\infty(\Omega)$ such that $1\prec p^-\leq p(x)\leq p^+\prec\infty$ where $p^-=\inf\limits_{x\in\overline{\Omega}}p(x)$ and $p^+=\sup\limits_{x\in\overline{\Omega}}p(x)$ a. e. in Ω. During the last decades, the high-order PDEs with variable exponent has undergone rapid development. From a mathematical point of view, equation (1.1) can be seen as a natural generalization of parabolic bi p-Laplace equation

$$\frac{\partial u}{\partial t}+\triangle\left(\operatorname{div}\left(|\triangle u|^{p-2}\nabla u\right)\right)=f$$

which falls within the framework of nonlinear PDEs where the exponent p is constant. One of our motivation for studying (1.1) comes from applications in area of elasticity, more precisely, it can be used

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in modelling of travelling waves in suspension bridges (see [12]- [13]. Other interesting applications are related to improve the visual quality of damaged and noisy images if $1 \prec p^- \leq p^+ \prec 2$ (see [14] and refereces cited therein) and the mathematical modeling of non-Newtonian fluid motions (see [15]). Note that in the stationary case and for $p(x) \equiv 2$ problem $(1.1) - (1.3)$ becames $\triangle^2 u = f$ which models the deformations of a thin homogeneous plate embedded along its beam and subjected to a distribution f of load normal to the plate (we refer to [7]). Among the most recent work concerning the p-Laplace equation, we can review Lazer et al. [13], where the authors tried to demonstrate the existence of periodic solutions for models of nonlinear supported bending beams and periodic flexing in floating beam. In [6] the authors used discontinuous Galekin method to approximate a biharmonic problem. They also gave an a priori analysis of the error in norm L^2. In [7] the author has studied a problem p-biharmonic using discontinuous Galerkin finite element Hessian. An imagery problem caused by p(.)-Lplace operator with $1 \leq p(.) \leq 2$ (only) has been considered in [14]. To solve the problem, the authors regularized the proposed PDE to be able to use a fixed point iterative method. Other related parabolic problems can be found in the references [15] - [17].

The paper is structured as follows: We present in section 2 some basic notations and material needed for our work. Section 3 is devoted to the time discretization and the existence of a weak solution u^i to the problem under investigation at each time step t_i in suitable Lebesgue Sobolev spaces with variable exponent using nonlinear monotone operators theory. Finally, in section 4, we show our main result with the help of Galerkin-Finit element method and some a priori estimates from which we can extract subsequences that converge to the weak solution.

2. Preliminaries

We define the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ as follows

$$L^{p(.)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R},\, u \text{ measurable and } \int_\Omega |u(x)|^{p(x)}\, dx \prec \infty \right\} \tag{2.1}$$

Note that $L^{p(.)}(\Omega)$ equipped with the Luxembourg norm

$$\|u\|_{L^{p(.)}(\Omega)} = \inf \left\{ \gamma \succ 0,\ \int_\Omega \left| \frac{u(x)}{\gamma} \right|^{p(x)} dx \leq 1 \right\} \tag{2.2}$$

is a Banach space. Note that all definitions and properties of Lebesgue and Sobelv spaces with variable exponent below are taken from references [1] - [5].

Definition 2.1. *Let $u : \Omega \longrightarrow \mathbb{R}$ be a measurable function then, the expression*

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx \tag{2.3}$$

is called modular of u.

Definition 2.2. *For some $p \in L^\infty_+(\Omega)$ and $m \in \mathbb{N}^*$, we introduce the exponent variable Sobolev space*

$$W^{m,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega)\, ;\, D^\alpha u \in L^{p(.)}(\Omega), \forall \alpha \in \mathbb{N}^n \text{ and } |\alpha| \leq m \right\} \tag{2.4}$$

equipped with the following norm

$$\|u\|_{m,\ p(.)} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^{p(.)}(\Omega)} \tag{2.5}$$

Remark 2.3. *1) Let p, q and $r \in L^\infty_+(\Omega)$, $u \in L^{p(.)}(\Omega)$, $v \in L^{q(.)}(\Omega)$ such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{r(x)}.$$

Then

$$\|uv\|_{L^{r(.)}(\Omega)} \leq \left(\frac{1}{\left(\frac{p}{r}\right)^{-}} + \frac{1}{\left(\frac{q}{r}\right)^{-}} \right) \|u\|_{L^{p(.)}(\Omega)} \|v\|_{L^{q(.)}(\Omega)} \tag{2.6}$$

2) *Suppose that* $p(x) \leq q(x)$ *a. e. in* Ω. *Then*

$$L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega) \tag{2.7}$$

3)

$$\|u\|_{L^{p(.)}(\Omega)} = k \Longleftrightarrow \rho_{p(.)}\left(\frac{u}{k}\right) = 1 \tag{2.8}$$

4)

$$\left(\|u_n - u\|_{L^{p(.)}(\Omega)} \underset{n\longrightarrow\infty}{\longrightarrow} 0\right) \Longleftrightarrow \left(\rho_{p(.)}(u_n - u) \underset{n\longrightarrow\infty}{\longrightarrow} 0\right) \tag{2.9}$$

5) *Let* $p, q \in L_{+}^{\infty}(\Omega)$ *and* $m \in \mathbb{N}^{*}$ *with* $p(x) \leq q(x)$ *a. e. in* Ω. *Then*

$$W^{m,q(.)}(\Omega) \hookrightarrow W^{m,p(.)}(\Omega) \tag{2.10}$$

6)

$$\min\left\{\|u\|_{L^{p(.)}(\Omega)}^{p^{-}},\ \|u\|_{L^{p(.)}(\Omega)}^{p^{+}}\right\} \leq \rho_{p(.)} \leq \max\left\{\|u\|_{L^{p(.)}(\Omega)}^{p^{-}},\ \|u\|_{L^{p(.)}(\Omega)}^{p^{+}}\right\} \tag{2.11}$$

Definition 2.4. *(see Definition 4.1.1 page 98 in [5]) A function* $\beta : \Omega \longrightarrow \mathbb{R}$ *is locally log-Hölder continuous on* Ω *if* $\exists C \succ 0$ *such that*

$$|\beta(x) - \beta(y)| \leq \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)},\ \forall x, y \in \Omega. \tag{2.12}$$

If

$$|\beta(x) - \beta_{\infty}| \leq \frac{C}{\log(e + |x|)},$$

for some $\beta_{\infty} \geq 1$, $C > 0$ *and all* $x \in \Omega$, *then we say* β *satisfies the log-Hölder decay condition (at infinity). We denote by* $P^{\log}(\Omega)$ *the class of variable exponents which are log-Hölder continuous, i.e. which satisfy the local log-Hölder continuity condition and the log-Hölder decay condition.*

Definition 2.5. *(See [5] Definition 11. 2. 1) Let* $p \in P^{\log}(\Omega)$. *We define also*

$$W_0^{2,p(.)}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{W^{2,p(.)}(\Omega)}$$

Remark 2.6. ***i)*** *Note That if* $p^{-} > 1$,*then spaces* $W^{2,p(.)}(\Omega)$ *and* $W_0^{2,p(.)}(\Omega)$ *are separable and reflexive Banach spaces.*

ii) (Poincaré inequality) *Let* $p \in L^{\infty}(\Omega)$ *with* $p^{-} \geq 1, \exists C(\Omega, p(.))$ *such that*

$$\|u\|_{p(.)} \leq C \|\nabla u\|_{p(.)},\ \forall u \in W_0^{1,p(.)}(\Omega) \tag{2.13}$$

Definition 2.7. *A function* u *is a weak solution of problem* $(1.1) - (1.3)$ *if it satisfies :*

i)

$$u \in L^{\infty}\left((0,T), W_0^{2,p(.)}(\Omega)\right) \cap W^{1,2}\left((0,T), L^2(\Omega)\right)$$

ii)

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial u}{\partial t} v dx dt + \int_0^T \int_\Omega \left(|\triangle u|^{p(x)-2} \triangle u\right) \triangle v dx dt &= \int_0^T \int_\Omega f v dx dt, \\ \forall v &\in L^{\infty}\left((0,T), W_0^{2,p(.)}(\Omega)\right) \end{aligned} \tag{2.14}$$

3. Semi discretized problem

Let us divide the interval $I=[0,T]$ to n subintervals where $\tau=\frac{T}{n}$, $t_i=i\tau$, $u^i(x)=u(t_i,x)$ and $\delta u^i(x)=\frac{u^i(x)-u^{i-1}(x)}{\tau}$. Then reccurent approximation scheme for $i=1,...,n$ is

$$\left(u^i-u^{i-1},v\right)+\tau\left(\left(\left|\triangle u^i\right|^{p(x)-2}\triangle u^i\right),\triangle v\right)=\tau\left(f^i,v\right),\quad \forall v\in W_0^{2,p(.)}(\Omega) \tag{3.1}$$

Let us show that problem(3.1) admits a weak solution at each time step t_i.

Theorem 3.1. *Let $f^i\in L^{q(.)}(\Omega)$, then for $i=1,...,n$ problem (3.1) admits a unique weak solution u^i in $W_0^{2,p(.)}(\Omega)$*

Proof. Let us introduce the operator $A:W_0^{2,p(.)}(\Omega)\longrightarrow\left(W_0^{2,p(.)}(\Omega)\right)^*$ defined by

$$Au^i=u^i+\tau\triangle_{p(x)}^2u^i \text{ where } \triangle_{p(x)}^2u:=\triangle\left(|\triangle u|^{p(x)-2}\triangle u\right)$$

We apply monotone operators theory, we should prove that A is hemicontinuous, coercive and monotone operator. Let us define the functional E on $W_0^{2,p(.)}(\Omega)$ as follows

$$E\left(u^i\right)=\int_\Omega\left(\frac{1}{2}\left(u^i\right)^2+\frac{\tau}{p(x)}\left|\triangle u^i\right|^{p(x)}\right)dx$$

we have

$$\begin{aligned}
\left(E'\left(u^i\right),v\right) &= \frac{d}{dt}\left\{J\left(u^i+tv\right)\right\}_{t=0}=\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega\left[\left(u^i+tv\right)\right]^2+\tau\int_\Omega\frac{1}{p(x)}\left|\triangle\left(u^i+tv\right)\right|^{p(x)}dx\right\}_{t=0}\\
&= \left\{\int_\Omega\left(u^i+tv\right)vdx+\tau\int_\Omega\frac{1}{p(x)}\triangle v.p(x)\left|\triangle\left(u^i+tv\right)\right|^{p(x)-1}dx\right\}_{t=0}\\
&= \int_\Omega u^ivdx+\tau\int_\Omega\left(\left|\triangle u^i\right|^{p(x)-2}\triangle u^i\right)\triangle vdx\\
&= \int_\Omega u^ivdx+\tau\int_\Omega\triangle\left(\left|\triangle u^i\right|^{p(x)-2}\triangle u^i\right)vdx\\
&= \left(u^i,v\right)+\tau\left(\triangle_{p(x)}^2u^i,v\right)=\left(A\left(u^i\right),v\right),\ \forall v\in W_0^{2,p(x)}(\Omega)
\end{aligned} \tag{3.2}$$

this implies that $E(.)$ is differentiable in Gateau sens and $E'=A$. Therefore A is hemicontinuous. On the other hand, from the inequality (see [8])

$$|b|^{p(x)}\geq|a|^{p(x)}+p|a|^{p(x)-2}a(b-a)+\frac{|b-a|^{p(x)}}{2^{p(x)-1}-1}\text{ for }p(x)\geq2\text{ and }a,b\in\mathbb{R}^n$$

we get

$$\begin{aligned}
\left(A\left(u^i\right)-A\left(v^i\right),u^i-v^i\right) &= \left(u^i-v^i,u^i-v^i\right)+\tau\left(\triangle^2_{p(x)}u^i-\triangle^2_{p(x)}v,u^i-v\right)\\
&= \left\|u^i-v^i\right\|^2++\tau\left(\triangle^2_{p(x)}u^i-\triangle^2_{p(x)}v,u^i-v\right)\\
&\geq \tau\left(\triangle^2_{p(x)}u^i-\triangle^2_{p(x)}v,u^i-v^i\right)\\
&= \tau\int_\Omega\left|\triangle u^i\right|^{p(x)-2}\triangle u^i\left(\triangle u^i-\triangle v^i\right)dx\\
&\quad -\tau\int_\Omega\left|\triangle v^i\right|^{p(x)-2}\triangle v\left(\triangle u^i-\triangle v^i\right)dx\\
&\geq \frac{2}{p(x)\left(2^{p(x)-1}-1\right)}\int_\Omega\left|\triangle u^i-\triangle v^i\right|^{p(x)}dx\\
&\geq \frac{2}{p^+\left(2^{p^+-1}-1\right)}\int_\Omega\left|\triangle u^i-\triangle v^i\right|^{p(x)}dx\\
&\geq \frac{2}{p^+\left(2^{p^+-1}-1\right)}\min\left\{\left\|\triangle u^i-\triangle v^i\right\|^{p^-}_{L^{p(.)}},\left\|\triangle u^i-\triangle v^i\right\|^{p^+}_{L^{p(.)}}\right\} \qquad (3.3)
\end{aligned}$$

Now, using Calderon-Zygmund and Poincaré inequalities we obtain that the norm $\|.\|_{W_0^{2,p(.)}(\Omega)}$ is equivalent to the semi norm $\|\triangle(.)\|_{L^{p(.)}(\Omega)}$ over the space $W_0^{2,p(.)}(\Omega)$.

This allows us to write

$$\left(A\left(u^i\right)-A\left(v^i\right),u^i-v^i\right)\geq C\left(p^+\right)\min\left\{\left\|u^i-v^i\right\|^{p^-}_{W_0^{2,p(.)}(\Omega)},\left\|u^i-v^i\right\|^{p^+}_{W_0^{2,p(.)}(\Omega)}\right\} \qquad (3.4)$$

from which we conclude the monotonicity of A.

Similarly

$$\left(A\left(u^i\right),u^i\right)\geq C\left(p^+\right)\min\left\{\left\|u^i\right\|^{p^-}_{W_0^{2,p(.)}(\Omega)},\left\|u^i\right\|^{p^+}_{W_0^{2,p(.)}(\Omega)}\right\} \qquad (3.5)$$

this proves the coercivity of A.

Finaly, by Hölder inequality we have

$$\left|\left(f^i,v\right)\right|=\left|\int_\Omega f^i v dx\right|\leq C\left\|f^i\right\|_{q(x)}\|v\|_{p(x)}$$

taking into account that $L^{q^+}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ and $W_0^{2,p(.)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ we arrive at

$$\left|\left(f^i,v\right)\right|\leq C\left\|f^i\right\|_{L^{q^+}(\Omega)}\|v\|_{W_0^{2,p(.)}(\Omega)} \qquad (3.6)$$

Hence, $f^i\in\left(W_0^{2,p(.)}(\Omega)\right)^*$.This achieves the proof. □

4. Galerkin-Finit element discretization and existence results

We consider a triangulation Υ_h made of triangles T whose edges are denoted by e. We assume that the intersection of tow different elements is either empty, a vertex, or a whole edge e and we assume also that this triangulation is regular in Ciarlet sens i. e.

$$\exists\sigma\succ 0;\frac{h_T}{\rho_T}\leq\sigma,\ \forall T\in\Upsilon_h$$

where h_T is the diameter of T and ρ_T is the diameter of its largest inscribed bull. We define $h=\max_{T\in\Upsilon_h}h_T$. Let us define the broken Laplace operator

$$\left(\triangle_h v\right)_{\backslash T}:=\triangle\left(v_{\backslash T}\right),\ \forall T\in\Upsilon_h \qquad (4.1)$$

For $h \succ 0$, we introduce the following spaces

$$X^h = \left\{ \phi \in C^0 \left(\overline{\Omega} \right); \ \forall T \in \Upsilon_h; \phi_{\backslash T} \in P^k \left(T \right), \ \phi_{\backslash \partial\Omega \cap \Upsilon_h} = 0 \text{ and } \nabla\phi_{\backslash \partial\Omega \cap \Upsilon_h} = 0 \right\} \tag{4.2}$$

Remark 4.1. *We will regularize the problem* (3.1) *to give the people who are interested in the numerical approximation a possibility of treating it by using the fixed point iterative methods.*

Let $u_h^0 = \Pi_{L^2} u^0$. For $i = 1, ..., n$, the dsiscret formulation of the regularized problem is to seek solution $u_h^i \in X^h$ such that

$$\left(\delta u_h^i, v \right) + \left(\left(k_\epsilon \left(\left| \triangle u_h^i \right| \right) \triangle u_h^i \right), \triangle v \right) = \left(f_h^i, v \right), \quad \forall v \in X^h \tag{4.3}$$

where

$$k_\epsilon \left(r \right) = \left(r^2 + \epsilon \right)^{\frac{p(x)-2}{2}}, \text{ for some } \epsilon \succ 0$$

Let us define Rothe function by the piecewise linear interpolation with respect to the time t

$$u_h^n \left(t \right) \equiv u_{\tau,h,\epsilon} =: u_h^{i-1} + \left(t - t_{i-1} \right) \delta u_h^i, \ i = 1, ..., n$$

together with step functions

$$\overline{u_h^n \left(t \right)} = u_h^i, \ i = 1, ..., n$$

and regularized energy functional

$$J_\epsilon \left(u^i \right) = \int_\Omega \left(\frac{1}{2} \left| u_h^i \right|^2 + \frac{\tau}{p\left(x \right)} \left[\left| \triangle u_h^i \right|^2 + \epsilon \right]^{\frac{p(x)}{2}} + \int_0^{u_h^i} f\left(s, x \right) ds \right) dx \tag{4.4}$$

Now, we are able to announce our main result

Theorem 4.2. *Suppose that $p^- \geq 2$ and $1 \prec r \prec p^-$, then problem* $(1.1) - (1.3)$ *admits a weak solution u in the sens of the definition* 2.5.

To show this theorem, we need the following lemmas

Lemma 4.3. *For $i = 1, ..., n$, the full discretized problem* (4.3) *admits a unique solution $u_h^i \in X^h$,*

Proof. Let us define the functional $J_\epsilon^\lambda : X^h \longrightarrow \mathbb{R}$ by

$$\begin{aligned} J_\epsilon^\lambda \left(v \right) &= \int_\Omega \left(\frac{1}{2\tau} \left| v - u_h^{i-1} \right|^2 + \frac{1}{p\left(x \right)} \left[\left| \triangle u_h^i \right|^2 + \epsilon \right]^{\frac{p(x)}{2}} \right. \\ & \left. + \frac{1}{\lambda} \int_0^{\lambda v + (1-\lambda) u_h^{i-1}} f\left(s, x \right) ds \right) dx \end{aligned} \tag{4.5}$$

where $0 \prec \lambda \prec 1$.
We have

$$\begin{aligned} \int_\Omega \left| \int_0^v f\left(s, x \right) ds \right| dx &\leq \int_\Omega \int_0^v \left(c \left| s \right|^r + g\left(x \right) \right) ds dx \\ &\leq \frac{c}{r} \int_\Omega \left| v \right|^r dx + \left\| g \right\|_\infty \int_\Omega \int_0^v ds dx \\ &\leq c \int_\Omega \left| v \right|^r dx + \left\| g \right\|_\infty \int_\Omega \left| v\left(s, x \right) \right| dx \\ &\leq c \left\| v \right\|_{L^r}^r + \left\| g \right\|_\infty \left\| v \right\|_{L^1} \end{aligned} \tag{4.6}$$

Now taking into account that for $r \succ 1$, $\left\| \triangle \left(. \right) \right\|_{L^r}$ is norm on $W_0^{2,r}$ equivalent to $\left\| . \right\|_{W^{2,r}}$ and using $\epsilon - Young$

inequality we arrive at

$$\begin{aligned}\int_\Omega \left|\int_0^v f(s,x)\,ds\right| dx &\leq c\|v\|_{L^r}^r + \frac{\epsilon^{r'}}{r'}\|g\|_\infty^{r'} + \frac{1}{r\epsilon^\epsilon}\|v\|_{L^1}^r \\ &\leq c\|\triangle v\|_{L^r}^r + \frac{\epsilon^{r'}}{r'}\|g\|_\infty^{r'} + \frac{c}{r\epsilon^\epsilon}\|v\|_{L^r}^r \\ &\leq c\|\triangle v\|_{L^r}^r + \frac{\epsilon^{r'}}{r'}\|g\|_\infty^{r'} + \frac{c}{r\epsilon^\epsilon}\|\triangle v\|_{L^r}^r \end{aligned} \tag{4.7}$$

where $r' = \frac{r}{r-1}$. Now choosing $\epsilon = \left(\frac{r'}{2}\right)^{\frac{1}{r'}}$ in (4.7) and we have

$$\int_\Omega \left|\int_0^v f(s,x)\,ds\right| dx \leq c\|\triangle v\|_{L^r}^r + \|g\|_\infty^{r'} \tag{4.8}$$

Since $r \prec p^-$, one can write

$$\begin{aligned}\int_\Omega \left|\int_0^v f(s,x)\,ds\right| dx &\leq c + c\int_\Omega |\triangle v|^r\,dx \leq c + c\int_\Omega \frac{p^+}{p(x)}\left(|\triangle v|^2\right)^{\frac{p(x)}{2}} dx \\ &\leq c + c\int_\Omega \frac{1}{2p(x)}\left(|\triangle v|^2 + \epsilon\right)^{\frac{p(x)}{2}} dx \end{aligned} \tag{4.9}$$

from which we can conclude the coercivity of J_ϵ^λ. Consequently, there exists a unique $u_h^i \in X^h$ minimize J_ϵ^λ . This achieves the proof. □

Lemma 4.4. *The solution u_h^i of* (4.3) *satisfies the following energy a priori estimate*

$$J_\epsilon\left(u^i\right) + \tau\sum_{j=1}^{i}\left\|\delta u_h^j\right\|^2 \leq J_\epsilon\left(u^0\right) \tag{4.10}$$

Proof. Testing with $v = u_h^i$ in (4.3) and using convexitry properties we can arrive at the desired reult. □

Lemma 4.5. *There exists $C \succ 0$ such that*

$$\|\partial_t u_h^n(t)\|_{L^2\left((0,T),W^{-2,q^+}(\Omega)\right)} + \|u_h^n(t)\|_{L^2((0,T),H^{-2}(\Omega))} \leq C \tag{4.11}$$

Proof. Note that the above notations allows us to rewrite (4.3) as

$$\int_0^T (\partial_t u_h^n, v)\,dt + \int_0^T \left(\left[\left|\overline{\triangle u_h^n}\right|^2 + \epsilon\right]^{\frac{p(x)-2}{2}} \overline{\triangle u_h^n}, \triangle v\right) = \int_0^T (f_h^n, v)\,, \quad \forall v \in X_0^h \tag{4.12}$$

It follows from theorem3 and theorem 4 in [18] that

$$\begin{aligned}\|\partial_t u_h^n\|_{W^{-2,q^+}(\Omega)} &= \sup_{v\in W_0^{2,p^-}} \frac{(\partial_t u_h^n, P_{L^2}v)}{\|v\|_{W^{2,p^-}}} \\ &\leq C\left\|\overline{\triangle u_h^n}\right\|_{L^{p^-}}^{p^--1} + C\end{aligned}$$

Estimate (4.10) concludes the proof. □

Proof. (of the theorem 4.1) The precident a priori estimates allow us to conclude that $u_h^n(t)$ has a subsequence still denoted by $u_h^n(t)$, and that there exists a function u for which we have when ϵ, h, and $\tau = \frac{T}{n} \longrightarrow 0$

$$\begin{aligned}
u_h^n &\rightharpoonup u \text{ in } L^\infty\left((0,T), W_0^{2,p(x)}(\Omega)\right) \cap W^{1,2}\left((0,T), L^2(\Omega)\right) \\
u_h^n &\longrightarrow u \text{ in } W^{1,2}\left((0,T), H_0^1(\Omega)\right) \\
f(u_h^n,.) &\longrightarrow f(u,.) \text{ in } L^{r'}\left((0,T),\Omega\right) \\
\left[\left|\overline{\triangle u_h^n}\right|^2+\epsilon\right]^{\frac{p(x)-2}{2}}\overline{\triangle u_h^n} &\rightharpoonup \eta \text{ in } L^\infty\left((0,T), L^{q(x)}(\Omega)\right) \\
\delta u_h^n &\rightharpoonup \partial_t u \text{ in } H^1\left((0,T), W^{-1,q^+}(\Omega)\right) \cap L^2\left((0,T), H_0^2(\Omega)\right)
\end{aligned} \tag{4.13}$$

relation $(4.13)_{ii}$ comes from the fact that

$$\int_0^T\int_\Omega \left|\overline{u_h^n(t)} - \overline{u_h^{n-1}(t)}\right|^2 dxdt \le \tau^2 \left\|\partial_t u_h^n\right\|^2_{L^2((0,T)\times\Omega)}$$

According to Kolmogorov compactness creterion we get the desired result.
Now, passing to the limit as ϵ, h, and $\tau = \frac{T}{n} \longrightarrow 0$ in (4.12) taking into account (4.13) we obtain

$$\int_0^T (\partial_t u, v)\,dt + \int_0^T (\eta, \triangle v)\,dt = \int_0^T (f, v)\,dt, \quad \forall v \in X_0^h \tag{4.14}$$

Let us prove that

$$\eta = |\triangle u|^{p(x)-2}\triangle u$$

choosing $v = \overline{u_h^n}$ in (4.13) and $v = u$ in (4.14) we find that

$$\int_0^T (u_h^n, \partial_t u_h^n)\,dt + \int_0^T \left(\left[\left|\overline{\triangle u_h^n}\right|^2+\epsilon\right]^{\frac{p(x)-2}{2}}\overline{\triangle u_h^n}, \overline{\triangle u_h^n}\right) = \int_0^T \left(f_h^n, \overline{u_h^n}\right) \tag{4.15}$$

and

$$\int_0^T \|\partial_t u\|^2\,dt + \int_0^T (\eta, \triangle u)\,dt = \int_0^T (f, u)\,dt \tag{4.16}$$

in view of the monotonicity of the operator $\triangle^2_{p(x)} u := \triangle\left(|\triangle u|^{p(x)-2}\triangle u\right)$ we can write

$$\int_0^T \left(\left[\left|\overline{\triangle u_h^n}\right|^2+\epsilon\right]^{\frac{p(x)-2}{2}}\overline{\triangle u_h^n} - |\triangle v|^{p(x)-2}\triangle v, \overline{\triangle u_h^n} - \triangle v\right) dt \ge 0,\ \forall v \in C_0^\infty((0,T)\times\Omega) \tag{4.17}$$

by virtue of (4.14) – (4.16) we will arrive at

$$\int_0^T \left(\eta - |\triangle v|^{p(x)-2}\triangle v, \triangle u - \triangle v\right) dt \ge 0,\ \forall v \in C_0^\infty((0,T)\times\Omega) \tag{4.18}$$

Taking $v = u - \lambda\theta$ for some $\lambda \succ 0$ and $\theta \in C_0^\infty((0,T)\times\Omega)$ we obtain

$$\int_0^T \left(\eta - |\triangle(u-\lambda\theta)|^{p(x)-2}\triangle(u-\lambda\theta), \triangle\theta\right) dt \ge 0 \tag{4.19}$$

Let λ tend towards 0, we arrive at

$$\int_0^T \left(\eta - |\triangle u|^{p(x)-2} \triangle u, \triangle\theta\right) dt \geq 0 \tag{4.20}$$

by repeating the same procedure with $v = u + \lambda\theta$, we conclude

$$\int_0^T \left(\eta - |\triangle u|^{p(x)-2} \triangle u, \triangle\theta\right) dt = 0,\ \forall\theta \in C_0^\infty\left((0,T)\times\Omega\right) \tag{4.21}$$

The desired result followes from the density of $C_0^\infty\left((0,T)\times\Omega\right)$. This achieves the proof. □

Remark 4.6. *Note that with additional assumptions on f and p, the results obtained in Theorem* 4.1 *can be extended to the case $p \equiv p(t,x)$ with a less regular solution in an anisotropic space.*

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