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Exponential stability for microtemperature Von Kármán beam with delay-time

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ABSTRACT: In this paper, we consider a one-dimensional Von kármán beam with delay term coupled to a microtemperature equation. Under suitable assumptions on the weight of delay and a microtemperature effect we prove the exponential stability.

Key Words: Von kármán, semi-group, delay time, exponential stability, Lyapunov functional.

Contents

1. Introduction

The phenomenon of vibrations arises basically in all mechanical structures in the field of engineering, and certain types of these vibrations are undesired because they have a negative impact on the functioning and lifespan of these structures. In which they have the potential to induce fractures, or even the destruction of these structures, Furthermore, they may pose a threat to the user himself. The external environment, atmosphere, and water, as well as a shock with other structures, are all sources of dynamic excitations that cause these vibrations.

Many constructions in various sectors of engineering are made up of one or more beams. Depending on the nature and type of vibrations, these beams have varied models. In contrast to other base models (such as the Timoshenko model or the Euler-Bernoulli), the Von kármán model is more suitable because it considers both transverse and longitudinal displacements when vibrating a narrow body with a significant deflection. As a result, numerous strategies are employed to eliminate or reduce the consequences of these vibrations, such as The Von kármán beam stability problem which has attracted the interest of many researchers, with a substantial range of literature addressing the issues of existence, uniqueness, and asymptotic behavior in time when damping effects are taken into account, as well as other relevant aspects, (See Refs. [2,3,14,8,17] and the references therein for more information). The controllability and stabilization of the Von kármán system were investigated by Horn and Lasiecka in 1995 (See [1]).

In 1990, Lagnese [6] investigated a one-dimensional Von kármán beam with internal damping.

$$\begin{cases} u_{tt} - \left[u_x + \frac{1}{2}(w_x)^2\right]_x = 0, \quad (x,t) \in (0,L) \times (0,\infty), \\ w_{tt} - \left[\left(u_x + \frac{1}{2}(w_x)^2\right) w_x\right]_x + w_{xxxx} + h w_{xxtt} = 0, \quad (x,t) \in (0,L) \times (0,\infty), \end{cases} \tag{1.1}$$

where h denotes the beam's rotational inertia. Using nonlinear boundary feedback, they were able to achieve model uniform stabilization. Furthermore, in 1998 Benabdallah and Teniou stabilized the system by linking it to two heat equations: one for the longitudinal component and the other for the transversal component (See [11]), In this regard we refer also to the work of Benabdallah and Lasiecka [2].

Green and Naghdi suggested three models that allow heat to be transmitted in the form of thermal waves at constrained speeds. When the beam is at a low temperature, this behavior occurs, the reader is directed to [7,8,9,10].

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Many articles have looked into the stabilization of systems using boundary damping (often in conjunction to internal damping), see Favini et al [14], Lagnese and Leugering [15], Puel and Tucsnak [16], and the references therein.

In 2013, the following Von kármán system was investigated by Djebabla and Tatar [12]:

$$\begin{cases} u_{tt} - D_1 \left[u_x + \frac{1}{2} (w_x)^2 \right]_x + \gamma \theta_{tx} = 0, \\ w_{tt} + K_1 w_t - D_1 \left[\left(u_x + \frac{1}{2} (w_x)^2 \right) w_x \right]_x + D_2 w_{xxxx} = 0, \\ \theta_{tt} - l\theta_{xx} + k_2 \theta_t + \gamma u_{tx} = 0, \end{cases} \tag{1.2}$$

in $\Omega \times (0, +\infty)$, where $\Omega = [0, L]$; and K_1, K_2, D_1, D_2, l , and γ are positive constants, with the boundary conditions

$$\begin{cases} u = 0, \quad w = 0, \quad \theta_x = 0, \quad x = 0, L, \quad t > 0, \\ w_x = 0, \quad x = 0, L, \quad t > 0, \end{cases} \tag{1.3}$$

and the initial data

$$\begin{cases} u(0, .) = u_0, \quad u_t(0, .) = u_1, \quad w(0, .) = w_0, \quad w_t(0, .) = w_1, \\ \theta(0, .) = \theta_0, \quad \theta_t(0, .) = \theta_1. \end{cases} \tag{1.4}$$

They achieved an exponential decay result of problem (1.2) for the full Von kármán system. The stability of the wave equation with delay has lately become a hot topic of research, with several authors demonstrating that delays can destabilize a system that is asymptotically stable without them (see [11] for more details).

Various sorts of dissipative mechanisms were examined by numerous writers in order to get stability results, temperature and microtemperature elements were included into the theory by Lesan [18], and Lesan and Quintanilla [19]. We can refer to some references in this topic [20].

Furthermore, in 2005 another work [4], demonstrated that mixing temperature and microtemperature cause exponential stability.

In the present paper, we shall prove that solutions decay to zero exponentially. The system is coupled with microtemperature, and internal delay acting on the first equation, so we will look into the following system

$$\begin{cases} \phi_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (\phi_x)^2 \right) \phi_x \right]_x + d_2 \phi_{xxxx} + \mu_1 \phi_t + \mu_2 \phi_t(x, t - \tau) = 0, \\ u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (\phi_x)^2 \right) \right]_x + d w_x = 0, \\ c w_t - k_1 w_{xx} + d u_{tx} + k_2 w = 0, \end{cases} \tag{1.5}$$

in $\Omega \times (0, \infty)$, where $\Omega = [0, L]$ and $d_1, d_2, d, \mu_1, c, k_1$, and k_2 are positive constants. We complement system (1.5) with boundary conditions

$$\begin{cases} \phi(0, t) = \phi(1, t) = u_x(0, t) = u_x(1, t), \quad t > 0 \\ \phi_x = 0 \quad at \quad x = 0, \ L \quad for \quad any \quad t > 0, \end{cases}$$

and the initial data

$$\begin{cases} u(0,.) = u_0, \quad u_t(0,.) = u_1, \\ \phi(0,.) = \phi_0, \quad \phi_t(0,.) = \phi_1, \quad w(x,0) = w_0(x), \\ \phi_t(x,t) = f_0(x,t) \;\; in \in (0,L), \end{cases} \tag{1.6}$$

here, we will prove the stability results for problems (1.5)–(1.6) under the assumption

$$|\mu_2| \le \mu_1, \tag{1.7}$$

Most phenomena are naturally dependent on their current state as well as previous events, which is why time delay occurs in so many applications. Introducing distributed delay,varying delay, or constant delay has been a important research topic in EDPs for decades, and has gotten a lot of attention (see, for example ([13,19]). We recall that the delay term became a source of instability when it was demonstrated that a slight delay in a boundary control may change a well-behaved hyperbolic system into a wild one, resulting in instability.

The goal of this study is to prove that the microtemperature effect is powerful enough to uniformly stabilize the system (1.5) even when time delay is present.

2. Preliminaries

To prove that systems (1.5)–(1.6) are well posed using the semigroup theory we introduce the following new variable:

$$z(x,\rho,t) = \phi_t(x, t-\tau\rho), x \in [0,L], \rho \in [0,1], t \in [0,\infty),$$

then, the above variable z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \;\; in \;(0,L) \times (0,1) \times (0,\infty).$$

Therefore, problem (1.5) takes the form

$$\begin{cases} \phi_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (\phi_x)^2 \right) \phi_x \right]_x + d_2 \phi_{xxxx} + \mu_1 \phi_t + \mu_2 z(x,1,t) = 0, \\ u_{tt} - d_1 \left[\left(u_x + \frac{1}{2} (\phi_x)^2 \right) \right]_x + d w_x = 0, \\ c w_t - k_1 w_{xx} + d u_{tx} + k_2 w = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \;\text{ in } (0,L) \times (0,1) \times (0,\infty), \end{cases} \tag{2.1}$$

and the initial conditions

$$\begin{cases} u(0,.) = u_0, \quad u_t(0,.) = u_1, \\ \phi(0,.) = \phi_0, \quad \phi_t(0,.) = \phi_1, \quad w(x,0) = w_0(x), \\ \phi_t(x,t) = f_0(x,t) \;\; in \in (0,L), \\ z(x,\rho,0) = f_0(x,\rho,t) \text{ in } (0,L) \times (0,1), \\ z(x,0,t) = \phi_t(x,t) \text{ in } (0,L) \times (0,\infty), \end{cases} \tag{2.2}$$

for $U = (\phi, \varphi, u, \psi, w, z)^T$, $\tilde{U} = \left(\tilde{\phi}, \tilde{\varphi}, \tilde{u}, \tilde{\psi}, \tilde{w}, \tilde{z} \right)^T$, equipped with the scalar product

$$\left\langle U,\widetilde{U}\right\rangle = \int_0^L \varphi\tilde{\varphi}dx+\int_0^L \psi\tilde{\psi}dx+c\int_0^L \phi\tilde{\phi}dx+d_2\int_0^L \phi_{xx}\tilde{\phi}_{xx}dx$$
$$+d_1\int_0^L u_x\tilde{u}_x dx+|\mu_2|\,\tau\int_0^L\int_0^1 z(x,\rho,t)\widetilde{z}(x,\rho,t)d\rho dx.$$

We introduce two new dependent variables

$$\varphi=\phi_t,\psi=u_t,$$

then the system (2.1)–(2.2) can be written as

$$\begin{cases} \dfrac{\partial U}{\partial t}=\mathcal{A}U+\mathcal{F}\left(U\right), \\ U\left(0\right)=U_0=\left(\phi_0,\phi_1,u_0,u_1,w_0,f_0\right)^T, \end{cases} \tag{2.3}$$

with the linear problem

$$\begin{cases} \dfrac{\partial U}{\partial t}=\mathcal{A}U, \\ U\left(0\right)=U_0=\left(\phi_0,\phi_1,u_0,u_1,w_0,f_0\right)^T. \end{cases} \tag{2.4}$$

So

$$\mathcal{A}U=\begin{pmatrix} \varphi \\ -d_2\phi_{xxxx}-\mu_1\phi_t-\mu_2 z(x,1,t) \\ \psi \\ d_1u_{xx}-dw_x \\ k_1w_{xx}-d\psi_x-k_2w \\ \dfrac{1}{\tau}z_\rho(x,\rho,t) \end{pmatrix} \tag{2.5}$$

and

$$\mathcal{F}\left(U\right)=\begin{pmatrix} 0 \\ d_1\left[\left(u_x+\frac{1}{2}\left(\phi_x\right)^2\right)\phi_x\right]_x \\ 0 \\ \dfrac{d_1}{2}\left(\phi_x\right)_x^2 \\ 0 \\ 0 \end{pmatrix} \tag{2.6}$$

with the domain

$$D(\mathcal{A}) = \Big\{(\phi,\varphi,u,\psi,w,z)^T\in\left[H^4\left(0,L\right)\cap H_0^2\left(0,L\right)\right]\times H_0^1\left(0,L\right)$$
$$\times\left[H^2\left(0,L\right)\cap H_0^2\left(0,L\right)\right]\times H_0^1\left(0,L\right)\times L^2\left(0,L\right)$$
$$\times L^2\left(\left(0,L\right)\times H_0^1\left(0,1\right)\right),\ \varphi=z\left(x,0\right)\Big\}.$$

Denote by H the Hilbert space

$$
\begin{aligned}
H &= \left[H^4(0,L)\cap H_0^2(0,L)\right]\times H_0^1(0,L)\times\left[H^2(0,L)\cap H_0^2(0,L)\right]\\
&\quad\times H_0^1(0,L)\times L^2(0,L)\times L^2\left((0,L)\times H_0^1(0,1)\right)\}.
\end{aligned}
$$

Clearly, $D(\mathcal{A})$ is dense in H, We have the following existence and uniqueness result.

Theorem 2.1 *([3])Let $U_0\in H$, then there exists a unique solution $U\in C(\mathbb{R}^+,H)$ of problem (2.4). Moreover, if $U_0\in D(\mathcal{A})$, then $U\in C(\mathbb{R}^+,D(\mathcal{A}))\cap C^1(\mathbb{R}^+,H)$.*

3. Stability result

In this section, we state and prove our stability result for the energy of the solution of system (2.1), To achieve our goal, we need the following lemmas.

Lemma 3.1 *Let (ϕ,u,w,z) be the solution of (2.1), then The energy functional $E(t)$, defined by*

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_\Omega\left\{\phi_t^2+u_t^2+cw^2+d_2\phi_{xx}^2+d_1\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)^2\right]dx\right.\\
&\quad+\frac{|\mu_2|}{2}\tau\int_0^1 z^2(x,\rho,t)d\rho dx
\end{aligned}
\tag{3.1}
$$

multiplying the first equation in (2.1) by ϕ_t, the second by u_t and the third by w and summing up, we find, after integration over $\Omega=[0,L]$,

$$
E'(t)\le -k_1\int_\Omega w_x^2dx-k_2\int_\Omega w^2dx-m_0\int_\Omega\phi_t^2,\quad\forall t\ge 0,
\tag{3.2}
$$

where $m_0=(\mu_1-|\mu_2|)$.

Lemma 3.2 *Let (ϕ,u,w,z) be the solution of (2.1). Then the functional*

$$
F_1(t)=\int_\Omega\left(u_tu+\frac{1}{2}\phi_t\phi+\frac{k_1}{4}\phi^2\right)dx,
$$

satisfies, the estimate

$$
\begin{aligned}
F_1'(t) &\le -d_1\int_\Omega\left(u_x+\frac{1}{2}(\phi_x)^2\right)^2dx-\frac{d_2}{2}\int_\Omega(\phi_{xx})^2dx+\int_\Omega\left(u_t^2+\frac{1}{2}\phi_t^2\right)dx+\varepsilon_1\int_\Omega u_x^2dx\\
&\quad+\frac{d}{4\varepsilon_1}\int_\Omega w^2dx+C_1\int_\Omega\phi_x^2dx+\frac{C_2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)dx.
\end{aligned}
\tag{3.3}
$$

Proof: Differentiating $F_1(t)$ using $(2.1)_1$ and $(2.1)_2$, gives

$$\begin{aligned}
F_1^{'}(t) &= \int_\Omega u_t^2 dx + \int_\Omega u_{tt} u dx + \frac{1}{2}\int_\Omega \phi_t^2 dx + \frac{1}{2}\int_\Omega \phi_{tt}\phi dx + \frac{k_1}{4}\int_\Omega \phi\phi_t dx, \\
&= \int_\Omega u_t^2 dx + \int_\Omega \left(d_1 \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\right]_x - dw_x \right) u dx + \frac{1}{2}\int_\Omega \phi_t^2 dx \\
&+ \frac{1}{2}\int_\Omega \left(d_1 \left[\left(u_x + \frac{1}{2}(\phi_x)^2\right)\phi_x\right]_x - d_2\phi_{xxxx} - \mu_1\phi_t - \mu_2 z(x,1,t) \right)\phi dx \\
&+ \frac{k_1}{4}\int_\Omega \phi\phi_t dx, \\
&= \int_\Omega u_t^2 dx - d_1 \int_\Omega \left(u_x + \frac{1}{2}(\phi_x)^2\right) u_x dx + d\int_\Omega w u_x dx + \frac{1}{2}\int_\Omega \phi_t^2 dx \\
&- \frac{d_1}{2}\int_\Omega \left(u_x + \frac{1}{2}(\phi_x)^2\right)\phi_x^2 dx - \frac{d_2}{2}\int_\Omega \phi_{xx}^2 dx - \frac{\mu_2}{2}\int_\Omega z(x,1,t)\phi dx,
\end{aligned}$$

by using Young's inequality, we find (3.3). □

Lemma 3.3 *Let (ϕ, u, w, z) be the solution of (2.1). Then the functional*

$$F_2(t) = \int_\Omega \left(\int_0^x w(t,y)\, dy\right) u_t dx,$$

satisfies, the estimate

$$F_2^{'}(t) \le -\left(\frac{d}{c} - \frac{k_1}{2c} - \frac{k_2\varepsilon_2}{c}\right)\int_\Omega u_t^2 dx + d_1\varepsilon_2 \int_\Omega \left(u_x + \frac{1}{2}(\phi_x)^2\right)^2 dx + \frac{k_1}{2}\int_\Omega w_x^2 dx + L(\varepsilon_2)\int_\Omega w^2 dx, \quad (3.4)$$

where, $L(\varepsilon_2) = \left(\frac{d_1}{4\varepsilon_2} + d + \frac{k_2}{4\varepsilon_2}\right)$

Proof: Differentiating $F_2(t)$ using $(2.1)_2$ and $(2.1)_3$, gives

$$F_2^{'}(t) = \int_\Omega \left(\int_0^x \frac{d}{dt} w(t,y)\, dy\right) u_t dx + \int_\Omega u_{tt}\left(\int_0^x w(t,y)\, dy\right) dx,$$

$$\begin{aligned}
\int_\Omega \left(\int_0^x \frac{d}{dt} w(t,y)\, dy\right) u_t dx &= \int_\Omega \left(\int_0^x w_t(t,y)\, dy\right) u_t dx \\
&= \int_\Omega \left(\int_0^x \left(\frac{k_1}{c} w_{xx} - \frac{d}{c} u_{tx} - \frac{k_2}{c} w\right) dy\right) u_t dx \\
&= \frac{k_1}{c}\int_\Omega u_t w_x dx - \frac{d}{c}\int_\Omega u_t^2 dx - \frac{k_2}{c}\int_\Omega u_t \int_0^x w dy dx, \quad (3.5)
\end{aligned}$$

applying Young's inequality, we obtain,

$$\int_\Omega \left(\int_0^x \frac{d}{dt} w(t,y)\, dy\right) u_t dx \le \frac{k_1}{2c}\int_\Omega u_t^2 + \frac{k_1}{2}\int_\Omega w_x^2 dx - \frac{d}{c}\int_\Omega u_t^2 dx + \frac{k_2\varepsilon_2}{c}\int_\Omega u_t^2 dx + \frac{k_2}{4\varepsilon_2}\left(\int_0^x w dy\right)^2 dx, \quad (3.6)$$

by recalling Cauchy–Schwarz inequality for the last term (3.6) we find

$$\begin{aligned}\int_0^x w(t,y)\,dy &\le \left(\int_0^x 1dy\right)\left(\int_0^x w^2(t,y)\,dy\right) \\ &\le x\left(\int_0^x w^2(t,y)\,dy\right) \\ &\le \left(\int_0^1 w^2dx\right),\end{aligned} \tag{3.7}$$

by substituting (3.7) in (3.6), we obtain,

$$\int_\Omega\left(\int_0^x \frac{d}{dt}w(t,y)\,dy\right)u_tdx \le \left(-\frac{d}{c}+\frac{k_1}{2c}+\frac{k_2\varepsilon_2}{c}\right)\int_\Omega u_t^2+\frac{k_1}{2}\int_\Omega w_x^2dx+\frac{k_2}{4\varepsilon_2}\int_\Omega w^2dx, \tag{3.8}$$

$$\begin{aligned}\int_\Omega u_{tt}\left(\int_0^x w(t,y)\,dy\right)dx &= \int_\Omega\left(d_1\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)\right]_x - dw_x\right)\left(\int_0^x w(t,y)\,dy\right)dx \\ &= -d_1\int_\Omega\left[\left(u_x+\frac{1}{2}(\phi_x)^2\right)\right]wdx + d\int_\Omega w^2dx,\end{aligned}$$

by using Young's inequality, we obtain,

$$\int_\Omega u_{tt}\left(\int_0^x w(t,y)\,dy\right)dx \le d_1\varepsilon_2\int_\Omega\left(u_x+\frac{1}{2}(\phi_x)^2\right)^2dx+\left(\frac{d_1}{4\varepsilon_2}+d\right)\int_\Omega w^2dx, \tag{3.9}$$

combining (3.8), (3.9), gives (3.4). □

Lemma 3.4 *Let (ϕ,u,w,z) be the solution of (2.1). Then the functional*

$$F_3(t) = \tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,$$

satisfies the estimate

$$F_3'(t) \le -m_1\int_0^1 z^2(x,1,t)dx - m_1\tau\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx + \int_0^1 \phi_t^2dx. \tag{3.10}$$

Proof: Differentiating $F_3(t)$, gives

$$\begin{aligned}F_3'(t) &= 2\tau\int_0^1\int_0^1 e^{-\tau\rho}z(x,\rho,t)z_t(x,\rho,t)d\rho dx \\ &= 2\tau\int_0^1\int_0^1 e^{-\tau\rho}z(x,\rho,t)\left(-\frac{1}{\tau}z_\rho(x,\rho,t)\right)d\rho dx \\ &= -2\int_0^1\int_0^1 e^{-\tau\rho}z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx \\ &= -\int_0^1\int_0^1\frac{d}{d\rho}\left(e^{-\tau\rho}z^2(x,\rho,t)\right)d\rho dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx \\ &= -\int_0^1 e^{-\tau}z^2(x,1,t)dx + \int_0^1 z^2(x,0,t)dx - \tau\int_0^1\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

integration by parts and using the fact that $z(x,0,t) = \phi_t(x,t)$ and $e^{-\tau} \leq e^{-\tau\rho} \leq 1$, we get for all $\rho \in [0,1]$

$$F_3'(t) \leq -\int_0^1 e^{-\tau} z^2(x,1,t)dx - \tau \int_0^1 \int_0^1 e^{-\tau\rho} z^2(x,\rho,t)d\rho dx + \int_0^1 \phi_t^2 dx.$$

Since $-e^{-\tau}$ is an increasing function, setting $m_1 = e^{-\tau}$ we obtain (3.10). □

Now, we defined the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1F_1 + N_2F_2 + N_3F_3, \tag{3.11}$$

where N,N_1,N_2 and N_3 are positive constants.

Lemma 3.5 *there exists two positive constants C_1 and C_2 such that the Lyapunov functional $L(t)$ satisfies*

$$C_1E(t) \leq L(t) \leq C_2E(t), \ \ \forall t \geq 0. \tag{3.12}$$

Proof: Let

$$L(t) = NE(t) + N_1F_1 + N_2F_2 + N_3F_3,$$

$$\begin{aligned}|L(t) - NE(t)| \leq{}& N_1 \int_\Omega |u_t u|\, dx + \frac{N_1}{2}\int_\Omega |\phi_t \phi|\, dx + \frac{N_1k_1}{4}\int_\Omega \phi^2 dx \\ &+ N_2 \int_\Omega \left| \left(\int_0^x w(t,y)\, dy \right) u_t \right| dx \\ &+ N_3\tau \int_0^1 \int_0^1 e^{-\tau\rho} z^2(x,\rho,t)d\rho dx.\end{aligned}$$

By exploiting Cauchy-Schwarz's and Young's and Poincaré inequalities, and the fact that $e^{-\tau\rho} \leq 1$, for all $\rho \in [0,1]$ we obtain,

$$\begin{aligned}L(t) \leq{}& C\int_\Omega \left(\phi_t^2 + u_t^2 + cw^2 + d_2\phi_{xx}^2 + d_1 \left[\left(u_x + \frac{1}{2}(\phi_x)^2 \right)^2 \right] \right) dx \\ &+ C\int_0^1 \int_0^1 z^2(x,\rho,t)d\rho dx, \\ &\leq CE(t).\end{aligned}$$

Which yields

$$(N - C)E(t) \leq L(t) \leq (N + C)E(t).$$

Consequently, by choosing N large enough We obtain estimate (3.12). □

Now, it's time to state and demonstrate the section's major result.

Theorem 3.1 *Let (ϕ, u, w, z) be the solution of (2.1). Then, the energy functional (3.1) decays exponentially, i.e., there exist two positive constants k_0 and k_1 independent of the initial data such that,*

$$E(t) \leq k_0 e^{-k_1 t}, \ \forall t \geq 0, \tag{3.13}$$

where k_0 and k_1 are two positive constants.

Proof: by differentiating (3.11) and recalling (3.2), (3.3), (3.4), (3.10), we obtain

$$\begin{aligned}
\frac{dL(t)}{dt} \leq & -\frac{N_1 d_2}{2}\int_\Omega (\phi_{xx})^2\,dx \\
& -\left(Nm_0 - \frac{N_1}{2} - N_3\right)\int_\Omega \phi_t^2 dx \\
& -\left(\frac{N_2 d}{c} - \frac{N_2 k_1}{2c} - \frac{N_2 k_2 \varepsilon_2}{c} - N_1\right)\int_\Omega u_t^2 dx \\
& -\left(Nk_2 - \frac{N_1 d}{4\varepsilon_1} - N_2\left(\frac{d_1}{4\varepsilon_2} + d + \frac{k_2}{4\varepsilon_2}\right)\right)\int_\Omega w^2 dx \\
& -(N_1 d_1 - N_2 d_1 \varepsilon_2)\int_\Omega \left(u_x + \frac{1}{2}(\phi_x)^2\right)^2 dx \\
& -\left(m_1 N_3 - \frac{N_1 C_2}{4\varepsilon_1}\right)\int_0^1 z^2(x,1,t)dx. \\
& -N_3 \tau m_1 \int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx. \\
& +\frac{N_2 k_1}{2}\int_\Omega w_x^2 dx.
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{d}{4N_1}, \quad \varepsilon_2 = \frac{c}{N_2},$$

we obtain,

$$\begin{aligned}
\frac{dL(t)}{dt} \leq & -\frac{N_1 d_2}{2}\int_\Omega (\phi_{xx})^2\,dx \\
& -\left(Nm_0 - \frac{N_1}{2} - N_3\right)\int_\Omega \phi_t^2 dx \\
& -\left(\frac{N_2 d}{c} - \frac{N_2 k_1}{2c} - k_2 - N_1\right)\int_\Omega u_t^2 dx \\
& -\left(Nk_2 - 1 - \frac{d_1}{4c} - N_2 d - \frac{k_2}{4c}\right)\int_\Omega w^2 dx \\
& -(N_1 d_1 - c d_1)\int_\Omega \left(u_x + \frac{1}{2}(\phi_x)^2\right)^2 dx \\
& -\left(m_1 N_3 - \frac{C_2}{d}\right)\int_0^1 z^2(x,1,t)dx \\
& -N_3 \tau m_1 \int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive.

We choose N_3 large enough such that

$$m_1 N_3 - \frac{C_2}{d} > 0,$$

then we choose N_1 large enough such that

$$\alpha_1 = N_1 d_1 - c d_1 > 0,$$

Once N_1 is fixed, we then choose N_2 large enough so that

$$\alpha_2 = \frac{N_2 d}{c} - \frac{N_2 k_1}{2c} - k_2 - N_1 > 0.$$

Finally, we choose N large enough (even larger so that (3.12) remains valid) so that

$$\alpha_3 = N m_0 - \frac{N_1}{2} - N_3 > 0,$$

$$\alpha_4 = N k_2 - 1 - \frac{d_1}{4c} - N_2 d - \frac{k_2}{4c},$$

where, $\alpha_0 = N_3 \tau m_1$, and $\alpha_5 = \frac{N_1 d_2}{2}$ we obtain

$$L^{'}(t) \leq -\int_{\Omega} \left(\alpha_5 (\phi_{xx})^2 dx + \alpha_3 \phi_t^2 + \alpha_2 u_t^2 + \alpha_4 w^2 + \alpha_1 \left(u_x + \frac{1}{2} (\phi_x)^2 \right)^2 \right) dx$$
$$- \alpha_0 \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx.$$

By (3.1), we obtain

$$L^{'}(t) \leq -\sigma_0 E(t) \quad , \ \forall t \geq 0, \tag{3.14}$$

for some $\sigma_0 > 0$. A combination of (3.12) and (3.14) gives

$$L^{'}(t) \leq -k_1 L(t) \quad , \ \forall t \geq 0, \tag{3.15}$$

where $k_1 = \alpha_0 / C_2$. A simple integration of (3.15) over $(0, t)$ yields

$$L(t) \leq L(0) e^{-k_1 t}, \ \forall t \geq 0. \tag{3.16}$$

Finally, by combining (3.12) and (3.16) we obtain (3.13) with $k_0 = \frac{C_2 E(0)}{C_1}$, which completes the proof. □

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