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On a $p(x)$-Biharmonic Singular Problem With Changing Sign Weight and With No-Flux Boundary Condition

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Abstract: In the present paper, we study $p(x)-$biharmonic problem involving $q(x)-$Hardy type potential with no-flux boundary condition. By using the mountain pass type theorem and Ekeland variatoinal principle, we obtain at least two nontrivial weak solutions.

Key Words: Variational methods, singular problem, p(.)-biharmonic operator, weak solutions.

Contents

1. Introduction

In recent years, the study of differential and partial differential equations with a variable exponent has received great attention, which contributed to the development of research and studies related to the problems of differential equations. We find important applications related elastic materials, image restoration, electrorheological fluids and mathematical biology, etc, (see [24,16]).

Singular elliptic problems have been intensively studied in the last decades. Among others, we mention works [17,22]. For instance, nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids and boundary layer phenomena for viscous fluids.

A $p(x)$-biharmonic problem with no flux boundary condition was treated, of our knowledge the first time by M.M. Bourneau, and all (see [2]). However they studied the problem

$$(P)\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u)+a(x)|u|^{p(x)-2}u=\lambda f(x,u) \text{ in } \Omega \\ u=constant,\ \Delta u=0 \qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega}\frac{\partial}{\partial n}\Big(|\Delta u|^{p(x)-2}\Delta u\Big)ds=0. \end{cases}$$

Where they proved that problem (P) admited at least two nontrivial weak solutions by using some assumptions of function f.

In the last years, there was much attention focused on the existence and multiplicity of solutions for $p(x)$-biharmonic equations under Dirichlet boundary conditions or navier boundary with hardy terms [20,10]. As far as we are aware, the $p(x)$-biharmonic problem with no-flux boundary condition involving the nonhomogenous Hardy inequality in variable exponent sobolev space have not yet been studied. that is why the present paper is a first contribution in this direction.

Inspired by the above results, in this paper we study the following $p(x)-$biharmonic involving $q(x)-$Hardy type potential and with changing sign weight under no-flux boundary condition.

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$$(P_\lambda)\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u)+a(x)|u|^{p(x)-2}u=\lambda m(x)\dfrac{|u|^{q(x)-2}u}{|x|^{q(x)}}+f(x,u) \text{ in } \Omega \\ u=constant,\ \Delta u=0 \qquad\qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega}\frac{\partial}{\partial n}\Big(|\Delta u|^{p(x)-2}\Delta u\Big)ds=0. \end{cases}$$

where $\Omega\subset\mathbb{R}^N(N>2)$ is a bounded domain containing the origin and with a smooth boundary $\partial\Omega$, λ is a positive parameter and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Caratheodory function satisfying certain conditions which will be stated later on.
The variable exponent $p(x),q(x)\in C_+(\overline{\Omega})=\big\{h\in C(\overline{\Omega}) \text{ and } h(x)>1 \text{ on } \overline{\Omega}\big\}$ and $p(x)$ is a log-Hölder continuous function in $\overline{\Omega}$, that is, there exists $k>0$ such that

$$|p(x)-p(y)|\leq\frac{k}{-ln|x-y|} \text{ for all } x,y\in\Omega, \text{ with } 0<|x-y|\leq\frac{1}{2}.$$

In this paper we assume the following conditions:
H(p,q) $\qquad 1<q^-<q^+<p^-<p^+<\frac{N}{2}.$
where $\quad q^-:=\min_{x\in\bar\Omega}q(x),\quad q^+:=\max_{x\in\bar\Omega}q(x).$
We will work under the following hypothese:
(a) $a\in L^\infty(\Omega)$ and there exists $a_0>0$ such that $a(x)\geq a_0$ for all $x\in\Omega$.
(m) $m\in L^\infty(\Omega)$, and there exists a ball $B_\eta(x_0)\subset\subset\Omega$ of centre $x_0\in\Omega$ and Radus $\eta>0$ such that $0\notin B_{\frac{\eta}{2}}(x_0)$ and $m(x)>0$ for a.e. x in $B_\eta(x_0)$.
(f_1) There exists $C>0$ and

$$|f(x,t)|\leq C(1+|t|^{r(x)-1}) \text{ for all } (x,t)\in\Omega\times\mathbb{R},$$

where $r(x)\in C_+(\overline{\Omega})$ and $p^+<r(x)<p_2^*(x)$, for all $x\in\overline{\Omega}$.
(f_2) There exist $T>0$ and $\theta>p^+$ such that

$$0<\theta F(x,t)\leq tf(x,t) \text{ for all } |t|>T \text{ and u.a. e } .x\in\Omega,$$

where $F(x,t)=\int_0^t f(x,s)ds \quad$ a.e $x\in\Omega$.
(f_3) $f(x,t)\geq C'|t|^{r(x)-1}$, $t\longrightarrow 0$, where $C'>0$.
(f_4) $f(x,t)=o(|t|^{p(x)-1})$ as $t\longrightarrow 0$ uniformly for $x\in\Omega$.

Example 1.1. *for $f(x,t)=|t|^{r(x)-1}$, we can see that $(f_1)-(f_4)$ are verified.*

Our main result are stated in the following theorem.

Theorem 1.2. *Assume hypotheses $H(p,q)$, (a), (m) and (f_1)-(f_4) are fulfilled. Then there exists a constant $\lambda^*>0$ such that for any $\lambda\in(0,\lambda^*)$, the problem (P_λ) has at least two distinct nontrivial weak solutions.*

2. Preliminaries

In this part, we recall some definitions and basic properties of the variable exponent Lebesgue and Sobolev spaces.but much more details can be found in the comprehensivec works.
We consider $p(x)$ to be log-Hölder continuous with $1<p^-\leq p^+<\infty$.
The Lebesgue space with variable exponent is defined by

$$L^{p(x)}(\Omega)=\left\{u:\Omega\to\mathbb{R} \text{ mesurable function such that }:\int_\Omega|u|^{p(x)}\,\mathrm{d}x<\infty\right\},$$

which is endowed with the lauxauborg norm,

$$|u|_{p(x)}=\inf\left\{\mu>0:\int_\Omega\left|\frac{u(x)}{\mu}\right|^{p(x)}dx\leq 1\right\}.$$

$\big(L^{p(x)}(\Omega),\ |.|_{p(x)}\big)$ is a separable and reflexive Banach space, (see [19], Theorem 2.5, Corollary 2.7). Moreover, we have the following continuous embedding result.

Proposition 2.1. *(see [19], Theorem 2.8). If $0 < |\Omega| < \infty$ and $p_1, p_2 \in C(\overline{\Omega}; \mathbb{R})$, $1 < p_i^- \leq p_i^+ < \infty (i = 1, 2)$, are such that $p_1 \leq p_2$ in Ω, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

The $p(.)$-modular of $L^{p(x)}(\Omega)$, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$\rho(u) = \int_\Omega |u|^{p(x)} dx \text{ , for all } u \in L^{p(x)}(\Omega).$$

And we have the following properties.

Proposition 2.2. *(see [11], Theorem 1.3 and Theorem 1.4). For $u,\ u_n \in L^{p(x)}(\Omega)$*

(i) $|u|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1 (= 1; > 1)$;

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$;

(iv) $|u_n|_{p(x)} \to 0 (\to \infty) \Leftrightarrow \rho(u_n) \to 0 (\to \infty)$;

(v) $|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho(u_n - u) \to 0$.

In addition, we get the Hölder type inequality.

Proposition 2.3. *(see [19], Theorem 2.1). For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder-type inequality*

$$\left| \int_\Omega u(x) v(x)\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2 |u|_{p(x)} |v|_{p'(x)}, \tag{2.1}$$

where $L^{p'(x)}(\Omega)$ is a conjugate space of $L^{p(x)}(\Omega)$.

We recall also the following proposition, which will be needed later.

Proposition 2.4. *(see [8], Lemma 2.1). Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$ such that $u \neq 0$. Then*

(i) $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|_{p(x)q(x)}^{p^+} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-}$

(ii) $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|_{p(x)q(x)}^{p^-} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}$.

The Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined by

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\},$$

where $D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_N^{\alpha_N}}$, with $\alpha = (\alpha_1, \alpha_2 ..., \alpha_N)$ is a multi-index such that $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$ endowed with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)},$$

becomes a separable and reflexive Banach space. We can refer to ([19]).
The log-Hölder continuity of the exponent $p(x)$ plays a decisive role in the following density results.

Theorem 2.5. *(see ([4], Section 6.5.3) and ([6], Theorem 3.7)). Assume that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary and p is log-Hölder continuous with $1 < p^- \leq p^+ < \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(x)}(\Omega)$.*

We also need the following embedding theorem.

Theorem 2.6. *(see ([4], Section 6) and ([11], Theorem 2.3)). Let us consider* $q \in C(\overline{\Omega})$ *such that* $1 < q^- \leq q^+ < \infty$ *and* $q(x) \leq p_k^*(x)$ *for all* $x \in \overline{\Omega}$*, where*

$$p_k^*(x) = \begin{cases} \dfrac{Np(x)}{N - kp(x)} & if \quad p(x) < \dfrac{N}{k}, \\ +\infty & if \quad p(x) \geq \dfrac{N}{k} \end{cases}$$

for any $x \in \overline{\Omega}$ *and* $k \geq 1$*. Then there is a continuous embedding* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$*. If we replace* $\leq$ *with* $<$ *the embedding is compact.*

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.
Taking into account the log-Hölder countinuity of the exponent $p(x)$, we have

$$W_0^{1,p(x)}(\Omega) = \left\{ u \in W^{1,p(x)}(\Omega) : u = 0 \text{ on } \partial\Omega \right\}.$$

As a consequence of the Poincaré inequality, $\|u\|_{W^{1,p(x)}(\Omega)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$ we can define an equivalent norm $\|u\|_{W_0^{1,p(x)}(\Omega)}$ such that

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = |\nabla u|_{p(x)},$$

and which makes $W_0^{1,p(x)}(\Omega)$ a separable and reflexive Banach space (see [13], Proposition 2.1). The space $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ equipped with the norm

$$\begin{aligned} \|u\|_{W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)} &= \|u\|_{W^{2,p(x)}(\Omega)} + \|u\|_{W_0^{1,p(x)}(\Omega)} \\ &= |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)} \end{aligned}$$

is a separable and reflexive Banach space.
Moreover, we know that $\|u\|_{W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)}$ and $|\Delta u|_{p(x)}$ are equivalent norms on $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, (see [23], Theorem 4.4).
In the sequel, we chose the norm:

$$\|u\|_a = \inf \left\{ \mu > 0 : \int_\Omega \left(\left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} + a(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\},$$

where a satisfies (a), this norm represents a norm on both $W^{2,p(x)}(\Omega)$ and $W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ and it is equivalent to the usual norm defined here, (see [9], Remark 2.1).
Throughout this work, we consider the space V defined by

$$V = \left\{ u \in W^{2,p(x)}(\Omega) : u|_{\partial\Omega} \equiv constant \right\},$$

that it can be viewed also as

$$V = \left\{ u + c : u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega),\ c \in \mathbb{R} \right\}.$$

V is a closed subspace of the separable and reflexive Banach space $W^{2,p(x)}(\Omega)$ equipped with the usual norm, so $\left(V, ||.||_{W^{2,p(x)}(\Omega)}\right)$ is a separable and reflexive Banach space (see [2], Theorem 4). The space V is the space where we will try to find weak solutions for our problem.
Therefore, we consider the modular $\Lambda : V \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_\Omega \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right] dx,$$

that can make an important connection to the norm $\|.\|_a$, precisely in the following inequalities.

Proposition 2.7. *(See [2], Proposition 1). For* $u,\ u_n \in W^{2,p(x)}(\Omega)$ *we have:*

(i) $\|u\|_a < 1(=1;>1) \Leftrightarrow \Lambda(u) < 1(=1;>1)$;

(ii) $\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \Lambda(u) \leq \|u\|_a^{p^+}$;

(iii) $\|u\|_a \leq 1 \Rightarrow \|u\|_a^{p^+} \leq \Lambda(u) \leq \|u\|_a^{p^-}$;

(iv) $\|u_n\|_a \to 0(\to \infty) \Leftrightarrow \Lambda(u_n) \to 0(\to \infty)$.

This following results will also help us.

Proposition 2.8. *(see [9], Proposition 2,5).*
The functional $L : V \to \mathbb{R}$ *defined by*

$$L(u) = \int_\Omega \left[\frac{1}{p(x)}|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right] dx.$$

verifies the following assumptions.

(i) L *is well defined and of class* C^1*, with derivative defined by*

$$< L'(u), \upsilon >= \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta \upsilon dx + \int_\Omega a(x)|u|^{p(x)-2}u\upsilon dx.$$

(ii) L *is (sequentially) weakly lower semicontinous, that is, for any* $u \in V$ *and any subsequence* $(u_n)_n \subset V$ *such that* $u_n \rightharpoonup u$ *in* V*, there holds*

$$L(u) \leq \lim_{n\to\infty} inf L(u_n).$$

(iii) The mapping $L' : V \to V'$ *is of type* (S_+)*, that is* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty} < L'(u_n)(u_n - u) >\leq 0 \text{ imply that } u_n \longrightarrow u.$$

3. Proof of the Theorem 1.2

First, in derection of hardy type , Mitidieri [21] showed that for any $p \in (1, \frac{N}{2})$, there holds

$$\int_\Omega |\Delta u|^p dx \geqslant \Big(\frac{N(p-1)(N-2p)}{p^2}\Big)^p \int_\Omega \frac{|u|^p}{|x|^{2p}} dx, \tag{3.1}$$

whenever $u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$.

Lemma 3.1. *Assume that* $H(p,q)$ *holds. then there exists positive constants* C_1 *and* C_2 *such that*

$$\int_\Omega \frac{|u|^{q(x)}}{|x|^{2q(x)}} dx \leqslant C_1(\|u\|_a^{q^+} + \|u\|_a^{q^-}),$$

and

$$\int_\Omega \frac{|u|^{q(x)}}{|x|^{q(x)}} dx \leqslant C_2(\|u\|_a^{q^+} + \|u\|_a^{q^-}).$$

Proof. Let us notice that

$$\begin{aligned}\int_\Omega \frac{|u|^{q(x)}}{|x|^{2q(x)}} dx &= \int_{|u|>x^2} \frac{|u|^{q(x)}}{|x|^{2q(x)}} dx + \int_{|u|\leqslant x^2} \frac{|u|^{q(x)}}{|x|^{2q(x)}} dx \\ &\leqslant \int_{|u|>x^2} \frac{|u|^{q^+}}{|x|^{2q^+}} dx + \int_{|u|\leqslant x^2} \frac{|u|^{q^-}}{|x|^{2q^-}} dx \\ &\leqslant \int_\Omega \frac{|u|^{q^+}}{|x|^{2q^+}} dx + \int_\Omega \frac{|u|^{q^-}}{|x|^{2q^-}} dx.\end{aligned}$$

In View of (3.1), we deduce that, there existe $C, C' > 0$ such that

$$\int_\Omega \frac{|u|^{q(x)}}{|x|^{2q(x)}}dx \leqslant C\int_\Omega |\Delta u|^{q^+}dx + C^{'}\int_\Omega |\Delta u|^{q^-}dx \tag{3.2}$$

by Theorem 2.6 , $H(p,q)$ and (3.2) there exist $C'', C_1 > 0$ such that

$$\begin{aligned}\int_\Omega \frac{|u|^{q(x)}}{|x|^{2q(x)}}dx &\leqslant C|\Delta u|^{q^+}_{q^+} + C^{'}|\Delta u|^{q^-}_{q^-}\\ &\leqslant C''(|\Delta u|^{q^+}_{p(x)} + |\Delta u|^{q^-}_{p(x)})\\ &\leqslant C_1(\|u\|^{q^+}_a + \|u\|^{q^-}_a).\end{aligned} \tag{3.3}$$

We take $B(0,1) = \{x \in \Omega; |x| < 1\}$,
According to assumptions (ii)-(iii) in Proposition 2.2, Theorem 2.6 and (3.3) we obtain

$$\begin{aligned}\int_\Omega \frac{|u|^{q(x)}}{|x|^{q(x)}}dx &= \int_{B(0,1)} \frac{|u|^{q(x)}}{|x|^{q(x)}}dx + \int_{\Omega\setminus B(0,1)} \frac{|u|^{q(x)}}{|x|^{q(x)}}dx\\ &\leqslant \int_{B(0,1)} \frac{|u|^{q(x)}}{|x|^{2q(x)}}dx + \int_{\Omega\setminus B(0,1)} |u|^{q(x)}dx\\ &\leqslant \int_\Omega \frac{|u|^{q(x)}}{|x|^{2q(x)}}dx + \int_\Omega |u|^{q(x)}dx\\ &\leqslant C_1(\|u\|^{q^+}_a + \|u\|^{q^-}_a) + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)}\\ &\leqslant C_2(\|u\|^{q^+}_a + \|u\|^{q^-}_a).\end{aligned}$$

□

Applying Green's formula and taking in account the fact that V is closed subspace of $\left(W^{2,p(x)}(\Omega), \|.\|_{W^{2,p(x)}(\Omega)}\right)$ together with the density result in Theorem 2.5 and the boundary conditions, we give the following definition.

Definition 3.2. *The function $u \in V$ is a weak solution of the boundary value problem (P_λ) if*

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx - \lambda\int_\Omega m(x)\frac{|u|^{q(x)-2}}{|x|^{q(x)}}uvdx - \int_\Omega f(x,u)vdx = 0,$$

for all $v \in V$.
The energy functional $J_\lambda : V \to \mathbb{R}$ corresponding to the problem (P_λ), is defined as

$$J_\lambda(u) = I_1(u) - \lambda I_2(u) - I_3(u),$$

$$\text{where} \qquad I_1(u) = L(u) \ \ , I_2(u) = \int_\Omega \frac{m(x)}{q(x)}\frac{|u|^{q(x)}}{|x|^{q(x)}}dx \quad \text{and} \quad I_3(u) = \int_\Omega F(x,u)dx.$$

According to assumptions (i)-(ii) in Proposition 2.8, we deduce that $I_1 \in C^1(V,\mathbb{R})$, weakly lower semicontinuous and

$$< I_1^{'}(u), v >= \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx \qquad \text{for all } v \in V.$$

Due to the properties fulfilled by f and proceeding similarly to ([18], Proposition 5), $I_3 \in C^1(V,\mathbb{R})$, weakly lower semicontinuous and

$$< I_3^{'}(u), v >= \int_\Omega f(x,u)vdx \qquad \text{for all } v \in V.$$

The inequality

$$\Big|\int_\Omega \frac{m(x)}{q(x)}\frac{|u|^{q(x)}}{|x|^{q(x)}}dx\Big| \leq \frac{\|m\|_\infty}{q^-}\int_\Omega \frac{|u|^{q(x)}}{|x|^{q(x)}}dx$$

and Lemma 3.1 allows us to state that I_2 is well defined. Moreover $I_2 \in C^1(V,\mathbb{R})$ and

$$< I_2^{'}(u), v >= \int_\Omega m(x)\frac{|u|^{q(x)-2}}{|x|^{q(x)}}uvdx \qquad \text{for all } v \in V.$$

We have the following lemma

Lemma 3.3. *$I_2(u)$ is weakly lower semi-continuous in V.*

Proof. Let (u_n) be sequence such that $u_n \rightharpoonup u$ in V. We have

$$\Big|\int_\Omega \frac{m(x)}{q(x)}\frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx\Big| \leq \frac{\|m\|_\infty}{q^-}\int_\Omega \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx. \tag{3.4}$$

Define the following Banach space

$$L^{q(x)}_{|x|^{-q(x)}}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable function such that } : \int_\Omega \left|\frac{u}{x}\right|^{q(x)} \mathrm{d}x < \infty\right\}$$

with the norme

$$|u|_{L^{q(x)}_{|x|^{-q(x)}}(\Omega)} = \inf\left\{\mu > 0 : \int_\Omega \frac{1}{|x|^{q(x)}}\left|\frac{u(x)}{\mu}\right|^{q(x)} dx \leq 1\right\}.$$

To prove our lemma, it suffices to show that there is a compact embedding $W^{2,p(x)}(\Omega) \to L^{q(x)}_{|x|^{-q(x)}}(\Omega)$. Indeed, we have

$$\frac{N-q(x)}{N}p_2^*(x) - q(x) = \frac{N(p(x)-q(x)) + p(x)q(x)}{N-2p(x)} > 0.$$

Then

$$1 \leq q(x) \leq \frac{N-q(x)}{N}p_2^*(x) \qquad \forall x \in \Omega. \tag{3.5}$$

The inequality (3.5) allows us to apply the corollary 2.1 in [14] to conclude the compact embedding $W^{2,p(x)}(\Omega) \to L^{q(x)}_{|x|^{-q(x)}}(\Omega)$ and using relation (3.4) we affirme that $I_2(u_n) \to I_2(u)$. □

According to previous results J_λ is of class C^1 and weakly lower semi-continuous with Gâteaux derivative defined by

$$\begin{aligned} < J_\lambda^{'}(u), v >= &\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta v dx + \int_\Omega a(x)|u|^{p(x)-2}uvdx \\ &- \lambda\int_\Omega m(x)\frac{|u|^{q(x)-2}}{|x|^{q(x)}}uvdx - \int_\Omega f(x,u)vdx, \quad \text{for all } v \in V. \end{aligned}$$

Thus we can infer that critical points of functional J_λ are exactly the weak solutions of problem (P_λ). For the existence of the first nontrivial solution we base on the celebrated mountain pass theorem:

Theorem 3.4. *(see [1], Theorem 2.1) Let $(X, \|.\|_X)$ be a Banach space. Assume that $\phi \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition, and has a mountain pass geometry, that is, there exists two positive constants τ and ρ such that*

1. *$\phi(u) \geq \rho$ if $\|u\|_X = \tau$,*

2. $\phi(0) < \rho$ *and there exists* $e \in X$ *such that* $\|e\|_X > \tau$ *and* $\phi(e) < \rho$.

Then ϕ *has a critical point* $u_0 \in X \setminus \{0, e\}$ *with critical value*

$$\phi(u_0) = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} \phi(u) \geq \rho > 0,$$

where Γ *denotes the class of the paths* $\gamma \in C([0,1]; X)$ *joining* 0 *to* e.

To obtain the second nontrivial solution we use the Ekeland's variational principle.

Theorem 3.5. *(see [15], Theorem 4.1) Let* (X, d) *be a complete metric space, and* $F : X \longrightarrow (-\infty, +\infty]$ *be lower semicontinuous function bounded from below. Then given any* $\epsilon > 0$ *there exists* $u_\epsilon \in X$ *such that* $F(u_\epsilon) \leq \inf_X F + \epsilon$ *and*

$$F(u_\epsilon) < F(u) + \epsilon d(u, u_\epsilon) \quad \forall u \in X \text{ with } \quad u \neq u_\epsilon.$$

Now we need the following lemmas.

Lemma 3.6. *There exists* $\psi \in V$, $\psi \neq 0$ *such that*

$$\lim_{t \to +\infty} J_\lambda(t\psi) = -\infty$$

Proof. Let $\psi \in V$ such that $\psi \neq 0$ and $t > 1$. By (f_2), it follows that there exists $c_1 > 0$ such that $F(x, s) \geq c_1 |s|^\theta$ for all $s \in \mathbb{R}$ with $|s| > T$.
We have

$$\begin{aligned} J_\lambda(t\psi) &= L(t\psi) - \lambda \int_\Omega \frac{m(x)}{q(x)} \frac{|t\psi|^{q(x)}}{|x|^{q(x)}} dx - \int_\Omega F(x, t\psi) dx \\ &\leq \frac{t^{p^+}}{p^-} \|\psi\|_a^{p(x)} + \frac{\lambda t^{q^+}}{q^-} \int_\Omega |m(x)| \frac{|\psi|^{q(x)}}{|x|^{q(x)}} dx - c_1 t^\theta \int_\Omega |\psi|^\theta dx. \end{aligned}$$

Since $q^+ < p^+ < \theta$, then $\lim_{t \to +\infty} J_\lambda(t\psi) = -\infty$. □

Lemma 3.7. *There exists* $\varphi \in V$ *such that* $\varphi > 0$ *and* $J_\lambda(t\varphi) < 0$ *for* $t > 0$ *small enough.*

Proof. Let $\varphi \in V$ such that $\varphi(x) = \begin{cases} 1 & \text{in } x \in B_{\frac{\eta}{2}}(x_0) \\ 0 \leqslant \varphi \leqslant 1 & \text{in } x \in B_\eta(x_0) \setminus B_{\frac{\eta}{2}}(x_0) \\ 0 & \text{in } x \in \Omega \setminus B_\eta(x_0) \end{cases}$,
By (m), (f_3) and for $t \in (0, 1)$, we have

$$\begin{aligned} J_\lambda(t\varphi) \leq & \frac{t^{p^-}}{p^-} \|\varphi\|_a^{p(x)} - \frac{\lambda t^{q^+}}{q^+} \int_{B_{\frac{\eta}{2}}(x_0)} \frac{m(x)}{|x|^{q(x)}} dx - \frac{C' t^{r^+}}{r^+} \int_\Omega |\varphi|^{r(x)} dx \\ \leq & \frac{t^{p^-}}{p^-} \|\varphi\|_a^{p(x)} - \frac{\lambda t^{q^+}}{q^+} \int_{B_{\frac{\eta}{2}}(x_0)} \frac{m(x)}{|x|^{q(x)}} dx. \end{aligned}$$

Since $q^+ < p^-$, then $J_\lambda(t\varphi) < 0$ for t small enough. The proof of lemma is complete. □

Lemma 3.8. *For* $\rho > 0$ *small enough, there exist* $\lambda^* > 0$ *and* $e > 0$ *such that for all* $u \in V$ *with* $||u||_a = \rho$, *we have* $J_\lambda(u) \geq e > 0$, *for any* $\lambda \in (0, \lambda^*)$.

Proof. For any $u \in V$ with $||u||_a = \rho$, we have

$$\begin{aligned} J_\lambda(u) &= L(u) - \lambda \int_\Omega \frac{m(x)}{q(x)} \frac{|u|^{q(x)}}{|x|^{q(x)}} dx - \int_\Omega F(x, u) dx \\ &\geq \frac{1}{p^+} \|u\|_a^{p(x)} - \frac{\lambda}{q^-} \|m\|_\infty \left| \int_\Omega \frac{|u|^{q(x)}}{|x|^{q(x)}} dx \right| - \int_\Omega F(x, u) dx. \end{aligned}$$

By (f_1) and (f_4), it follows that for any $\epsilon > 0$ there exists $C_\epsilon = C(\epsilon) > 0$ depending on ϵ such that

$$|F(x,t)| \leq \frac{\epsilon}{p(x)}|t|^{p(x)} + \frac{C_\epsilon}{r(x)}|t|^{r(x)} \text{ for all } (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.6}$$

by Lemma 3.1 and for $\|u\|_a = \rho$ small enough, we obtain

$$\int_\Omega \frac{|u|^{q(x)}}{|x|^{q(x)}} dx \leq C_2(\|u\|_a^{q^+} + \|u\|_a^{q^-}) \leq 2C_2\|u\|_a^{q^-} \tag{3.7}$$

Therefore by (3.6) and (3.7), we deduce that

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|_a^{p(x)} - \frac{2\lambda C_2}{q^-}\|m\|_\infty\|u\|_a^{q^-} - \frac{\epsilon}{p^-}\int_\Omega |u|^{p(x)}dx - \frac{C_\epsilon}{r^-}\int_\Omega |u|^{r(x)}dx. \tag{3.8}$$

Hence for ϵ sufficiently small and due to embedding $V \hookrightarrow L^{r(x)}(\Omega)$, there exist $c_2, > 0$ such that

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|_a^{p^+} - \frac{2\lambda C_2}{q^-}\|m\|_\infty\|u\|_a^{q^-} - \frac{C_\epsilon}{r^-}c_2\|u\|_a^{r^-}$$

Since $q^- < p^+ < r^-$, then for ρ small enough, we can find $\lambda^* > 0$ and $e > 0$ such that $J_\lambda(u) \geq e > 0$ for any $\lambda \in (0;\lambda^*)$ and for $||u||_a = \rho$. □

Lemma 3.9. *The functional J_λ satisfies the (P.S) condition in V. that is, any sequence $(u_n) \subset V$ which satisfies the properties*

$$|J_\lambda(u_n)| \leq c \text{ and } J'_\lambda(u_n) \longrightarrow 0 \text{ in } V^* \text{ as } n \longrightarrow \infty,$$

prossesses a convergent subsequence in V, where V^* is the dual space of V.

Proof. Let $(u_n) \subset V$ be a sequence such that

$$J_\lambda(u_n) \longrightarrow \overline{c} > 0 \text{ and } J'_\lambda(u_n) \longrightarrow 0 \text{ in } V^* \text{ as } n \longrightarrow \infty, \tag{3.9}$$

We first prove that (u_n) is bounded in V. Indeed, we assume the contrary.
Then, passing eventually to a subsequence still denoted (u_n), we may assume that $||u_n||_a \longrightarrow \infty$ as $n \longrightarrow \infty$.
we deduce from (3.9) that

$$\begin{aligned}
\overline{c} + 1 + ||u_n||_a &\geq J_\lambda(u_n) - \frac{1}{\theta} < J'_\lambda(u_n), u_n > \\
&\geq \frac{1}{p^+}\|u_n\|_a^{p(x)} - \lambda\int_\Omega \frac{m(x)}{q(x)}\frac{|u_n|^{q(x)}}{|x|^{q(x)}}dx - \int_\Omega F(x,u_n)dx - \frac{1}{\theta}\|u_n\|_a^{p(x)} \\
&+ \frac{\lambda}{\theta}\int_\Omega m(x)\frac{|u_n|^{q(x)}}{|x|^{q(x)}}dx + \frac{1}{\theta}\int f(x,u_n)u_n dx \\
&\geq (\frac{1}{p^+} - \frac{1}{\theta})\|u_n\|_a^{p(x)} + \int_\Omega \frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)dx - \frac{\lambda}{q^-}\|m\|_\infty\int_\Omega \frac{|u_n|^{q(x)}}{|x|^{q(x)}}dx \\
&- \frac{\lambda}{\theta}\|m\|_\infty\int_\Omega \frac{|u_n|^{q(x)}}{|x|^{q(x)}}dx
\end{aligned}$$

by (3.7) and for $||u_n||_a \longrightarrow \infty$, we obtain

$$\int_\Omega \frac{|u_n|^{q(x)}}{|x|^{q(x)}}dx \leq 2C_2\|u_n\|_a^{q^+}. \tag{3.10}$$

then using (f_2) we have

$$\overline{c} + 1 + ||u_n||_a \geq (\frac{1}{p^+} - \frac{1}{\theta})\|u_n\|_a^{p(x)} - \frac{2\lambda C_2}{q^-}\|m\|_\infty\|u_n\|_a^{q^+} - \frac{2\lambda C_2}{\theta}\|m\|_\infty\|u_n\|_a^{q^+}$$

Since $\theta > p^+$ and $1 < q^+ < p^-$, wich contraduct the fact that $||u_n||_a \longrightarrow +\infty$. Then (u_n) is bounded in V.
Therefore, there exists a subsequence still denoted by (u_n) and $u \in V$ such that

$$u_n \rightharpoonup u \text{ in } V \text{ as } n \longrightarrow \infty, \tag{3.11}$$

then

$$\lim_{n\longrightarrow\infty} < J'_\lambda(u_n), u_n - u >= 0.$$

More exactly, we have

$$\begin{aligned} 0 = \lim_{n\longrightarrow\infty} \Big(\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n-u)dx + \int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx \\ - \int_\Omega f(x,u_n)(u_n-u)dx - \lambda \int_\Omega m(x)\frac{|u_n|^{q(x)-2}}{|x|^{q(x)}}u_n(u_n-u)dx \Big). \end{aligned}$$

By Hölder inequality (2.1), Proposition 2.4 and (3.7), we obtain

$$\Big| \int_\Omega m(x)\frac{|u_n|^{q(x)-2}}{|x|^{q(x)}}u_n(u_n-u)dx \Big| \le 2\|m\|_\infty \Big| \frac{|u_n|^{q(x)-1}}{|x|^{q(x)-1}} \Big|_{q'(x)} \Big| \frac{u_n-u}{|x|} \Big|_{q(x)},$$

by using Lemma (3.1) we obtain

$$\Big| \frac{|u_n|^{q(x)-1}}{|x|^{q(x)-1}} \Big|_{q'(x)} \le \Big(\|u_n\|_a^{q^+-1} + \|u_n\|_a^{q^--1} \Big)^{\frac{q^--1}{q(x)}} + \Big(\|u_n\|_a^{q^+-1} + \|u_n\|_a^{q^--1} \Big)^{\frac{q^+-1}{q(x)}}.$$

Since (u_n) bounded in V, then there exist C' such that,

$$\Big| \frac{|u_n|^{q(x)-1}}{|x|^{q(x)-1}} \Big|_{q'(x)} \le C'.$$

Thus to show that $\lim_{n\longrightarrow\infty} \Big| \int_\Omega m(x)\frac{|u_n|^{q(x)-2}}{|x|^{q(x)}}u_n(u_n-u)dx \Big| = 0$,
we prove that $\lim_{n\longrightarrow\infty} \Big| \frac{u_n-u}{|x|} \Big|_{q(x)} = 0$, for every $\epsilon > 0$ we have

$$\begin{aligned} \int_\Omega \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx &= \int_{B(0,\epsilon)} \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx + \int_{\Omega\setminus B(0,\epsilon)} \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx \\ &\le \int_{B(0,\epsilon)} \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx + \Big(\frac{1}{\epsilon^{q^+}} + \frac{1}{\epsilon^{q^-}} \Big) \int_\Omega |u_n-u|^{q(x)}dx \end{aligned}$$

and due to embedding $V \hookrightarrow L^{q(x)}(\Omega)$, we get $\Big(\frac{1}{\epsilon^{q^+}} + \frac{1}{\epsilon^{q^-}}\Big)|u_n-u|_{q(x)}^{q(x)} \to 0$ as $n \to +\infty$.
On the other hand

$$\int_{B(0,\epsilon)} \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx = \int_\Omega \chi_{B(0,\epsilon)} \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx,$$

we have $q^+ < p^-$ then there existe α such that $q^+ < \alpha < p^-$.
By Hölder inequality, Proposition 2.4 and Lemma 3.1, we deduce

$$\begin{aligned} \int_\Omega \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx &\le 2(mes(B(0,\epsilon))^{\frac{\alpha}{\alpha-q(x)}} \Big| \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}} \Big|_{\frac{\alpha}{q(x)}} + o(n) \\ &\le 2(mes(B(0,\epsilon))^{\frac{\alpha}{\alpha-q^-}} \Big(\Big| \frac{|u_n-u|}{|x|} \Big|_\alpha^{q^+} + \Big| \frac{|u_n-u|}{|x|} \Big|_\alpha^{q^-} \Big) + o(n) \\ &\le C\epsilon^{\frac{N\alpha}{\alpha-q^-}} (\|u_n-u\|_a^{q^+} + \|u_n-u\|_a^{q^-}) + o(n). \end{aligned}$$

Since (u_n) is bounded in V, then there exists $M>0$ such that $\|u_n-u\|_a\leq M$ for evry $n\in\mathbb{N}$. Therefore

$$\int_\Omega \frac{|u_n-u|^{s(x)}}{|x|^{q(x)}}dx \leq C\epsilon^{\frac{N\alpha}{\alpha-q^-}}(M^{q^+}+M^{q^-})+o(n),$$

hence for every $\epsilon>0$,

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n-u|^{q(x)}}{|x|^{q(x)}}dx \leq C\epsilon^{\frac{N\alpha}{\alpha-s^-}}(M^{q^+}+M^{q^-}).$$

Which implies that

$$\lim_{n\longrightarrow\infty}\left|\frac{u_n-u}{|x|}\right|_{q(x)}=0$$

Then

$$\lim_{n\longrightarrow\infty}\left|\int_\Omega m(x)\frac{|u_n|^{q(x)-2}}{|x|^{q(x)}}u_n(u_n-u)dx\right|=0.$$

From (f_1), Hölder inequality and Proposition 2.4 we have

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| &\leq \int_\Omega |f(x,u_n)||(u_n-u)|dx\\ &\leq C\|u_n-u\|_{L^1}+2C\left||u_n|^{r(x)-1}\right|_{r'(x)}|u_n-u|_{r(x)}\\ &\leq C\|u_n-u\|_{L^1}+2C\left(|u_n|_{r(x)}^{r^+-1}+|u_n|_{r(x)}^{r^--1}\right)|u_n-u|_{r(x)}.\end{aligned}$$

and due to the compact embeddings $V\hookrightarrow L^1(\Omega)$ and $V\hookrightarrow L^{r(x)}(\Omega)$, we get

$$\lim_{n\longrightarrow\infty}\int_\Omega f(x,u_n)(u_n-u)dx=0.$$

By (a) and (2.1),we deduce that

$$\left|\int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx\right|\leq 2\|a\|_\infty\left||u_n|^{p(x)-1}\right|_{p'(x)}|u_n-u|_{p(x)}.$$

and due to the compact embeddings $V\hookrightarrow L^{p(x)}(\Omega)$, we get

$$\lim_{n\longrightarrow\infty}\int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx=0.$$

Therefore, we arrive at

$$\lim_{n\longrightarrow\infty}\int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n\Delta(u_n-u)dx=0.$$

And we know that $\Delta_p^2(x)$ is (S^+) type. Then $u_n\longrightarrow u$ in V as $n\longrightarrow+\infty$.
With this we conclude that J_λ verefie the palais-smale condition. □

Proof of Theorem 1.2. By Lemma 3.6, Lemma 3.8 and Lemma 3.9, all assumptions of Theorem 3.5 are satisfied. Then there exists $u_1\in V$ such that $J_\lambda(u_1)=\overline{c}>0$ for $\lambda\in(0,\lambda^*)$, as a nontrivial weak solution of problem (P_λ).
Let us show now, the existence of second nontrivial weak solution $u_2\neq u_1$ in V.
By Lemma 3.8, it follows that on the boundary of the ball centered at the origin and of the radius ρ in V, denoted by $B_\rho(0)$, we have

$$\underset{\partial B_\rho(0)}{inf}\ J_\lambda>0 \qquad \text{for} \qquad \lambda\in]0,\lambda^*[.$$

On the other hand, Lemma 3.7 yields the existence of $\varphi \in V$ such that $J_\lambda(t\varphi) < 0$ for $t > 0$ small enough. In addition, since the relation (3.8) for all $u \in B_\rho(0)$

$$J_\lambda(u) \geq \frac{1}{p^+}\|u\|_a^{p^+} - \frac{2\lambda C_2}{q^-}\|m\|_\infty \|u\|_a^{p^-} - \frac{C_\epsilon}{r^-} c_2 \|u\|_a^{r^-}$$

it follows that

$$-\infty < \underline{c} = \underset{\overline{B_\rho(0)}}{inf}\, J_\lambda < 0.$$

Applying Theorem 3.4 to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{aligned} &\underline{c} \leq J_\lambda(u_\epsilon) \leq \underline{c} + \epsilon \\ &0 < J_\lambda(u) - J_\lambda(u_\epsilon) + \epsilon||u - u_\epsilon||_a, \qquad u \neq u_\epsilon. \end{aligned}$$

Let us choose ϵ such that

$$0 < \epsilon < \underset{\partial B_\rho(0)}{inf}\, J_\lambda(u) - \underset{B_\rho(0)}{inf}\, J_\lambda(u).$$

Since $J_\lambda(u_\epsilon) \leq \underset{\partial B_\rho(0)}{inf}\, J_\lambda(u)$ and thus $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $\Psi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Psi_\lambda(u) = J_\lambda(u) + \epsilon||u - u_\epsilon||_a$. It is clear that u_ϵ is a minimum point of Ψ_λ and thus

$$\frac{\Psi_\lambda(u_\epsilon + tv) - \Psi_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and all $v \in B_1(0)$. The above relation yields

$$\frac{J_\lambda(u_\epsilon + tv) - J_\lambda(u_\epsilon)}{t} + \epsilon||v||_a \geq 0.$$

Letting $t \longrightarrow 0^+$, it follows that $< J'_\lambda(u_\epsilon), v > \geq -\epsilon||v||_a$, it should be noticed that $-v$ also belong to $B_1(0)$, so replacing v by $-v$, we get

$$< J'_\lambda(u_\epsilon), v > \leq \epsilon||v||_a,$$

which helps us to deduce that $|J'_\lambda(u_\epsilon)|_{V^*} \leq \epsilon$.
Therefore, there exists a sequence $(u_n) \subset B_\rho(0)$ such that

$$J_\lambda(u_n) \longrightarrow \underline{c} < 0 \text{ and } J'_\lambda(u_n) \longrightarrow 0 \text{ in } V^*. \tag{3.12}$$

From Lemma 3.9, (u_n) converge strongly to u_2 in V.
Since $J_\lambda \in C^1(V, \mathbb{R})$, by (3.12) it follows that $J_\lambda(u_2) = \underline{c}$ and $J'_\lambda(u_2) = 0$. Thus u_2 is a nontrivial weak solution of problem (P_λ). Finally, since

$$J_\lambda(u_1) = \overline{c} > 0 > \underline{c} = J_\lambda(u_2),$$

it is clear that $u_1 \neq u_2$. The proof is complete. □

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