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Applications of Fractional Difference Operators for New Version of Brudno-Mazur Orlicz Bounded Consistency Theorem *

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ABSTRACT: In this paper, we intend to prove that the modulus $\mathcal{A}-$lacunary statistical convergence of fractional difference double sequences and modulus lacunary fractional matrix of four-dimensions taken over the space of modulus $\mathcal{A}-$lacunary fractional difference uniformly integrable real sequences are equivalent. We represent another version of the Brudno-Mazur Orlicz bounded consistency theorem by using modulus function, lacunary sequence, and fractional difference operator. We show that the four-dimensional $RH-$regular matrices $\mathcal{A}$ and $\mathcal{B}$ are modulus lacunary fractional difference consistent over the multipliers space of modulus fractional difference $\mathcal{A}-$summable sequences and an algebra Z.

Key Words: Lacunary sequence, modulus function, fractional double difference operator, statistical convergence, four-dimensional $RH-$regular summability matrix, multiplier space.

Contents

1. Introduction

In [8] Dutta and Baliarsingh introduced fractional difference operators $\Delta^{\alpha}, \Delta^{(\alpha)}, \Delta^{-\alpha}$, $\Delta^{(-\alpha)}$ and discussed some topological results concerning the spaces thus formed. Baliarsingh et al. [3] studied approximation theorems and statistical convergence in fractional difference sequence spaces. Recently, Choudhary and Raj [7] investigated some interesting results on fractional difference double sequence space.
The set $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. Let ℓ^2_{∞} denotes the space of bounded double sequences. The fractional difference operator $\Delta^{(\alpha)}$ for a positive proper fraction α on a single sequence is defined as

$$\Delta^{(\alpha)}(x_i) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} x_{i-m},$$

where $\Gamma(\alpha)$ denotes generalized factorial function. The double difference operator of fractional order α is defined as

$$\Delta_2^{(\alpha)}(x_{i,j}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{m+n} \frac{\Gamma(\alpha+1)^2}{m!n!\Gamma(\alpha-m+1)\Gamma(\alpha-n+1)} x_{i-m,j-n}.$$

The above-defined infinite series can be reduced to finite series if α is a positive integer (see [4]). The generalized version of difference operator was studied by Kadak [30]. For more details on fractional difference operator see ([2], [11] and [35]).
In operator, spectral, and matrix theories, the investigation of sequence spaces performs an effectual role. As a matter of fact, the theory of difference sequence spaces plays an important role in enveloping the classical theory of fractional calculus and numerical analysis. The theory of fractional calculus deals with the examination of derivatives and integrations of a function with arbitrary orders. The application of fractional derivatives becomes more apparent in modeling mechanical and electrical properties of real

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materials as well as in the description of rheological properties of rocks and in numerous different fields. More investigations on fractional calculus and its several applications to real-world problems including ordinary and partial differential equations in applied mathematics and fluid mechanics. Specifically, the theory of fractional derivatives has been broadly utilised in the study of fractal theory, the theory of control of dynamic systems, the theory of visco-elasticity, electrochemistry, diffusion processes etc. The concept of modulus function was introduced by Nakano [20]. For definition and results one can see in ([1], [26], [28]). The space of lacunary strongly convergent sequence was defined by Freedman et al. [9] as follows:

$$N_\theta = \Big\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} |x_i - L| = 0, \ \text{ for some } \ L \Big\}.$$

To know more about lacunary sequence spaces one can refer to ([18], [24], [25], [32], [33]). Móricz [17] extended convergent and null single sequence spaces to double sequence spaces. Taş and Orhan [31] gave the characterization of $q-$Cesaro convergence for double sequences. In [22] Orhan gave some inequalities between functionals on bounded sequences.

Let $\mathcal{A} = (a_{klij})$ be an infinite four-dimensional matrix of real or complex numbers a_{klij}, where $i,\ j,\ k,\ l \in \mathbb{N}$. The $\mathcal{A}$ transform of $x = (x_{ij})$ is written as $\mathcal{A}x$ and $\mathcal{A}x = \{(\mathcal{A}x)_{kl}\}$ defined by $(\mathcal{A}x)_{kl} = \sum_{i,j} a_{klij} x_{ij}$ converges for each $k, l \in \mathbb{N}$. A four-dimensional matrix $\mathcal{A} = (a_{klij})$ is said to be $RH-$regular or bounded regular (see [10], [27]) if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robinson-Hamilton conditions state that the four-dimensional matrix $\mathcal{A} = (a_{klij})$ is $RH-$regular if and only if

(RH$_1$) $P\text{-}\lim_{k,l} a_{klij} = 0$ for each $(i,j) \in \mathbb{N}^2$,

(RH$_2$) $P\text{-}\lim_{k,l} \sum_{i,j\in\mathbb{N}^2} a_{klij} = 1$,

(RH$_3$) $P\text{-}\lim_{k,l} \sum_{j} |a_{klij}| = 1$ for each i,

(RH$_4$) $P\text{-}\lim_{k,l} \sum_{i} |a_{klij}| = 1$ for each j,

(RH$_5$) $\sum_{(i,j)\in\mathbb{N}^2} |a_{klij}|$ is $P-$convergent $\forall\ k,\ l$,

(RH$_6$) there exist finite positive integers r and s such that $\sum_{i,j>s} |a_{klij}| < r$ holds for every $(k,l) \in \mathbb{N}^2$.

A four dimensional matrix $\mathcal{A} = (a_{klij})$ is said to be inner finite matrix, if there exist $P(k,l)$ and $S(k,l)$ such that $a_{klij} = 0$ whenever $i > P(k,l)$ or $j > S(k,l)$. A four-dimensional matrix $\mathcal{A}$ is said to be inner-rectangular if $P(k,l) = k$ and $S(k,l) = l$. For an inner rectangular matrix $\mathcal{A} = (a_{klij})$, the inner matrix $[\mathcal{A}(k,l)_{i,j}]$ can be represented as follows:

$$[\mathcal{A}(k,l)_{i,j}] = \begin{pmatrix} a_{kl11} & a_{kl12} & \cdots & a_{kl1l} & 0 & 0 & \cdots \\ a_{kl21} & a_{kl22} & \ddots & a_{kl2l} & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ a_{klk1} & a_{klk2} & \cdots & a_{klkl} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

For $[\mathcal{A}(k,l)_{i,j}] = (a_{klij})$ an inner-rectangular matrix,$[\mathcal{A}(k,l)_{i,j}]^{(n)}$ represents the matrix in which exactly $n-$terms of the first $kl-$terms are zero.

In 1945, Brudno [6] stated that if $\mathcal{A}$ and $\mathcal{B}$ are regular summability matrix methods such that every bounded sequence summed by $\mathcal{A}$ is also summed by $\mathcal{B}$, then it is summed by $\mathcal{B}$ to the same value as $\mathcal{A}$. Mohiuddine [19] studied statistical weighted version of $A-$summability. Initiated by Mazur and Orlicz [13], several mathematicians obtained the variant of Brudno theorem ([5], [14], [29]). In [23] Patterson obtained multidimensional analog of Brudno theorem for double sequences using four-dimensional matrix method to give accessible proof of this theorem. Khan and Orhan [12] showed another version of the Brudno-Mazur-Orlicz theorem by characterizing the set of multipliers of $\mathcal{A}$, over an algebra. Moreover, Miller and Miller Van-Wieren [16] presented the matrix characterizations of statistical convergence of double sequences. In [21] Orhan and Ünver gave Brudno-Mazur Orlicz bounded consistency theorem.
Inspired essentially by the above mentioned study, we represent another version of the Brudno-Mazur Orlicz bounded consistency theorem by using modulus function, lacunary sequence, and fractional difference operator. We show that the four-dimensional $RH-$ regular matrices $\mathcal{A}$ and $\mathcal{B}$ are modulus lacunary fractional difference consistent over the multipliers space of modulus fractional difference $\mathcal{A}-$summable sequences and an algebra Z. We obtain certain matrix characterization of modulus $\mathcal{A}-$lacunary statistical convergence of fractional difference double sequences and uniformly integrable real sequences.

Now we give certain new definitions and notions that are used in this paper.

Definition 1.1. *A double sequence $x = (x_{ij})$ is said to be modulus lacunary fractional difference $\mathcal{A}-$summable if*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) a_{klij} = L$$

and the modulus lacunary fractional difference $\mathcal{A}-$summable limit of $x = (x_{ij})$ is denoted by $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)$ and is defined as follows:

$$\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) a_{klij}.$$

By $\mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ we denote the space of modulus lacunary fractional difference $\mathcal{A}-$summable sequences.

Definition 1.2. *Let $\mathcal{F} = (f_{ij})$ be a double sequence of modulus functions and $\mathcal{A} = (a_{klij})$ be a non-negative RH-regular summability matrix. The space of modulus lacunary fractional difference $\mathcal{A}-$bounded double sequences is denoted by $\mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ and is defined as follows:*

$$\mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} = \Big\{(x_{ij}) : \sup_{k,l\geq G} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|a_{klij}| < \infty, \text{ for some } G > 0\Big\}.$$

Definition 1.3. *A double sequence $x = (x_{ij})$ is said to be modulus $\mathcal{A}-$lacunary statistically fractional difference convergent to L, if for any $\epsilon > 0$,*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}: f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|\geq\epsilon} a_{klij} = 0.$$

By $\mathcal{S}_{\theta,\mathcal{A}}^{2,\mathcal{F},\Delta_2^{(\alpha)}}$, we denote the set of modulus $\mathcal{A}-$lacunary statistically fractional difference convergent sequences and is defined as follows:

$$\mathcal{S}_{\theta,\mathcal{A}}^{2,\mathcal{F},\Delta_2^{(\alpha)}} = \Big\{(x_{ij}) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \Big|\{(i,j) \in I_{r,s} : |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}| - L)a_{klij}| < \epsilon\}\Big| = 0\Big\}.$$

Proposition 1.4. *([15]) Let $\mathcal{A} = (a_{klij})$ be a non-negative inner rectangular matrix of four-dimensions such that*

$$\sum_{i,j} a_{klij} = 1, \text{ for all } k, l.$$

If $\mathcal{A}-$statistical sequence $x = (x_{ij})$ converges to L, then there will be a $K \subset \mathbb{N}^2$ such that

$$\delta_{\mathcal{A}}^{(2)}(K) = P - \lim_{k,l} \sum_{i,j \in K} a_{klij} = 0$$

and $P - \lim_{(i,j) \in K^c} x_{ij} = L$.

Definition 1.5. *A sequence $x = (x_{ij})$ is called modulus $\mathcal{A}-$lacunary strongly fractional difference converging to L, if*

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|)| \, a_{klij} = 0.$$

Definition 1.6. *A sequence $x = (x_{ij})$ is said to be modulus $\mathcal{A}-$lacunary fractional difference uniformly integrable if for $\epsilon > 0$, there exist $Q = Q(\epsilon)$ and $t = t(\epsilon)$, such that for all $q > t$*

$$\sup_{k,l > Q} \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} : |\Delta_2^{(\alpha)} x_{ij}| > q} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)| \, |a_{klij}| < \epsilon.$$

By $\mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, we denote the space of modulus $\mathcal{A}-$lacunary fractional difference uniformly integrable real sequences.

A sequence $x = (x_{ij})$ is said to be $\mathcal{A}-$strongly summable if it is Pringsheim $\mathcal{A}-$uniformly integrable and $\mathcal{A}-$statistically convergent(see [34]).
Let X and Y be two double sequence spaces and Z be a sequence space and an algebra that is for all $(x_{ij}), (y_{ij}) \in Z$, $(x_{ij}y_{ij}) = (z_{ij}) \in Z$. The set

$$M_Z^2(X, Y) = \{x \in Z : \text{ for each } y \in X, xy \in Y\}$$

is known as the multiplier space over Z. By $M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$, we denote the multiplier space $M_Z^2(\mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}, Z)$.

2. Main Results

Theorem 2.1. *Let $\mathcal{A} = (a_{klij})$ and $\mathcal{B} = (b_{ijkl})$ be RH−regular four-dimensional summability matrices. If*

$$\mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \subseteq \mathcal{L}_{\infty,\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}},$$

then $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = \mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)$, for all $x \in \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$.

Proof. Let $\mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \subseteq \mathcal{L}_{\infty,\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ and a double sequence $x = (x_{ij}) \in \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ such that $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) \neq \mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)$. Without loss of generality suppose that $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = 0$ and $\mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = 1$. Now it is given that $x \in \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$. Then there exists a $G \in \mathbb{N}$ such that

$$\frac{1}{h_{r,s}} \sum_{i,j \in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)| \, |a_{klij}|$$

is convergent for all $k, l > G$. Hence, for all $\epsilon > 0$ and for all $k, l > G$, there exist positive integers $Q = Q(k, l), R = R(k, l)$ such that

$$\frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} : i \leq Q, j > R} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)| \, |a_{klij}| + \frac{1}{h_{r,s}} \sum_{(i,j) \in I_{r,s} : i > Q, j \leq R} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)| \tag{2.1}$$

$$|a_{klij}| + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>Q,j>R} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| < \epsilon.$$

Let $P(0) = S(0) = 0$ and $M(1), N(1) > G$. One can choose $P(1), S(1)$ from (2.1) such that

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(1),j>S(1)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(1),j\le S(1)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}|$$

$$+\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(1),j>S(1)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \quad < \quad 1$$

whenever $G < k \le M(1), G < l \le N(1)$. Now, by $P - \lim_{k,l} a_{klij} = P - \lim_{k,l} b_{ijkl} = 0$, we can choose $M(2) > M(1)$ and $N(2) > N(1)$ such that

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(1),j\le S(1)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| < 1$$

whenever $k > M(2)$ and $l > N(2)$. By means of the similar argument we can select the indices

$$M(1) < M(2) < \cdots < M(n), \qquad N(1) < N(2) < \cdots < N(n)$$

$$P(1) < P(2) < \cdots < P(n-1), \qquad S(1) < S(2) < \cdots < S(n-1)$$

for some $n \ge 1$. Now, select $P(n) > P(n-1)$ and $S_n > S_{n-1}$ such that

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(n),j>S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(n),j\le S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(n),j>S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| < \frac{1}{n} \tag{2.2}$$

whenever $G < k \le M(n)$ and $G < l \le N(n)$. Now choose $M(n+1) > M(n)$ and $N(n+1) > N(n)$ so that

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(n),j\le S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| < \frac{1}{n} \tag{2.3}$$

whenever $k > M(n+1)$ and $l > N(n+1)$. Define the index sets by

$$\mathcal{I}(p) = \{(i,j) \in I_{r,s} : 1 \le i \le P(p+1), 1 \le j \le S(p+1)\}, \quad \mathcal{I}(-1) = \phi$$

and

$$\mathcal{T}(p,u) = \mathcal{I}(u) \setminus \mathcal{I}(p)$$

for all $p, u = 0, 1, 2, 3, \cdots$. By using (2.2), we have for $G < k \le M(n)$ and $G < l \le N(n)$,

$$\sum_{p>n-1} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \tag{2.4}$$

$$\begin{aligned} &= \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(n),j>S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \\ &\quad +\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(n),j\le S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \\ &\quad +\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i>P(n),j>S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \\ &< \frac{1}{n}. \end{aligned}$$

From (2.3), we have

$$\sum_{0\le p<n} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\,|a_{klij}| \tag{2.5}$$

$$= \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:i\le P(n),j\le S(n)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\,|a_{klij}| < \frac{1}{n},$$

for all $k > M(n+1)$ and $l > N(n+1)$.
Also, for all $k, l \in \mathbb{N}$, we have

$$(Ax)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} = \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \tag{2.6}$$

$$\begin{aligned} &= \sum_{p=0}^{\infty} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\ &= \sum_{0\le p<n-1} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\,|a_{klij}| \\ &\quad + \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(n-2,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\ &\quad + \sum_{p>n} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\,|a_{klij}|. \end{aligned}$$

Thus, from (2.4), (2.5), and (2.6), we have

$$(Ax)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} = O\left(\frac{1}{n}\right) + \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(n-2,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij}, \tag{2.7}$$

for all $M(n) < k \le M(n+1)$ and $N(n) < l \le N(n+1)$. In the same manner, one can easily get

$$(Bx)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} = O\left(\frac{1}{n}\right) + \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(n-2,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)b_{ijkl}.$$

Define a sequence $z = (z_{ij})$ by $f_{ij}|\Delta_2^{(\alpha)} z_{ij}| = f_{ij}|\Delta_2^{(\alpha)} x_{ij}|\psi_p$, for $(i,j) \in \mathcal{T}(p-1, p-2)$ where $\psi = (\psi_n) \in (0,1]$ for all n and $\lim_n(\psi_{n+1} - \psi_n) = 0$. Note that $z \in \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ by construction. Since ψ is a bounded sequence,

$$\mathcal{T}(n-2,n-1) = \mathcal{T}(n-2,n-1) \cup \mathcal{T}(n-2,n) \text{ and } \mathcal{T}(n-2,n-1) \cap \mathcal{T}(n-1,n) = \phi.$$

So, we have

$$(Az)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} = \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} z_{ij}|)a_{klij} \tag{2.8}$$

$$\begin{aligned} &= \sum_{p=0}^{\infty} \psi_{p+1} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\ &= \sum_{0\le p<n-1} \frac{1}{h_{r,s}} \sum_{(i,j)\in\mathcal{T}(p-1,p)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \end{aligned}$$

$$
\begin{aligned}
&+\psi_n \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-2,n-1)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
&+\psi_{n+1} \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-1,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
&+\sum_{p>n} \psi_{n+1} \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(p-1,p)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
= \ & O\Big(\frac{1}{n}\Big) + \psi_n \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-2,n-1)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
&+\psi_{n+1} \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-1,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
= \ & O\Big(\frac{1}{n}\Big) + \psi_n \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-2,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \\
&+(\psi_{n+1}+\psi_n) \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-1,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij},
\end{aligned}
$$

for all $n \in Q$ and $M(n) < k \le M(n+1), N(n) < l \le N(n+1)$. By using (2.7), we have $\psi_n \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{T}(n-2,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij}$ is the order of $\psi_n\big((\mathcal{A}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} + O(\frac{1}{n})\big)$. Hence, $\psi_n\big((\mathcal{A}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} + O(\frac{1}{n})\big)$ goes to zero since $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = 0$. Also, $x \in \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ is given. Then there exists $\mathcal{N} > 0$ such that

$$
\begin{aligned}
&\sup_{k,l>\mathcal{N}} \left| \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{I}(n,n)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij} \right| \qquad (2.9) \\
&\le \sup_{k,l>\mathcal{N}} \frac{1}{h_{r,s}} \sum_{(i,j)\in \mathcal{I}(n,n)} \big|(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)a_{klij}\big| \\
&< \infty.
\end{aligned}
$$

In (2.8), the end term approaches zero since ψ is slowly oscillating. Hence, $(\mathcal{A}z)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}}$ goes to zero as $k, l \to \infty$. In the same manner, we have

$$
(\mathcal{B}z)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} = O\Big(\frac{1}{n}\Big) + \psi_n\bigg((\mathcal{B}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} + O\Big(\frac{1}{n}\Big)\bigg), \qquad (2.10)
$$

for all $n \in Q, M(n) < k \le M(n+1)$ and $N(n) < l \le N(n+1)$. Since $\mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = 1$ and ψ oscillates between 0 and 1. Then from (2.10), we have $P - \lim_{k,l} (\mathcal{B}z)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}}$ does not exist which is a contradiction. □

Theorem 2.2. *Let $\Delta_2^{(\alpha)}$ be a fractional double difference operator, $\mathcal{A} = (a_{klij})$ and $\mathcal{B} = (b_{ijkl})$ be RH−regular four-dimensional summability matrices and let $\ell_\infty^2 \subseteq Z \subseteq \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap \mathcal{L}_{\infty,\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}}$. If $\mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap Z \subseteq \mathcal{C}_{\theta,\mathcal{B}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap Z$, then $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = \mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)$, for all $x \in M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$.*

Lemma 2.3. *Let $\Delta_2^{(\alpha)}$ be a fractional double difference operator. If $\mathcal{A} = (a_{klij})$ is RH-regular summability matrix, then there will be an inner-rectangular summability matrix $D = (d_{klij})$ of four-dimension such that for all k, l*

$$
\sum_{i,j} d_{klij} = 1
$$

and $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = D_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x),\ \forall\, x \in \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$.

Proof. Since $\mathcal{A} = (a_{klij})$ is an RH-regular summability matrix then $\sum_{i,j} |a_{klij}|$ is $P-$convergent for each k ad l. Now there exist $P(k,l), S(k,l)$ such that

$$\sum_1 |a_{klij}| + \sum_2 |a_{klij}| + \sum_3 |a_{klij}| < \epsilon_{kl}, \tag{2.11}$$

where $\sum_1$ denotes the sum over $\{(i,j) \in I_{r,s} : 1 \leq i \leq P(k,l), j > S(k,l)\}$, $\sum_2$ denotes the sum over $\{(i,j) \in I_{r,s} : i > P(k,l), 1 \leq j \leq S(k,l)\}$, $\sum_3$ denotes the sum over $\{(i,j) \in I_{r,s} : i > P(k,l), j > S(k,l)\}$, $\epsilon_{kl} > 0$, for all k,l and $\epsilon_{kl} \to 0$ as $k,l \to \infty$. Since $x \in \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ for all $\epsilon > 0$ there exist $Q = Q(\epsilon)$ and $t = t(\epsilon)$ such that for all $q > t$

$$\sup_{k,l>Q} \frac{1}{h_{r,s}} \sum_{|\Delta_2^{(\alpha)} x_{ij}|>q} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| < \frac{\epsilon}{3}. \tag{2.12}$$

Define a four-dimensional inner matrix $\mathcal{E} = (e_{klij})$ as follows:

$$e_{klij} = \begin{cases} a_{klij}, & i \leq P(k,l) \text{ and } j \leq S(k,l); \\ 0, & \text{otherwise}. \end{cases}$$

Now from (2.12), we have

$$|(\mathcal{A}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} - (\mathcal{E}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}}| \tag{2.13}$$

$$\begin{aligned}
&= \left| \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) a_{klij} - \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) e_{klij} \right| \\
&= \left| \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) a_{klij} - \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}: 1\leq i\leq P(k,l), 1\leq j\leq S(k,l)} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) a_{klij} \right| \\
&\leq \frac{1}{h_{r,s}} \sum_4 |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + \frac{1}{h_{r,s}} \sum_5 |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + \frac{1}{h_{r,s}} \sum_6 |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)| \\
&\quad |a_{klij}| \\
&\leq 3 \sup_{k,l\geq Q} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}: |\Delta_2^{(\alpha)} x_{ij}|>q} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| + q \sum_4 |a_{klij}| + q \sum_5 |a_{klij}| \\
&\quad + q \sum_6 |a_{klij}| \\
&\leq \epsilon + q \left\{ \sum_4 |a_{klij}| + \sum_5 |a_{klij}| + \sum_6 |a_{klij}| \right\},
\end{aligned}$$

where $\sum_4$ denotes the sum over $\{(i,j) \in I_{r,s} : 1 \leq i \leq P(k,l), j > S(k,l)\}$, where $\sum_5$ denotes the sum over $\{(i,j) \in I_{r,s} : i > P(k,l), 1 \leq j \leq S(k,l)\}$ and where $\sum_6$ denotes the sum over $\{(i,j) \in I_{r,s} : i > P(k,l), j > S(k,l)\}$. Hence, from (2.11) and (2.13), we have

$$P - \lim_{k,l} |(\mathcal{A}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} - (\mathcal{E}x)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}}| = 0.$$

Hence, $\mathcal{A}$ and $\mathcal{E}$ are equivalent over $\mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$. Define $\gamma_{kl} = \sum_{i,j} e_{klij} = \sum_{1\leq i\leq P(k,l), 1\leq j\leq S(k,l)} a_{klij}$. As we know that $\mathcal{A}$ is an RH-regular, thus we have $P-\lim_{k,l}\gamma_{k,l} = 1$. Let us suppose that $\gamma_{k,l} \neq 0$. Now, define $\mathcal{B} = (b_{klij})$ by

$$b_{klij} = \begin{cases} \frac{1}{\gamma_{k,l}} a_{klij}, & i \leq P(k,l) \text{ and } j \leq S(k,l); \\ 0, & \text{elsewhere.} \end{cases}$$

Since $\mathcal{E}$ and $\mathcal{B}$ are equivalent, thus for all k, l we obtain that

$$\sum_{i,j} b_{klij} = 1.$$

Let

$$\xi(n) = \max\{\max_{1\leq p\leq n+1} P(p, n+1), \max_{1\leq p\leq n+1} P(n+1, p)\},$$

$$\phi(n) = \max\{\max_{1\leq p\leq n+1} S(p, n+1), \max_{1\leq p\leq n+1} S(n+1, p)\},$$

$$\psi(n) = \max\{\xi(n), \phi(n)\}, \text{ and } \xi(0) = \phi(0) = 0.$$

If $\psi(n) > n + 1 + \psi(n-1)$, $\forall\, n = 1, 2, \cdots$, then the matrix $[D(k + \psi(n) - 1, l + \psi(n) - 1)_{i,j}]$ will be the matrix $[\mathcal{B}(k,l)_{i,j}]$ $\forall\, k, l$, where $k, l \geq \psi(n-1) + n + 1$. From $U(n)$, we can arbitrarily select the other inner matrices of D where

$$U(n) = V(n) \setminus V(n-1), V(n) = \{[\mathcal{B}(k,l)_{i,j}] : 1 \leq k, l \leq n+1\}$$

and $V(0) = \phi$. Since the inner matrices of $\mathcal{B}$ on repetitions provides the inner matrices of D in the finite rows and columns. Therefore, we conclude that $\mathcal{B}$ and D are equivalent. Hence, $\sum_{i,j} d_{klij} = 1$ holds. □

Theorem 2.4. *Let $\mathcal{A} = (a_{klij})$ be RH-regular four-dimensional matrix and Z be an algebra such that $s = (s_{ij}) \in Z$ and $s_{ij} = 1$ for all i, j. Then the statements given below are true:*
(1) $M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$ is an algebra.
(2) If $\ell_\infty^2 \subseteq Z \subseteq \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, then $\mathcal{A}$ is multiplicative over $M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$ that is

$$\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(xy) = \mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(y),$$

for all $x, y \in M_{\theta,}^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$.
(3)If $\ell_\infty^2 \subseteq Z \subseteq \mathcal{L}_{\infty,\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$ and $\mathcal{A}$ is non-negative then

$$M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z) \subseteq \{x \in Z : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|)|^r a_{klij} = 0, \textit{ for some } L \textit{ and each } r \geq 1\}.$$

(4)If $\ell_\infty^2 \subseteq Z \subseteq \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, then

$$\{x \in Z : P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|)|\, |a_{klij}| = 0, \textit{ for some } L\} \subseteq M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z).$$

Proof. One can easily proof the first three parts by using Theorem 2.2 in the same manner as in Theorem 2.1 in [12]. To prove (4), we use Lemma 2.3. Consider an inner rectangular matrix $\mathcal{A}$ such that $\forall\, k, l$,

$\sum_{i,j} a_{klij} = 1,.$

Assume that $x = (x_{ij}) \in Z$ ad $y = (y_{ij}) \in \mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}} \cap Z$ such that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|)|\, |a_{klij}| = 0$$

and

$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}$

$$= L(\mathcal{A}y)_{kl}^{\theta,\mathcal{F},\Delta_2^{(\alpha)}} + \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij} - L|) \Delta_2^{(\alpha)} y_{ij} a_{klij}.$$

Without loss of generality let us assume $L = 0$ and define $\mathcal{B} = (b_{klij})$ by $b_{klij} = |a_{klij}|$, for all i, j, k, l. It can be seen x is $\mathcal{B}-$statistically convergent to zero. Thus, by Proposition 1.4, we can find a subset $H \subset \mathbb{N}^2$ having the property that $\delta_{\mathcal{B}}^2(H) = 0$ and $P - \lim_{(i,j)\in H^C} (f_{ij}|\Delta_2^{(\alpha)} x_{ij}|) = 0$. Since $xy \in \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, for all $\epsilon > 0$, there exist $Q(\epsilon)$ and $G(\epsilon)$ such that for all $d > G(\epsilon)$

$$\sup_{k,l>Q} \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}:|\Delta_2^{(\alpha)} x_{ij}|>d} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|)|\, |a_{klij}| < \epsilon.$$

For all $k, l > Q$ and for fixed $R > 0$

$$\left| \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} (f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij} \right| \tag{2.14}$$

$$\begin{aligned} &\leq \frac{1}{h_{r,s}} \sum_{7} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}| + \frac{1}{h_{r,s}} \sum_{8} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}| \\ &\leq \sup_{k,l>Q} \frac{1}{h_{r,s}} \sum_{7} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}| + R \sum_{i,j\in H} |a_{klij}| + \frac{1}{h_{r,s}} \sum_{9} |(f_{ij} \\ &\quad |\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}|, \end{aligned}$$

where $\sum_{7}$ denotes the sum over $\{(i,j) \in I_{r,s} : |\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}| > R\}$, $\sum_{8}$ denotes the sum over $\{(i,j) \in I_{r,s} : |\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}| \leq R\}$ and $\sum_{9}$ denotes the sum over $\{(i,j) \in I_{r,s} \cap H^C : |\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}| \leq R\}$. Since $(xy) \in \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, the first term will be small enough by considering R large enough. In Pringsheim's sense, $R \sum_{k,l\in H} |a_{klij}|$ goes to zero since $\delta_{\mathcal{B}}^2(H) = 0$. Now, we have

$$\frac{1}{h_{r,s}} \sum_{9} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}| \tag{2.15}$$

$$\begin{aligned} &\leq \frac{1}{h_{r,s}} \sum_{10} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij} \Delta_2^{(\alpha)} y_{ij}|) a_{klij}| + S \sum_{(i,j)\in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \\ &\leq \frac{R}{S} \frac{1}{h_{r,s}} \sum_{11} |(f_{ij}|\Delta_2^{(\alpha)} y_{ij}|) a_{klij}| + S \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} |(f_{ij}|\Delta_2^{(\alpha)} x_{ij}|)|\, |a_{klij}| \end{aligned}$$

for all k,l and $S>0$ where $\sum_{10}$ denotes the sum over $(i,j)\in I_{r,s}\cap H^C:|\Delta_2^{(\alpha)}x_{ij}\Delta_2^{(\alpha)}y_{ij}|\leq R$ and $|\Delta_2^{(\alpha)}y_{ij}|>S$ and $\sum_{11}$ denotes the sum over $(i,j)\in I_{r,s}\cap H^C:|\Delta_2^{(\alpha)}y_{ij}|>S$.

Now the first part is small enough for large S, since $y\in\mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$. From (2.14), (2.15) and hypothesis, we get $xy\in\mathcal{C}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$. Hence, we have $x\in M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)$. □

Corollary 2.5. *Let $\Delta_2^{(\alpha)}$ be a fractional double difference operator, $\mathcal{A}=(a_{klij})$ be a non-negative RH-regular four-dimensional summability matrix, Z be algebra and a sequence $s\in Z$. If $\ell_\infty^2\subseteq Z\subseteq\mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$, then $M_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(Z)=\mathcal{S}_{\theta,\mathcal{A}}^{2,\mathcal{F},\Delta_2^{(\alpha)}}\cap Z$.*

Theorem 2.6. *Let $\Delta_2^{(\alpha)}$ be a fractional double difference operator. If $\mathcal{A}=(a_{klij})$ is a non-negative RH-regular four-dimensional summability matrix then there will be a non-negative RH-regular inner-rectangular matrix $\mathcal{B}=(b_{klij})$ of four-dimensions such that $\mathcal{S}_{\theta,\mathcal{A}}^{2,\mathcal{F},\Delta_2^{(\alpha)}}-\lim x=\mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x)$, $\forall\, x\in\mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}$.*

Proof. From Lemma 2.3 let $\mathcal{A}$ be a non-negative inner rectangular matrix such that $\sum_{k,l}a_{klij}=1$, $\forall\, k,l$. Firstly, we construct a matrix $D=(d_{klij})$. For all $n=1,2,\cdots$ define $\varpi=\varpi(n)$, by

$$\varpi(n)=\frac{n^2+\frac{(-1)^{n+1}+1}{2}}{2}$$

and the index sets

$$\mathcal{I}(n)=\{(k,l):1\leq k,l\leq Q_n\}\text{ and }\mathcal{T}(n)=\mathcal{I}(n)\setminus\mathcal{I}(n-1)$$

where $\mathcal{I}(-1)=\emptyset$ and $Q_n=\sum_{j=1}^{n}2^{\varpi(j)}$.

Also, define a set

$$K_{uv}=\{(k,l):Q_{u-1}<k\leq Q_u,Q_{v-1}<l\leq Q_v\}$$

where $Q_0=0$. Note that the cardinality of K_{uv} is equal to $2^{\varpi(u)+\varpi(v)}$ and $\mathcal{T}(n)=\left(\bigcup_{u=1}^{n}K_{un}\right)\bigcup\left(\bigcup_{v=1}^{n-1}K_{nv}\right)$. The inner matrix $\binom{uv}{0}$ of D will be $[\mathcal{A}(u,v)_{i,j}]$ and $\binom{uv}{1}$ inner matrices of D will contain in $[\mathcal{A}(u,v)]^{(1)}$, $\binom{uv}{2}$ inner-matrices of D will contain in $[\mathcal{A}(u,v)]^{(2)}$ and so on. Hence, for $\mathcal{T}(n)$, we consider the similar argument for $K_{u_1},K_{u_2},\cdots,K_{u_n}$ and $K_{n1},K_{n2},\cdots,K_{(n-1)v}$. Note that there are $\sum_{j=0}^{uv}\binom{uv}{j}=2^{uv}$ possibilities. Since K_{uv} consists of $2^{\varpi(u)+\varpi(v)}$ elements, we will write ζ for the other $2^{\varpi(u)+\varpi(v)}-2^{uv}$ inner matrices, where ζ is the infinite matrix with all entries as zeros. Define $\mathcal{B}=(b_{klij})$ by

$$b_{klij}=(1-\vartheta_{wp})a_{klij}+d_{wpij}$$

where $\vartheta_{wp}=\sum_{i\leq w,j\leq p}d_{wpij}$, for $(w,p)\in K_{kl}$. Since D and $\mathcal{A}$ have similar inner matrices. Hence, $0\leq\vartheta_{wp}\leq 1$, for all w,p. Since $P-\lim_{w,p}d_{wpij}=0$ and ϑ is bounded

$$P-\lim_{w,p}b_{wpij}=P-\lim_{w,p}((1-\vartheta_{wp})a_{klij}+d_{wpij})=0$$

and since $w \geq k$ and $p \geq l$

$$\begin{aligned}\sum_{i\leq w,j\leq p} b_{wpij} &= \sum_{i\leq w,j\leq p}((1-\vartheta_{wp})a_{klij}+d_{wpij})\\ &= (1-\vartheta_{wp})\sum_{i\leq w,j\leq p} a_{klij} + \sum_{i\leq w,j\leq p} d_{wpij}\\ &= (1-\vartheta_{wp})+\vartheta_{wp}\\ &= 1.\end{aligned}$$

Thus, the conditions $RH1$ and $RH2$ are satisfied. Now the conditions $RH3$, $RH4$, $RH5$ and $RH6$ is obvious since D and $\mathcal{A}$ have similar inner matrices with the exception that exactly the finite number of ζ matrices have been added.

Assume that x be a modulus $\mathcal{A}-$lacunary fractional difference uniformly integrable that is modulus $\mathcal{A}-$lacunary statistically fractional difference convergent to L and $y_{ij} = x_{ij} - L$. Then the double sequence y_{ij} is modulus $\mathcal{A}-$lacunary statistically fractional difference convergent to zero. Hence, from Proposition 1.4, $P - \lim_{(i,j)\in K^C} f_{ij}|\Delta_2^{(\alpha)} y_{ij}| = 0$. For any $(w,p) \in K_{kl}$, we have

$$\begin{aligned}&\left|\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)b_{wpij}\right|\\ &\leq \left|\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}:|\Delta_2^{(\alpha)}y_{ij}|>R}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)b_{wpij}\right| + \left|\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}:|\Delta_2^{(\alpha)}y_{ij}|\leq R}(f_{ij}|\Delta_2^{(\alpha)} y_{ij}|)b_{wpij}\right|\\ &\leq \sup_{k,l\geq G}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}:|\Delta_2^{(\alpha)}y_{ij}|>R}|(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)|b_{wpij} + R\sum_{i,j\in K} b_{wpij}\\ &\quad + \left|\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}\cap K^C:|\Delta_2^{(\alpha)}y_{ij}|\leq R}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)b_{wpij}\right|\\ &\leq 2\sup_{k,l\geq G}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}:|\Delta_2^{(\alpha)}y_{ij}|>R}|(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)|\,a_{klij} + 2R\sum_{(i,j)\in K} a_{klij}\\ &\quad + \left|\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}\cap K^C:|\Delta_2^{(\alpha)}y_{ij}|\leq R}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)b_{wpij}\right|.\end{aligned}$$

The last term approaches zero since the matrix $\mathcal{B}$ is RH regular and the sequence which is summed up in the summation given in the end term is bounded as well as convergent. Also, the second term tends to zero in Pringsheim's sense since as w, p get large, k, l also gets large. We take R large enough so that the first term became arbitrarily small. Thus, $\mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(y) = 0$. We can conclude that $\mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = L$.

To prove the converse part, let us assume that $x \in \mathcal{V}_{\theta,\mathcal{A}}^{\mathcal{F},\Delta_2^{(\alpha)}}, \mathcal{B}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(x) = L$ and $y_{ij} = x_{ij} - L$. Hence, we have $[\mathcal{A}(w,p)_{i,j}]$ corresponds to the first pair of K_{wp} and $\mathcal{A}_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(y) = 0$. Hence, $D_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(y) = 0$. Now choose $(w,p) \in K_{kl}$ such that the negative signs of the $k \times l$ terms of y are precisely where the terms of the inner matrix $[\mathcal{A}(k,l)_{i,j}]$ are not replaced by zero. Since $D_\theta^{\mathcal{F},\Delta_2^{(\alpha)}}(y) = 0$, we have

$$P - \lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)\chi_{(\Delta_2^{(\alpha)}y_{ij}<0)}(j)a_{klij} = 0,$$

for a characteristic function χ. In the similar manner, for positive terms, we have

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)\chi_{(\Delta_2^{(\alpha)}y_{ij}\geq 0)}(j)a_{klij}=0.$$

Hence,

$$P-\lim_{r,s}\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}|(f_{ij}|\Delta_2^{(\alpha)}y_{ij}|)|\,a_{klij}=0.$$

Thus, y is modulus $\mathcal{A}-$lacunary strongly fractional difference summable to zero that is, it is modulus $\mathcal{A}-$lacunary statistically fractional difference convergent to zero. □

3. Conclusion

In the present work, first we prove that the modulus $\mathcal{A}-$lacunary statistical convergence of fractional difference double sequences and modulus lacunary fractional matrix of four-dimensions taken over the space of modulus $\mathcal{A}-$lacunary fractional difference uniformly integrable real sequences are equivalent. We discuss another form of the Brudno-Mazur Orlicz bounded consistency theorem by means of modulus function, lacunary sequence, and fractional difference operator. Further, we show that the four-dimensional $RH-$ regular matrices $\mathcal{A}$ and $\mathcal{B}$ are modulus lacunary fractional difference consistent over the multipliers space of modulus fractional difference $\mathcal{A}-$summable sequences and an algebra Z. Researchers may also study similar results for generalized difference operator on double sequence spaces by Orlicz function.

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