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On Some New Results for the q-Series

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ABSTRACT: This paper deals with some new results for the q-series. The paper is divided into four sections. In section two, we aim to establish three interesting results closely related to the q-analogue of the Kummer's second summation theorem obtained earlier by Srivastava and Jain. As limiting cases, we recover known results obtained earlier by Kim, et al.. In third section , we obtain a q-contiguous quadratic transformation formula closely related to q-quadratic transformation formula obtained earlier by Jain. As limiting case, we recover a known result obtained earlier by Kim and Rathie.

In fourth section, an attempt has been made to obtain a q-contiguous result closely related to Reed Dawson identity obtained earlier by Andrews. As limiting case, we recover a result due to Choi, et al. . An application of our results are given in section fifth.

Key Words: Hypergeometric Series, q-Series, Summation Theorems, Identities, Reed Dawson Identity, Transformation Formula.

Contents

1. Introduction

We begin by recalling the definition of generalized hypergeometric function ${}_rF_s$ as follows [3]

$$ {}_rF_s\left[\begin{array}{cccc} a_1, & \ldots, & a_r & \\ & & & ; \quad z \\ b_1, & \ldots, & b_s & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_n}{\prod_{j=1}^{s}(b_j)_n}\frac{z^n}{n!} \tag{1.1}$$

where

$$(a_j)_n = \begin{cases} a_j(a_j+1)\ldots(a_j+n-1) = \prod_{i=0}^{n-1}(a_j+i) & ; n \neq 0 \\ 1 & ; n = 0 \end{cases}$$

where $(a_j)_n$ is the shifted factorial function.

For more details about this function including convergence conditions and properties, we refer the standard texts [3,13].

In a similar manner, the basic or q- hypergeometric series ${}_r\phi_s$ as follows [1]

$$ {}_r\phi_s\left[\begin{array}{cccc} a_1, & \ldots, & a_r & \\ & & & ; \quad q, \ z \\ b_1, & \ldots, & b_s & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i;q)_n}{(q;q)_n \ \prod_{j=1}^{s}(b_j;q)_n}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+s-r} z^n \tag{1.2}$$

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where

$$(a;q)_n = \begin{cases} (1-a)(1-aq)\dots(1-aq^{n-1}) & ; n \neq 0 \\ 1 & ; n = 0 \end{cases}$$

For a detail results about this q-series, we refer the standard text [1,3].

In the theory of hypergeometric and generalized hypergeometric function, the Kummer's second theorem [13] is recorded as follows:

$$e^{-\frac{1}{2}x} \; {}_1F_1\left[\begin{array}{c} a \\ \\ 2a \end{array} ; \; x\right] = {}_0F_1\left[\begin{array}{c} - \\ \\ a+\frac{1}{2} \end{array} ; \; \frac{x^2}{16}\right] \tag{1.3}$$

Kummer [13] established this result from the theory of differential equations. Bailey [2] re-derived this result by employing Gauss's second summation theorem. Later on Rathie and Choi [14] re-derived this result by using classical Gauss's summation theorem.

From (1.3), it is not difficult to establish the following two interesting terminating results for the series ${}_2F_1$ record in [13]

$${}_2F_1\left[\begin{array}{cc} -2n, & a \\ \\ 2a & \end{array} ; \; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n} \tag{1.4}$$

and

$${}_2F_1\left[\begin{array}{cc} -2n-1, & a \\ \\ 2a & \end{array} ; \; 2\right] = 0 \tag{1.5}$$

The q- analogue of Kummer's second theorem (1.3) was given by Srivastava and Jain [20] in the following form

$${}_2\phi_1\left[\begin{array}{cc} a, & -a \\ \\ a^2 & \end{array} ; \; q, \; x\right] = (-x;q)_\infty \; {}_2\phi_1\left[\begin{array}{cc} 0, & 0 \\ \\ a^2q & \end{array} ; \; q^2, \; x^2\right] \tag{1.6}$$

In 1995, Rathie and Nagar [16] established the following two interesting results closely related to (1.3) viz.

$$\begin{aligned} e^{-\frac{1}{2}x} \; {}_1F_1\left[\begin{array}{c} a \\ \\ 2a+1 \end{array} ; \; x\right] &= {}_0F_1\left[\begin{array}{c} - \\ \\ a+\frac{1}{2} \end{array} ; \; \frac{x^2}{16}\right] \\ &\quad - \frac{x}{2(2a+1)} \; {}_0F_1\left[\begin{array}{c} - \\ \\ a+\frac{3}{2} \end{array} ; \; \frac{x^2}{16}\right] \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} e^{-\frac{1}{2}x} \; {}_1F_1\left[\begin{array}{c} a \\ \\ 2a-1 \end{array} ; \; x\right] &= {}_0F_1\left[\begin{array}{c} - \\ \\ a-\frac{1}{2} \end{array} ; \; \frac{x^2}{16}\right] \\ &\quad + \frac{x}{2(2a-1)} \; {}_0F_1\left[\begin{array}{c} - \\ \\ a+\frac{1}{2} \end{array} ; \; \frac{x^2}{16}\right] \end{aligned} \tag{1.8}$$

From (1.7) and (1.8), it follows that

$$ {}_2F_1\left[\begin{matrix} -2n, \quad a \\ 2a+1 \end{matrix} ; 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n} \tag{1.9} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, \quad a \\ 2a+1 \end{matrix} ; 2\right] = \frac{\left(\frac{3}{2}\right)_n}{(2a+1)\ \left(a+\frac{3}{2}\right)_n} \tag{1.10} $$

$$ {}_2F_1\left[\begin{matrix} -2n, \quad a \\ 2a-1 \end{matrix} ; x\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a-\frac{1}{2}\right)_n} \tag{1.11} $$

$$ {}_2F_1\left[\begin{matrix} -2n-1, \quad a \\ 2a-1 \end{matrix} ; x\right] = -\frac{\left(\frac{3}{2}\right)_n}{(2a-1)\ \left(a+\frac{1}{2}\right)_n} \tag{1.12} $$

In 1999, Kim, et al. [11] gave certain different methods to derive the results (1.7) and (1.8).

In 2010, Kim, et al. [10] obtained explicit expressions of

$$ e^{-\frac{1}{2}x}\ {}_1F_1\left[\begin{matrix} a \\ 2a+i \end{matrix} ; x\right] $$

for $i = 0, \pm1, \ldots, \pm5$, of which we shall utilize here the following two results for $i = 2$ and $i = -2$ viz.

$$ \begin{aligned} &e^{-\frac{1}{2}x}\ {}_1F_1\left[\begin{matrix} a \\ 2a+2 \end{matrix} ; x\right] \\ &= {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{x^2}{16}\right] - \frac{x}{4(a+1)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{3}{2} \end{matrix} ; \tfrac{x^2}{16}\right] \\ &+ \frac{ax^2}{16(a+1)(2a+1)(2a+3)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{5}{2} \end{matrix} ; \tfrac{x^2}{16}\right] \end{aligned} \tag{1.13} $$

and

$$ \begin{aligned} &e^{-\frac{1}{2}x}\ {}_1F_1\left[\begin{matrix} a \\ 2a-2 \end{matrix} ; x\right] \\ &= {}_0F_1\left[\begin{matrix} - \\ a-\frac{3}{2} \end{matrix} ; \tfrac{x^2}{16}\right] + \frac{x}{4(a-1)}\ {}_0F_1\left[\begin{matrix} - \\ a-\frac{1}{2} \end{matrix} ; \tfrac{x^2}{16}\right] \\ &+ \frac{(a-2)x^2}{16(a-1)(2a-1)(2a-3)}\ {}_0F_1\left[\begin{matrix} - \\ a+\frac{1}{2} \end{matrix} ; \tfrac{x^2}{16}\right] \end{aligned} \tag{1.14} $$

Also, the well-known and useful quadratic transformation due to Kummer recorded in Rainville [13] is

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ 2b \end{matrix} ; 2x\right] = (1-x)^{-a}\ {}_2F_1\left[\begin{matrix} \frac{a}{2}, \quad \frac{a+1}{2} \\ b+\frac{1}{2} \end{matrix} ; \tfrac{x^2}{(1-x)^2}\right] \tag{1.15} $$

Its q- analogue was given by Jain [9] in the following form

$$ {}_3\phi_2\left[\begin{matrix} a, & b, & -b \\ b^2, & ax & \end{matrix} ; q, -x\right] = \frac{(x;q)_\infty}{(ax;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} a, & aq \\ b^2q & \end{matrix} ; q^2, x^2\right] \tag{1.16}$$

In 2001, Kim and Rathie [15] have established the following two interesting results contiguous to (1.15) viz.

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, & b \\ 2b+1 & \end{matrix} ; 2x\right] &= (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2} \\ b+\frac{1}{2} & \end{matrix} ; \frac{x^2}{(1-x)^2}\right] \\ &- \frac{ax}{2(2b+1)}(1-x)^{-a}\, {}_2F_1\left[\begin{matrix} \frac{a+1}{2}, & \frac{a}{2}+1 \\ b+\frac{3}{2} & \end{matrix} ; \frac{x^2}{(1-x)^2}\right] \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} {}_2F_1\left[\begin{matrix} a, & b \\ 2b-1 & \end{matrix} ; 2x\right] &= (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} \frac{a}{2}, & \frac{a+1}{2} \\ b-\frac{1}{2} & \end{matrix} ; \frac{x^2}{(1-x)^2}\right] \\ &+ \frac{ax}{2(2b-1)}(1-x)^{-a}\, {}_2F_1\left[\begin{matrix} \frac{a}{2}, & \frac{a}{2}+1 \\ b+\frac{1}{2} & \end{matrix} ; \frac{x^2}{(1-x)^2}\right] \end{aligned} \tag{1.18}$$

On the other hand, the well known and useful Reed Dawson identity is given by

$$\sum_{k=0}^{\infty}(-1)^k \binom{n}{k} 2^{-k} \binom{2k}{k} = \begin{cases} 2^{-2\nu-1}\binom{2\nu}{\nu} & \text{, if } n=2\nu \\ 0 & \text{, if } n=2\nu+1 \end{cases} \tag{1.19}$$

Its q- analogue is given by Andrews [1] in the following form

$$\sum_{k=0}^{\infty}(-1)^k \binom{n}{k}_q \frac{1}{(-q)_k} \binom{2k}{k}_q q^{\frac{(n-k)(n-k-1)}{2}} = \begin{cases} \binom{2\nu}{\nu}_q \frac{q^{2\nu^2}}{(-q)_\nu^2} & \text{, if } n=2\nu \\ 0 & \text{, if } n=2\nu+1 \end{cases} \tag{1.20}$$

In 2004, Choi, et al. [4] established the following identity clearly related to (1.12) viz.

$$\sum_{k=1}^{n}(-1)^k \binom{n}{k} 2^{-k} \binom{2k-1}{k} = \begin{cases} 2^{-2\nu-1}\binom{2\nu}{\nu} & \text{, if } n=2\nu \\ 0 & \text{, if } n=2\nu+1 \end{cases} \tag{1.21}$$

Remark 1.1 *It is not out of place to mention here that the Reed Dawson identity (1.19) is also known as Knuth's old sum. For several proofs of Reed Dawson identity (1.19), we refer a survey paper by Prodinger [12].*

The paper is organised as follows. In section 2, we aim to establish three interesting results closely related to the q- analogue of Kummer's second theorem (1.6) obtained earlier by Srivastava and Jain. In section 3, we obtained a q- contiguous quadratic transformation formula closely related to q- quadratic transformation formula (1.16) due to Jain. In section 4, an attempt has been made to obtain a q- contiguous result closely related to q- Reed Dawson identity (1.20) due to Andrew. As limiting cases, we recover some known result of hypergeometric function. An application of our results are also given in section 5.

2. Results on q-contiguous Kummer's second theorem

In this section, the following three results closely related to the q-analogue of Kummer's second theorem (1.6) will be established.

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2q & \end{matrix} ; q, x\right] \\
&= (-x;q)_\infty \left\{ {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q & \end{matrix} ; q^2, x^2\right] \right. \\
&\left. - \frac{a^2x}{1-a^2q}\, {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q^3 & \end{matrix} ; q^2, x^2\right]\right\}
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2q^2 & \end{matrix} ; q, x\right] \\
&= (-x;q)_\infty \left\{ {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q & \end{matrix} ; q^2, x^2\right] \right. \\
&- \frac{a^2x}{1-a^2q}\left(1+\frac{q(1-a^2)}{1-a^2q^2}\right) {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q^3 & \end{matrix} ; q^2, x^2\right] \\
&\left. + \frac{a^4q^3(1-a^2)x^2}{(1-a^2q)(1-a^2q^2)(1-a^2q^3)}\, {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q^5 & \end{matrix} ; q^2, x^2\right]\right\}
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} a, & -a \\ \frac{a^2}{q} & \end{matrix} ; q, x\right] \\
&= (-x;q)_\infty\, {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ \frac{a^2}{q} & \end{matrix} ; q^2, x^2\right] \\
&+ \frac{a^2x}{(q-a^2)}\, (-qx;q)_\infty\, {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q & \end{matrix} ; q^2, q^2x^2\right]
\end{aligned} \tag{2.3}
$$

Proof: In order to establish our first result (2.1), we shall first establish the following relation involving three ${}_2\phi_1$ viz.

$$
\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2q & \end{matrix} ; q, x\right] \\
&= {}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2 & \end{matrix} ; q, x\right] - \frac{a^2x}{(1-a^2q)}\, {}_2\phi_1\left[\begin{matrix} aq, & -aq \\ a^2q^2 & \end{matrix} ; q, x\right].
\end{aligned} \tag{2.4}
$$

For this, denoting the left-hand side of (2.4) by S, we have

$$S = {}_2\phi_1\left[\begin{matrix} a, & -a \\ & \\ a^2q & \end{matrix} \; ; \quad q, \; x \right] = \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n(a^2q;q)_n} x^n.$$

Writing this in the form

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n(a^2q;q)_n} x^n \left\{ \frac{1-a^2q^n - a^2(1-q^n)}{1-a^2} \right\}.$$

Which can be written as

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n(a^2q;q)_n} x^n \left\{ \frac{1-a^2q^n}{1-a^2} - \frac{a^2(1-q^n)}{1-a^2} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n(1-a^2q^n)}{(q;q)_n(a^2q;q)_n(1-a^2)} x^n \\ &\quad - \frac{a^2}{1-a^2} \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n(1-q^n)}{(q;q)_n(a^2q;q)_n} x^n. \end{aligned}$$

Using the identity

$$(a^2q;q)_n \; (1-a^2) = (a^2;q)_{n+1} = (a^2;q)_n \; (1-a^2q^n). \tag{2.5}$$

We have

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n(1-a^2q^n)}{(q;q)_n(a^2;q)_n(1-a^2q^n)} x^n \\ &\quad - \frac{a^2}{(1-a^2)} \sum_{n=1}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_{n-1}(a^2q;q)_n} x^n \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n(a^2;q)_n} x^n \\ &\quad - \frac{a^2}{(1-a^2)} \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}(-a;q)_{n+1}}{(q;q)_n(a^2q;q)_{n+1}} x^{n+1}. \end{aligned}$$

Again using (2.5) in the second term, we have

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n(a^2;q)_n} x^n \\ &\quad - \frac{a^2x}{(1-a^2q)} \sum_{n=0}^{\infty} \frac{(aq;q)_n(-aq;q)_n}{(q;q)_n(a^2q^2;q)_n} x^n. \end{aligned}$$

Finally, summing up both the series, we easily arrive at the right-hand side of (2.4). Now, we are ready to establish our first result (2.1). For this, replacing a by aq in (1.6), we have

$${}_2\phi_1\left[\begin{matrix} aq, & -aq \\ & \\ a^2q^2 & \end{matrix} \; ; \quad q, \; x \right] = (-x;q)_\infty \; {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ & \\ a^2q^3 & \end{matrix} \; ; \quad q^2, \; x^2 \right]. \tag{2.6}$$

We now observe that the first and the second ${}_2\phi_1$ on the right-hand side of (2.4) can be evaluated with the help of (1.6) and (2.6) respectively and we easily obtain our first result (2.1). This completes the proof of our first result (2.1).

Next, in order to establish the result (2.2), we shall use the following result which can be established on similar lines that of (2.4).

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2q^2 & \end{matrix} ; q, x\right] &= {}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2q & \end{matrix} ; q, x\right] \\ &- \frac{a^2q(1-a^2)x}{(1-a^2q)(1-a^2q^2)}\, {}_2\phi_1\left[\begin{matrix} aq, & -aq \\ a^2q^3 & \end{matrix} ; q, x\right]. \end{aligned} \tag{2.7}$$

Again, from (2.1), by replacing a by aq, we have

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix} aq, & -aq \\ a^2q^3 & \end{matrix} ; q, x\right] &= (-x;q)_\infty \left\{ {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q^3 & \end{matrix} ; q^2, x^2\right] \right. \\ &\left. - \frac{a^2q^2x}{(1-a^2q^3)}\, {}_2\phi_1\left[\begin{matrix} 0, & 0 \\ a^2q^5 & \end{matrix} ; q^2, x^2\right]\right\}. \end{aligned} \tag{2.8}$$

The second result (2.2) follows from (2.7), (2.1) and (2.8). We omit the details.

Lastly, in order to establish our third result (2.3), first we shall establish the following relation involving three ${}_2\phi_1$, viz.

$$\begin{aligned}&{}_2\phi_1\left[\begin{matrix} a, & -a \\ \frac{a^2}{q} & \end{matrix} ; q, x\right] \\ &= {}_2\phi_1\left[\begin{matrix} \frac{a}{q}, & -\frac{a}{q} \\ \frac{a^2}{q^2} & \end{matrix} ; q, x\right] + \frac{a^2x}{(q-a^2)}\, {}_2\phi_1\left[\begin{matrix} a, & -a \\ a^2 & \end{matrix} ; q, qx\right]. \end{aligned} \tag{2.9}$$

For this, let us denote the left-hand side of (2.9) by S, we have

$$
\begin{aligned}
S = {}_2\phi_1 &\left[\begin{matrix} a, & -a \\ & & ; & q, & x \\ \frac{a^2}{q} \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n \left(\frac{a^2}{q};q\right)_n} x^n \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n \left(\frac{a^2}{q};q\right)_n} x^n \frac{(1-a^2q^{2n-2})}{(1-a^2q^{2n-2})} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n \left(\frac{a^2}{q};q\right)_n} x^n \left\{\frac{1-a^2q^{n-2}+a^2q^{n-2}(1-q^n)}{(1-a^2q^{2n-2})}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n}{(q;q)_n \left(\frac{a^2}{q};q\right)_n} x^n \left\{\frac{1-a^2q^{n-2}}{(1-a^2q^{2n-2})} + \frac{a^2q^{n-2}(1-q^n)}{(1-a^2q^{2n-2})}\right\} \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n(1-a^2q^{n-2})}{(q;q)_n \left(\frac{a^2}{q};q\right)_n (1-a^2q^{2n-2})} x^n \\
&+ \frac{a^2}{q} \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n(1-q^n)q^{n-1}}{(q;q)_n \left(\frac{a^2}{q};q\right)_n (1+aq^{n-1})(1-aq^{n-1})} x^n.
\end{aligned}
$$

Using the results

$$
(a;q)_n = (a;q)_{n-1}(1-aq^{n-1}),
$$

and

$$
\frac{(1-a^2q^{n-2})}{(1-a^2q^{2n-2})} = \frac{\left(1-\frac{a^2q^n}{q^2}\right)}{(1-aq^{n-1})(1+aq^{n-1})}.
$$

We have

$$
\begin{aligned}
S &= \sum_{n=0}^{\infty} \frac{\left(\frac{a}{q};q\right)_n \left(-\frac{a}{q};q\right)_n}{(q;q)_n \left(\frac{a^2}{q^2};q\right)_n} x^n + \frac{a^2}{q} \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n q^n}{(q;q)_n \left(\frac{a^2}{q};q\right)_{n+1}} x^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{\left(\frac{a}{q};q\right)_n \left(-\frac{a}{q};q\right)_n}{(q;q)_n \left(\frac{a^2}{q^2};q\right)_n} x^n + \frac{a^2x}{q\left(1-\frac{a^2}{q}\right)} \sum_{n=0}^{\infty} \frac{(a;q)_n(-a;q)_n q^n}{(q;q)_n \, (a^2;q)_n} x^n.
\end{aligned}
$$

Finally, summing up the two series, we easily arrive at the right-hand side of (2.9). Now, we are ready to establish our third result (2.3). For this, in (1.6), replacing x by qx, we have

$$
{}_2\phi_1 \left[\begin{matrix} a, & -a \\ & & ; & q, & qx \\ a^2 \end{matrix}\right] = (-qx;q)_\infty \, {}_2\phi_1 \left[\begin{matrix} 0, & 0 \\ & & ; & q^2, & q^2x^2 \\ a^2q \end{matrix}\right], \tag{2.10}
$$

and in (1.6), replacing a by $\frac{a}{q}$, we have

$$
{}_2\phi_1 \left[\begin{matrix} \frac{a}{q}, & -\frac{a}{q} \\ & & ; & q, & x \\ \frac{a^2}{q^2} \end{matrix}\right] = (-x;q)_\infty \, {}_2\phi_1 \left[\begin{matrix} 0, & 0 \\ & & ; & q^2, & x^2 \\ \frac{a^2}{q} \end{matrix}\right]. \tag{2.11}
$$

Finally, using (2.10) and (2.11) in (2.9), we get at once our third result (2.3). This completes the proof of (2.3). □

2.1. Limiting Cases

In the results (2.1), (2.2) and (2.3), if we take $q \to 1$, we recover the results (1.7), (1.13) and (1.8) respectively recovered in [10].

3. On a q- contiguous quadratic transformation formula

The result contiguous to q- analogue of quadratic transformation due to Kummer (1.16) to be proved in this section is the following:

$$
{}_3\phi_2\left[\begin{array}{ccc} a, & b, & -b \\ b^2q, & ax & \end{array} ; \; q, \; -x\right] = \frac{(x;q)_\infty}{(ax;q)_\infty}\left\{ {}_2\phi_1\left[\begin{array}{cc} a, & aq \\ b^2q & \end{array} ; \; q^2, \; x^2\right] \right.
$$
$$
\left. + \frac{b^2(1-a)}{(1-b^2q)} x \; {}_2\phi_1\left[\begin{array}{cc} aq, & aq^2 \\ b^2q & \end{array} ; \; q^2, \; x^2\right]\right\} \tag{3.1}
$$

Proof: In order to establish the result (3.1), first we shall prove the following relation involving three ${}_3\phi_2$ viz.

$$
{}_3\phi_2\left[\begin{array}{ccc} a, & b, & -b \\ b^2q, & ax & \end{array} ; \; q, \; -x\right] = {}_3\phi_2\left[\begin{array}{ccc} a, & b, & -b \\ b^2, & ax & \end{array} ; \; q, \; -x\right]
$$
$$
+ \frac{b^2(1-a)x}{(1-b^2q)(1-ax)} \; {}_3\phi_2\left[\begin{array}{ccc} aq, & bq, & -bq \\ b^2q^2, & aqx & \end{array} ; \; q, \; -x\right]. \tag{3.2}
$$

For proving (3.2), let us denote the left-hand side of (3.2) by S, we have

$$
\begin{aligned}
S &= {}_3\phi_2\left[\begin{array}{ccc} a, & b, & -b \\ b^2q, & ax & \end{array} ; \; q, \; -x\right] \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n \, (b;q)_n \, (-b;q)_n}{(q;q)_n \, (b^2q;q)_n \, (ax;q)_n} (-x)^n.
\end{aligned}
$$

Writing this in the following form

$$
= \sum_{n=0}^{\infty} \frac{(a;q)_n \, (b;q)_n \, (-b;q)_n}{(q;q)_n \, (b^2q;q)_n \, (ax;q)_n} (-x)^n \left\{ \frac{1-b^2q^n - b^2(1-q^n)}{1-b^2} \right\}.
$$

Now, separating into two terms, we have

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n(1-b^2q^n)}{(q;q)_n\,(b^2q;q)_n\,(ax;q)_n(1-b^2)}(-x)^n \\
&- \frac{b^2}{(1-b^2)} \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n(1-q^2)}{(q;q)_n\,(b^2q;q)_n\,(ax;q)_n}(-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n(1-b^2q^n)}{(q;q)_n\,(b^2q;q)_n\,(ax;q)_n(1-b^2)}(-x)^n \\
&- \frac{b^2}{(1-b^2)} \sum_{n=1}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n}{(q;q)_{n-1}\,(b^2q;q)_n\,(ax;q)_n}(-x)^n \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n(1-b^2q^n)}{(q;q)_n\,(b^2q;q)_n\,(ax;q)_n(1-b^2)}(-x)^n \\
&- b^2 \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}\,(b;q)_{n+1}\,(-b;q)_{n+1}}{(q;q)_n\,(b^2q;q)_{n+1}\,(ax;q)_{n+1}(1-b)(1+b)}(-x)^{n+1}.
\end{aligned}$$

Using the identity

$$(a;q)_n = (a;q)_{n-1}(1-aq^{n-1}),$$

and after some calculation, we have

$$\begin{aligned}
S &= \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n\,(-b;q)_n(1-b^2q^n)}{(q;q)_n\,(b^2q;q)_n\,(ax;q)_n(1-b^2)}(-x)^n \\
&+ \frac{b^2(1-a)x}{(1-b^2q)(1-ax)} \sum_{n=0}^{\infty} \frac{(aq;q)_n\,(bq;q)_n\,(-bq;q)_n}{(q;q)_n\,(b^2q^2;q)_n\,(aqx;q)_n}(-x)^n.
\end{aligned}$$

Finally, summing up the two series on the right-hand side, we easily arrive at the right- hand side of (3.2). This completes the proof of (3.2).

Now, we are ready to derive our result (3.1). For this, replacing a by aq in (1.16), we have

$$ {}_3\phi_2 \left[\begin{matrix} aq, & b, & -b \\ b^2, & aqx & \end{matrix} ; \; q, \; -x \right] = \frac{(x;q)_\infty}{(aqx;q)_\infty} \, {}_2\phi_1 \left[\begin{matrix} aq, & aq^2 \\ b^2q & \end{matrix} ; \; q^2, \; x^2 \right]. \tag{3.3}$$

It is now observed that if we use the result (1.16) on the first ${}_3\phi_2$ appearing on the right-hand side of (3.2) and using the result (3.3) on the second ${}_3\phi_2$ appearing on the right-hand side of (3.2), after little simplification, we get the desired result (3.1). This completes the proof of (3.1). □

3.1. Limiting Cases

If in our result (3.1), we take $q \to 1$, we recover a known result (1.17) due to Rathie and Kim [15].

4. A Result Closely Related to q-analogue of Reed Dawson Identity

In this section, the following result closely related to the q-analogue of Reed Dawson identity [1,4,5, 6,7,8,17,18,19] will be established.

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k}_q \frac{1}{(-q)_{k-1}} \binom{2k-1}{k}_q q^{\frac{(n-k)(n-k-1)}{2}} = \begin{cases} \binom{2\nu}{\nu}_q \frac{q^{2\nu^2}}{(-q)_\nu \; (-q)_\nu} & \text{, if } \; n = 2\nu \\ 0 & \text{, if } \; n = 2\nu+1 \end{cases} \tag{4.1}$$

Proof: In order to establish the result (4.1), let us denote it by S, we have

$$S=\sum_{k=1}^{n}(-1)^k\binom{n}{k}_q\frac{1}{(-q)_{k-1}}\binom{2k-1}{k}_q q^{\frac{(n-k)(n-k-1)}{2}}$$

Replacing k by $n-k$, we have

$$S=\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{n-k}_q\frac{1}{(-q)_{n-k-1}}\binom{2n-2k-1}{n-k}_q q^{\frac{k(k-1)}{2}}$$

Using the identity

$$\binom{n}{k}_q=\binom{n}{n-k}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \tag{4.2}$$

We have

$$S=\sum_{k=1}^{n}(-1)^{n-k}\frac{(q)_n}{(q)_k(q)_{n-k}}\frac{1}{(-q)_{n-k-1}}\frac{(q)_{2n-2k-1}}{(q)_{n-k}(q)_{n-k-1}}q^{\frac{k(k-1)}{2}}$$

Using the identities

$$(a;q)_{n-k}=\frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-a^{-1}q)^k q^{\binom{k}{2}-nk} \tag{4.3}$$

and

$$(q;q)_{2n-2k-1}=(q;q^2)_{n-k}(q^2;q^2)_{n-k-1} \tag{4.4}$$

We have after some algebra

$$S=(-1)^n\sum_{k=1}^{n}\frac{(q^{-n})_k(q;q^2)_{n-k}}{(q)_k(q)_{n-k}}q^{nk}$$

But using the identity

$$(q;q^2)_{n-k}=\frac{(q;q^2)_n}{(q^{-2n+1};q^2)_k}(-1)^k q^{k(k-1)-2nk+1} \tag{4.5}$$

and (4.3), it is easy to see that

$$\frac{q^{nk}(q;q^2)_{n-k}}{(q;q)_{n-k}}=\frac{(q;q^2)_n(q^{-n})_k}{(q)_n(q^{-2n+1};q^2)_k}q^{\frac{k(k+1)}{2}}$$

Therefore, we have

$$S=\frac{(-1)^n(q;q^2)_n}{(q)_n}\sum_{k=1}^{n}\frac{(q^{-n})_k(q^{-n})_k}{(q)_k(q^{-2n+1};q^2)_k}q^{\frac{k(k+1)}{2}}$$

Summing up the series, we have

$$S=\frac{(-1)^n(q;q^2)_n}{(q)_n}\,{}_2\phi_2\left[\begin{matrix}q^{-n}, & q^{-n} \\ q^{\frac{1}{2}(-2n+1)}, & -q^{\frac{1}{2}(-2n+1)}\end{matrix}\;;\;q,\;-q\right]$$

Using the summation theorem due to Andrews [1] viz.

$$ {}_2\phi_2\left[\begin{array}{cc} a, & b \\ \sqrt{qab}, & -\sqrt{qab} \end{array} ; \; q, \; -q\right] = \frac{(-q;q)_\infty (aq;q^2)_\infty (bq;q^2)_\infty}{(qab;q^2)_\infty} \tag{4.6} $$

We have

$$ S = \frac{(-1)^n \; (q;q^2)_n (-q)_\infty (q^{-n+1};q^2)_\infty (q^{-n+1};q^2)_\infty}{(q)_n (q^{-2n+1};q^2)_\infty} \tag{4.7} $$

Now, two cases arises.
Case-1 For $n = 2\nu$, we have

$$ S_1 = \frac{(q;q^2)_{2\nu} (-q)_\infty (q^{-2\nu+1};q^2)_\infty (q^{-2\nu+1};q^2)_\infty}{(q)_{2\nu} (q^{-4\nu+1};q^2)_\infty} $$

Which on simplification gives,

$$ S_1 = \frac{q^{2\nu^2}}{(-q)_\nu (-q)_\nu} \binom{2\nu}{\nu}_q . \tag{4.8} $$

Also, when $n = 2\nu + 1$, then since it is easily seen that the value of $(q^{-n+1};q^2)_\infty$ is zero, so we have

$$ S_2 = 0 \tag{4.9} $$

Finally, from (4.8) and (4.9), our result (4.1) is completely established. □

4.1. Limiting Case

In (4.1), if we take $q \to 1$, we immediately recover the result (1.21) obtained earlier by Choi, et al. [4].

5. Application

In this section, we shall mention, as an application of the result (1.6) and establish the following two interesting results which are believed to be new.

$$ {}_3\phi_1\left[\begin{array}{ccc} q^{-2n}, & a, & -a \\ a^2 & & \end{array} ; \; q, \; -q^{-2n}\right] = \frac{(q;q^2)_n}{(a^2q;q^2)_n}, \tag{5.1} $$

and

$$ {}_3\phi_1\left[\begin{array}{ccc} q^{-2n-1}, & a, & -a \\ a^2 & & \end{array} ; \; q, \; -q^{-2n-1}\right] = 0. \tag{5.2} $$

Proof: In order to establish the results (5.1) and (5.2), we start with the result (1.6) which can be written as

$$ \frac{1}{(-x;q)_\infty} \; {}_2\phi_1\left[\begin{array}{cc} a, & -a \\ a^2 & \end{array} ; \; q, \; x\right] = {}_2\phi_1\left[\begin{array}{cc} 0, & 0 \\ a^2q & \end{array} ; \; q^2, \; x^2\right]. \tag{5.3} $$

Since

$$ \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}. $$

And expressing both ${}_2\phi_1$ as a series, we have from (5.3)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a;q)_m(-a;q)_m(-1)^n x^{n+m}}{(q;q)_m(q;q)_n(a^2;q)_m}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(q^2;q^2)_n(a^2q;q^2)_n}.$$

Now, replacing n by $n-m$ and using a known result recovered in Rainville [13].

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(n,k)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k).$$

We have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(a;q)_m(-a;q)_m(-1)^{n-m}x^{n}}{(q;q)_m(q;q)_{n-m}(a^2;q)_m}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(q^2;q^2)_n(a^2q;q^2)_n}.$$

Using the identity

$$(q;q)_{n-m}=(-1)^m\frac{(q;q)_n}{(q^{-n};q)_m}q^{\frac{m(m-1)}{2}-mn}.$$

We have, after some simplification

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(q;q)_n}\sum_{m=0}^{n}\frac{(q^{-n};q)_m(a;q)_m(-a;q)_m}{(q;q)_m(a^2;q)_m}\left\{(-1)^m q^{\frac{m(m-1)}{2}-mn}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{x^{2n}}{(q^2;q^2)_n(a^2q;q^2)_n}.$$

Summing up the inner series, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(q;q)_n}\,{}_3\phi_1\left[\begin{matrix}q^{-n}, & a, & -a \\ & a^2 & \end{matrix}\ ;\ q,\ -q^{-n}\right]=\sum_{n=0}^{\infty}\frac{x^{2n}}{(q^2;q^2)_n(a^2q;q^2)_n}.$$

Separating into even and odd powers of x, we have

$$\sum_{n=0}^{\infty}\frac{x^{2n}}{(q;q)_{2n}}\,{}_3\phi_1\left[\begin{matrix}q^{-2n}, & a, & -a \\ & a^2 & \end{matrix}\ ;\ q,\ -q^{-2n}\right]$$
$$-\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(q;q)_{2n+1}}\,{}_3\phi_1\left[\begin{matrix}q^{-2n-1}, & a, & -a \\ & a^2 & \end{matrix}\ ;\ q,\ -q^{-2n-1}\right]$$
$$=\sum_{n=0}^{\infty}\frac{x^{2n}}{(q^2;q^2)_n(a^2q;q^2)_n}.$$

Now, equating the coefficients of x^{2n} and x^{2n+1} both sides, we have

$$\frac{1}{(q;q)_{2n}}\,{}_3\phi_1\left[\begin{matrix}q^{-2n}, & a, & -a \\ & a^2 & \end{matrix}\ ;\ q,\ -q^{-2n}\right]=\frac{1}{(q^2;q^2)_n(a^2q;q^2)_n},$$

and

$$\frac{1}{(q;q)_{2n+1}}\,{}_3\phi_1\left[\begin{matrix}q^{-2n-1}, & a, & -a \\ & a^2 & \end{matrix}\ ;\ q,\ -q^{-2n-1}\right]=0.$$

Which on simplification gives the required results (5.1) and (5.2). This completes the proof of the results (5.1) and (5.2).

5.1. Limiting Cases

In (5.1) and (5.2), if we take $q \to 1$, we recover the well known results (1.4) and (1.5) respectively. $\square$

Conflict of Interests

The authors declare that they have no any conflict of interests.

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