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Topological invariants for the triangular graph of even degree vertices and its line graph

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ABSTRACT: Graph theory has an important role in the field of computer science, for example, networks of communication; in electronic and electrical engineering, used in, the industrial processes, electronic instrumentation; in chemistry, for example, the study of molecules and its properties, etc. The triangular graph with even degree vertices, TG_k, associated with a number $k \in Z^+$ is the graph $TG_k = (V, E)$ with $|V| = n = \frac{k(k+1)}{2}$ and $|E| = m = \frac{3k(k-1)}{2}$, $k \in Z^+$, where k denotes the number of vertices in the base graph. A topological index (T_I) is a real number attached with the networks graph and correlates the chemical networks with many chemical and physical properties. In this paper, our aim is to compute the degree-based indices of the TG_k and its line graph. In [10], the M-polynomial is defined as $M(G; x, y) = \sum_{i \le j} m_{ij}(G)x^i y^j$, where G is a graph, $x, y \in V(G)$ and $m_{ij}(G), i, j \ge 1$ is the number of edges of G. To compute degree-based indices, we firstly computed the M-polynomials and then from the M-polynomial we recover eleven degree-based topological indices of the TG_k and its line graph.

Key Words: Line graph, M-polynomial, Degree-based topological descriptors.

Contents

1. Introduction

Chemical graph theory is an important area of research in mathematical chemistry which deals with topology of molecular structure such as the mathematical study of isomerism and the development of topological descriptors or indices TIs. In fact, TIs are real numbers attached with graph networks and graph of chemical compounds and have applications in quantitative structure-property relationships. TIs remain invariant up to graph isomorphism and help to predict many properties of chemical compounds, networks and nanomaterials, for example, boiling points, viscosity, radius of gyrations, etc. [14], [15], [17], [35]. Deutsch and Klavžar [10] introduced M-polynomial in 2015 and plays a similar role in computation of numerous degree-based TIs [25]-[28], [1]. The M-polynomial contains precious information about degree-based TIs and many TIs can be computed from this simple algebraic polynomial. The first TI was defined in 1947 by Weiner, during studying boiling point of alkanes [34]. This index is now known as Weiner index. Thus Weiner established the framework of TIs and the Wiener index is initially the first and most concentrated TI [11], [20]. The other oldest TI is Randić index (RI), given by Milan Randić [31] in 1975. After the success of Randić index, the generalized version of Randić index was introduced [2], [6]. This version attracts both the mathematicians and chemists [23]. Numerous numerical properties of this simple TI are studied in [8]. For comprehensive study about this index, the book [21] can be of great help. After Randić index, the most studied TIs were 1st Zagreb index (ZI) and 2nd ZI [9], [16], [19], [30], [32]. The modified 2nd ZI was defined in [24]. Another TI are the symmetric division index (SDI) [18], Harmonic index (HI) [12], [13], augmented ZI [3].

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2. Methodology

In this paper, we focus on degree-based topological indices. To compute degree-based topological indices of the triangular graph with even degree vertices and its line graph, firstly we drawn its graphs and then we divide the edge set of these graphs into classes based on the degree of the end vertices and compute its cardinality. From this edge partition, we compute our desired results. First we compute M-polynomials of the families of graphs in study. Then by applying calculus and using table 1, we compute several degree-based topological indices.

The relationship between M-polynomial and indices are presented in Table 1 [34].

Topological Index	Derivation from $M(G;x,y)$
First Zagreb Index = M_1	$(D_x + D_y)(M(G;x,y)) \mid_{x=y=1}$
Second Zagreb Index = M_2	$(D_x D_y)(M(G;x,y)) \mid_{x=y=1}$
Forgotten topological index = F	$(D_x^2 + D_y^2)(M(G;x,y)) \mid_{x=y=1}$
Redefined Third Zagreb = $ReZG_3$	$D_x D_y(D_x + D_y)(M(G;x,y)) \mid_{x=y=1}$
Second modified Zagreb index = ${}^mM_2$	$(S_x S_y)(M(G;x,y)) \mid_{x=y=1}$
General Randic index = R_α	$(D_x D_y)(M(G;x,y)) \mid_{x=y=1}$
General inverse Randic index = $R_{\alpha\alpha}$	$(S_x S_y)(M(G;x,y)) \mid_{x=y=1}$
Symmetric division deg index =SSD	$(D_x S_y + S_x D_y)(M(G;x,y)) \mid_{x=y=1}$
Harmonic index = H	$(2S_x J)(M(G;x,y)) \mid_{x=y=1}$
Inverse sum index = I	$(S_x J D_x D_y)(M(G;x,y)) \mid_{x=y=1}$
Augmented Zagreb index = A	$(S_x^3 Q_{-2} J D_x^3 D_y^3)(M(G;x,y)) \mid_{x=y=1}$

Table 1

Where

$$D_x(g(x,y)) = \frac{x\,\partial(g(x,y))}{\partial x} \qquad D_y(g(x,y)) = \frac{y\,\partial(g(x,y))}{\partial y}$$

$$S_x(g(x,y)) = \int_0^x \frac{g(t,y)}{t}\,dt \qquad S_y(g(x,y)) = \int_0^y \frac{g(x,t)}{t}\,dt$$

$$J(g(x,y)) = g(x,x) \qquad Q_\alpha(g(x,y)) = x^\alpha g(x,y)$$

3. M-polynomial of the triangular graph with even degree vertices

In this section, we study the triangular graph with even degree vertices. The triangular graph with even degree vertices TG_k, associated with a number $k \in Z^+$ is the graph $TG_k = (V,E)$ with $|V| = n = \frac{k(k+1)}{2}$ and $|E| = m = \frac{3k(k-1)}{2}$, $k \in Z^+$, where k denotes the number of vertices in the base graph and contains results on the triangular graph with even degree vertices, shown in Figure 1.

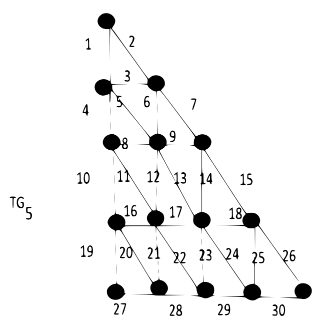

Figure 1: TG_5 the triangular graph with even degree vertices.

Theorem 3.1 *If TG_k is the triangular graph with even degree vertices, where k denotes the number of vertices in the base graph, then we have*

$$MTG_k(x,y) = 6x^2y^4 + 3(k-2)x^4y^4 + 6(k-3)x^4y^6 + \sum_{i=4}^{k} 3(i-4)x^6y^6.$$

Proof: Using Figure 1, we have:

$$|V(TG_k)| = \frac{k(k+1)}{2} \text{ and } |E(TG_k)| = \frac{3k(k-1)}{2},\ k \in Z^+.$$

We can divide the vertex set of TG_k in the following three types depending on the degree:

1. $V_1(TG_k) = \{v \in V(TG_k) : dv = 2\}$,

2. $V_2(TG_k) = \{v \in V(TG_k) : dv = 4\}$,

3. $V_3(TG_k) = \{v \in V(TG_k) : dv = 6\}$.

Such that

$$|V_1(TG_k)| + |V_2(TG_k)| + |V_3(TG_k)| = |V(TG_k)|.$$

We can divide the edge set of TG_k in the following four classes depending on each edge at the degree of the end vertices:

1. $E_1(TG_k) = \{xy \in E(TG_k) : d(x) = 2, d(y) = 4\}$,

2. $E_2(TG_k) = \{xy \in E(TG_k) : d(x) = 4, d(y) = 4\}$,

3. $E_3(TG_k) = \{xy \in E(TG_k) : d(x) = 4, d(y) = 6\}$,

4. $E_4(TG_k) = \{xy \in E(TG_k) : d(x) = 6, d(y) = 6\}$.

Now

$$|E_1(TG_k)| = 6, \qquad |E_2(TG_k)| = 3(k-2),$$

$$|E_3(TG_k)| = 6(k-3), \qquad |E_4(TG_k)| = \sum_{i=4}^{k} 3(i-4)$$

By definition of M-polynomial,

$$\begin{aligned} MTG_k(x,y) &= \sum_{\delta \le i \le j \le \Lambda} m_{i,j} x^i y^j \\ &= \sum_{xy \in E_1(TG_k)} m_{2,4} x^2 y^4 + \sum_{xy \in E_2(TG_k)} m_{4,4} x^4 y^4 \\ &\quad + \sum_{xy \in E_3(TG_k)} m_{4,6} x^4 y^6 + \sum_{xy \in E_4(TG_k)} m_{6,6} x^6 y^6 \end{aligned}$$

Therefore, we obtain

$$MTG_k(x,y) = 6x^2y^4 + 3(k-2)x^4y^4 + 6(k-3)x^4y^6 + \sum_{i=4}^{k} 3(i-4)x^6y^6.$$

□

Proposition 3.1 *Let TG_k be the triangular graph with even degree vertices, where k denotes the number of vertices in the base graph, then we have:*

1. $M_1(TG_k) = 36\sum_{i=4}^{k} i + 84k - 336,$

2. $M_2(TG_k) = 108\sum_{i=4}^{k} i + 192k - 912,$

3. $F(TG_k) = 216\sum_{i=4}^{k} i + 408k - 1872,$

4. $ReZG_3(TG_k) = 1296\sum_{i=4}^{k} i + 1824k - 9984,$

5. ${}^mM_2(TG_k) = \frac{1}{12}\sum_{i=4}^{k} i + \frac{7}{16}k - 17/8,$

6. $$\begin{aligned} R_\alpha(TG_k) = \;& 3^{2\alpha+1}.2^{2\alpha}\sum_{i=4}^{k}(i-4) + 3.2^{4\alpha}(k-2) \\ & +3^{\alpha+1}.2^{3\alpha+1}(k-3) + 3.2^{3\alpha+1} \end{aligned}$$

7. $$\begin{aligned} RR_\alpha(TG_k) = \;& 3^{1-2\alpha}.2^{1-2\alpha}\sum_{i=4}^{k}(i-4) + 3.2^{-4\alpha}(k-2) \\ & +3^{1-2\alpha}.2^{1-2\alpha}(k-3) + 3.2^{1-3\alpha} \end{aligned}$$

8. $SSD(TG_k) = 6\sum_{i=4}^{k} i + 19k - 60$

9. $H(TG_k) = \frac{1}{2}\sum_{i=4}^{k} i + \frac{39}{20}k - \frac{51}{10},$

10. $I(TG_k) = 9\sum_{i=4}^{k} i + \frac{102}{5}k - \frac{416}{5},$

11. $A(TG_k) = \sum_{i=4}^{k} \frac{17496}{125}i + \frac{1970}{9}k - \frac{1250606}{1125}.$

Proof: Using Theorem 3.1,

$$\begin{aligned} f(x,y) = MTG_k(x,y) = \;& 6x^2y^4 + 3(k-2)x^4y^4 + 6(k-3)x^4y^6 \\ & +\sum_{i=4}^{k} 3(i-4)x^6y^6 \end{aligned}$$

And then, doing the respective calculus, we obtain:

1.- $$D_x(f(x,y)) = 12x^2y^4 + 12(k-2)x^4y^4 + 24(k-3)x^4y^6 + \sum_{i=4}^{k} 18(i-4)x^6y^6$$

2.- $$D_y(f(x,y)) = 24x^2y^4 + 12(k-2)x^4y^4 + 36(k-3)x^4y^6 + \sum_{i=4}^{k} 18(i-4)x^6y^6.$$

3.- $$(D_x + D_y)(f(x,y)) = 36x^2y^4 + 24(k-2)x^4y^4 + 60(k-3)x^4y^6 + \sum_{i=4}^{k} 36(i-4)x^6y^6.$$

4.- $$(D_x^2)(f(x,y)) = 24x^2y^4 + 48(k-2)x^4y^4 + 96(k-3)x^4y^6 + \sum_{i=4}^{k} 108(i-4)x^6y^6.$$

5.- $$(D_y^2)(f(x,y)) = 96x^2y^4 + 48(k-2)x^4y^4 + 216(k-3)x^4y^6 + \sum_{i=4}^{k} 108(i-4)x^6y^6.$$

6.- $$(D_x^2 + D_y^2)(f(x,y)) = 120x^2y^4 + 96(k-2)x^4y^4 + 312(k-3)x^4y^6 + \sum_{i=4}^{k} 216(i-4)x^6y^6.$$

7.- $$D_yD_x(f(x,y)) = 48x^2y^4 + 48(k-2)x^4y^4 + 144(k-3)x^4y^6 + \sum_{i=4}^{k} 108(i-4)x^6y^6.$$

8.- $$D_xD_y(f(x,y)) = 48x^2y^4 + 48(k-2)x^4y^4 + 144(k-3)x^4y^6 + \sum_{i=4}^{k} 108(i-4)x^6y^6.$$

9.- $$D_y(D_x + D_y)(f(x,y)) = 144x^2y^4 + 96(k-2)x^4y^4 + 360(k-3)x^4y^6 + \sum_{i=4}^{k} 216(i-4)x^6y^6.$$

10.- $$D_xD_y(D_x + D_y)(f(x,y)) = 288x^2y^4 + 384(k-2)x^4y^4 + 1440(k-3)x^4y^6 + \sum_{i=4}^{k} 1296(i-4)x^6y^6$$

11.- $$S_y(f(x,y)) = \frac{3}{2}x^2y^4 + \frac{3(k-2)}{4}x^4y^4 + (k-3)x^4y^6 + \sum_{i=4}^{k} \frac{(i-4)}{2}x^6y^6.$$

12.- $$S_y^{\alpha}(f(x,y)) = \frac{3}{2^{2\alpha-1}}x^2y^4 + \frac{3(k-2)}{2^{2\alpha}}x^4y^4 + \frac{(k-3)}{6^{\alpha-1}}x^4y^6 + \sum_{i=4}^{k}\frac{(i-4)}{3^{\alpha-1}2^{\alpha}}x^6y^6.$$

13.- $$S_xS_y(f(x,y)) = \frac{3}{4}x^2y^4 + \frac{3(k-2)}{16}x^4y^4 + \frac{(k-3)}{4}x^4y^6 + \sum_{i=4}^{k}\frac{(i-4)}{12}x^6y^6.$$

14.- $$S_x^{\alpha}S_y^{\alpha}(f(x,y)) = \frac{3}{2^{3\alpha-1}}x^2y^4 + \frac{3(k-2)}{2^{4\alpha}}x^4y^4 + \frac{3^{1-\alpha}(k-3)}{2^{3\alpha-1}}x^4y^6 + \sum_{i=4}^{k}\frac{3^{1-2\alpha}(i-4)}{2^{2\alpha}}x^6y^6$$

15.- $$D_y^{\alpha}(f(x,y)) = 6.4^{\alpha}x^2y^4 + 3.4^{\alpha}(k-2)x^4y^6 + 6^{\alpha+1}.4^{\alpha}(k-3)x^4y^6 + \sum_{i=4}^{k}3.6^{\alpha}(i-4)x^6y^6$$

16.- $$D_x^{\alpha}D_y^{\alpha}(f(x,y)) = 3.2^{3\alpha+1}x^2y^4 + 3.2^{4\alpha}(k-2)x^4y^4 + 6^{\alpha+1}\,2^{2\alpha}(k-3)x^4y^6 + \sum_{i=4}^{k}3.6^{2\alpha}(i-4)x^6y^6$$

17.- $$S_yD_x(f(x,y)) = 3x^2y^4 + 3(k-2)x^4y^4 + 4(k-3)x^4y^6 + \sum_{i=4}^{k}3(i-4)x^6y^6$$

18.- $$S_xD_y(f(x,y)) = 12x^2y^4 + 3(k-2)x^4y^4 + 9(k-3)x^4y^6 + \sum_{i=4}^{k}3(i-4)x^6y^6$$

19.- $$(S_yD_x + S_xD_y)(f(x,y)) = 15x^2y^4 + 6(k-2)x^4y^4 + 13(k-3)x^4y^6 + \sum_{i=4}^{k}6(i-4)x^6y^6$$

20.- $$J(f(x,y)) = 6x^6 + 3(k-2)x^8 + 6(k-3)x^{10} + \sum_{i=4}^{k}3(i-4)x^{12}$$

21.- $$S_xJ(f(x,y)) = x^6 + \frac{3(k-2)}{8}x^8 + \frac{3(k-3)}{5}x^{10} + \sum_{i=4}^{k}\frac{(i-4)}{4}x^{12}$$

22.- $$JD_xD_y(f(x,y)) = 48x^6 + 48(k-2)x^8 + 144(k-3)x^{10} + \sum_{i=4}^{k}108(i-4)x^{12}$$

23.- $S_xJD_xD_y(f(x,y)) = 8x^6 + 6(k-2)x^8 + \frac{72(k-3)}{5}x^{10} + \sum_{i=4}^{k} 9(i-4)x^{12}.$

24.- $D_x^\alpha D_y^\alpha(f(x,y)) = 3.2^{3\alpha+1}x^2y^4 + 3.2^{4\alpha}(k-2)x^4y^4 + 6^{\alpha+1}2^{2\alpha}(k-3)x^4y^6 + \sum_{i=4}^{k} 3.6^{2\alpha}(i-4)x^6y^6$

Taking $\alpha = 3$, we obtain:

25.- $D_x^3D_y^3(f(x,y)) = 3.2^{10}x^2y^4 + 3.2^{12}(k-2)x^4y^4 + 3^42^{10}(k-3)x^4y^6 + \sum_{i=4}^{k} 2^63^7(i-4)x^6y^6$

26.- $JD_x^3D_y^3(f(x,y)) = 3.2^{10}x^6 + 3.2^{12}(k-2)x^8 + 3^42^{10}(k-3)x^{10} + \sum_{i=4}^{k} 3^72^6(i-4)x^{12}$

27.- $Q_{-2}JD_x^3D_y^3(f(x,y)) = 3.2^{10}x^4 + 3.2^{12}(k-2)x^6 + 3^42^{10}(k-3)x^8 + \sum_{i=4}^{k} 3^72^6(i-4)x^{10}$

28.- $S_x^3Q_{-2}JD_x^3D_y^3(f(x,y)) = 3.2^4x^4 + \frac{2^9(k-2)}{9}x^6 + 2.3^4(k-3)x^8 + \sum_{i=4}^{k} \frac{2^23^7}{5^3}(i-4)x^{10}$ $\square$

Now using the above calculus, we obtain the Topological Indices, described in Table 1.

(1) First Zagreb index

$$M_1(TG_k) = (D_x + D_y)(M(G_k;x,y))|_{x=y=1} = 36\sum_{i=4}^{k} i + 84k - 336$$

(2) Second Zagreb index

$$M_2(TG_k) = (D_xD_y)(M(G_k;x,y))|_{x=y=1} = 108\sum_{i=4}^{k} i + 192k - 912$$

(3) Forgotten topological index

$$F(TG_k) = (D_x^2 + D_y^2)(MG_k(x,y))|_{x=y=1} = 216\sum_{i=4}^{k} i + 408k - 1872$$

(4) Forgotten topological index

$$F(TG_k) = (D_x^2 + D_y^2)(MG_k(x,y))|_{x=y=1} = 216\sum_{i=4}^{k} i + 408k - 1872$$

(5) Redefined third Zagreb index

$$\begin{aligned} ReZG_3(TG_k) &= D_xD_y(D_x+D_y)(M_k(G_k;x,y))|_{x=y=1} \\ &= 1296\sum_{i=4}^{k} i + 1824k - 9984 \end{aligned}$$

(6) Modified Second Zagreb index

$${}^mM_2(TGk) = (S_xS_y)(M(G_k;x,y))|_{x=y=1} = \frac{1}{12}\sum_{i=4}^{k} i + \frac{7}{16}k - \frac{11}{24}$$

(7) Generalized Randic Index

$$\begin{aligned} R_\alpha(TG_k) = &\ (D_x^\alpha D_y^\alpha)(M(G_k;x,y))|_{x=y=1} \\ = &\ 3.2^{3\alpha+1} + 3.2^{4\alpha}(k-2) + 3^{\alpha+1}.2^{3\alpha+1}(k-3) \\ &+3^{2\alpha+1}.2^{2\alpha}\sum_{i=4}^{k}(i-4). \end{aligned}$$

(8) Inverse Randic Index

$$\begin{aligned} RR_\alpha(TG_k) = &\ (S_x^\alpha S_y^\alpha)(M(G_k;x,y))|_{x=y=1} \\ = &\ \frac{3}{2^{2\alpha-1}} + \frac{3(k-2)}{2^{4\alpha}} + \frac{3^{1-\alpha}(k-3)}{2^{3\alpha-1}} + \sum_{i=4}^{k}\frac{3^{1-2\alpha}(i-4)}{2^{2\alpha}}. \end{aligned}$$

(9) Symmetric Division Deg index

$$SSD(TG_k) = (S_yD_x + S_xD_y)(M(G_k;x,y))|_{x=y=1} = 6\sum_{i=4}^{k} i + 19k - 60$$

(10) Harmonic index

$$H(TG_k) = 2S_xJ(M(G_k;x,y))|_{x=1} = \frac{1}{2}\sum_{i=4}^{k} i + \frac{39}{20}k - \frac{51}{10}$$

(11) Inverse Sum index

$$I(TG_k) = S_xJD_xD_y(M(G_k;x,y))|_{x=1} = 9\sum_{i=4}^{k} i + \frac{102}{5}k - \frac{416}{5}$$

(12) Augmented Zagreb index

$$A(TG_k) = S_x^3Q_2JD_x^3D_y^3(M(G;x,y))|_{x=1} = \sum_{i=4}^{k}\frac{2^33^7i}{5^3} + \frac{1970k}{9} - \frac{1250606}{1125}$$

The line graph of a graph TG_k which is defined as the graph whose vertices are the edges of TG_k, with two vertices are adjacent if the corresponding edges have one common vertex in TG_k.

The line graph of the triangular graph with even degree vertices in TG_k is the graph $LTG_k = (V,E)$ with $|VTG_k| = \frac{3k(k-1)}{2}$ and

$$|ETG_k| = \sum_{i=4}^{k} 15(i-2) + 3, \quad k \in Z^+$$

4. M-polynomial of line graph of the triangular graph with even degree vertices

This section contains results on the line graph of the triangle graph with even degree vertice and is denoted by LTG_k, shown in Figura 2.

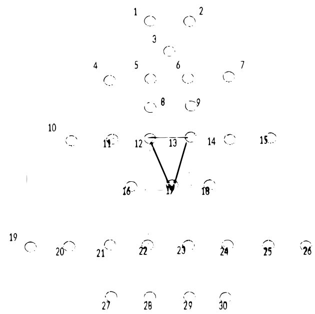

Figure 2: LTG_5 the line graph of the triangle graph with even degree vertices

Theorem 4.1 *If LTG_k is the line graph of the triangular graph with even degree vertices, then:*

$$MLTG_k(x,y) = \begin{cases} f(x,y), & if \quad k=3 \\ g(x,y)+h(x,y), & if \quad k>3, \end{cases}$$

Where

$$f(x,y) = 3x^4y^4\left(2+4y^2+x^2y^2\right)$$

$$g(x,y) = 3x^4y^4\left(1+4y^2+2y^4+(k-2)x^2y^2+4(k-3)x^2y^4\right)$$

$$h(x,y) = 3x^8y^8\left((2k-3)+8(k-4)y^2+\frac{5k^2-43k+92}{2}x^2y^2\right)$$

Proof: We analyze the following cases:

Case 1: (When $k=3$). We can divide the vertex set of LTG_k in the following two types depending on the degree:

$$V_1(LTG_3) = \{v \in V(LTG_3) : dv = 4\}$$

$$V_2(LTG_3) = \{v \in V(LTG_3) : dv = 6\}$$

Such that $|V_1(LTG_3)| + |V_2(LTG_3)| = |V(LTG_3)|$. We can divide the edge set of LTG_3 in the following three classes depending on each edge at the degree of end vertices:

$$E_1(LTG_3) = \{xy \in E(LTG_3) : d(x)=4, d(y)=4\}$$

$$E_2(LTG_3) = \{xy \in E(LTG_3) : d(x)=4, d(y)=6\}$$

$$E_3(LTG_3) = \{xy \in E(LTG_3) : d(x)=6, d(y)=6\}.$$

Now

$$|E_1(LTG_3)| = 6, \qquad |E_2(LTG_3)| = 12, \qquad |E_3(LTG_3)| = 3$$

By definition of M-polynomial,

$$MLTG_k(x,y) = \sum_{4\le i\le j\le 6} m_{i,j}x^iy^j,$$

then for $k=3$, we have

$$\begin{aligned} MLTG_3(x,y) = & \sum_{4\le i\le j\le 6} m_{i,j}x^iy^j \\ = & \sum_{xy\in E_1(LTG_3)} m_{4,4}x^4y^4 + \sum_{xy\in E_2(LTG_3)} m_{4,6}x^4y^6 \\ & + \sum_{xy\in E_3(LTG_3)} m_{6,6}x^6y^6 = 6x^4y^4+12x^4y^6+3x^6y^6. \end{aligned}$$

Therefore,

$$MLG_3(x,y) = 3x^4y^4\left(2+4y^2+x^2y^2\right) = f(x,y).$$

Case 2: (When $k > 3$). We can divide the vertex set of LTG_k in the following four types, depending on the degree:

$$V_1(LTG_k) = \{v \in V(LTG_k) : dv = 4\}$$
$$V_2(LTG_k) = \{v \in V(LTG_k) : dv = 6\}$$
$$V_3(LTG_k) = \{v \in V(LTG_k) : dv = 8\}$$
$$V4(LTG_k) = \{v \in V(LTG_k) : dv = 10\}$$

Such that

$$|V_1(LTG_k)| + |V_2(LTG_k)| + |V_3(LTG_k)| + |V_4(LTG_k)| = |V(LTG_k)|.$$

We can divide the edge set of LTG_k in the following eight classes depending on each edge at the degree of end vertices:

$$E_1(LTG_k) = \{xy \in E(LTG_k) : d(x) = 4, d(y) = 4\},$$
$$E_2(LTG_k) = \{xy \in E(LTG_k) : d(x) = 4, d(y) = 6\},$$
$$E_3(LTG_k) = \{xy \in E(LTG_k) : d(x) = 4, d(y) = 8\},$$
$$E_4(LTG_k) = \{xy \in E(LTG_k) : d(x) = 6, d(y) = 6\},$$
$$E_5(LTG_k) = \{xy \in E(LTG_k) : d(x) = 6, d(y) = 8\},$$
$$E_6(LTG_k) = \{xy \in E(LTG_k) : d(x) = 8, d(y) = 8\},$$
$$E_7(LTG_k) = \{xy \in E(LTG_k) : d(x) = 8, d(y) = 10\},$$
$$E_8(LTG_k) = \{xy \in E(LTG_k) : d(x) = 10, d(y) = 10\}.$$

Now

$$\begin{array}{ccc} |E_1(LTG_k)| = 3 & |E_3(LTG_k)| = 6 & |E_5(LTG_k)| = 12(k-3) \\ |E_2(LTG_k)| = 12 & |E_4(LTG_k)| = 3(k-2) & |E_6(LTG_k)| = 3(2k-3) \end{array}$$
$$|E_7(LTG_k)| = 24(k-4) \quad |E_8(LTG_k)| = \tfrac{3}{2}(5k-23)(k-4)$$

By definition of M-polynomial,

$$
\begin{aligned}
MLTG_k(x,y) = & \sum_{4\le i\le j\le 6} m_{i,j}x^iy^j \\
= & \sum_{xy\in E_1(LTG_k)} m_{4,4}x^4y^4 + \sum_{xy\in E_2(LTG_k)} m_{4,6}x^4y^6 \\
& + \sum_{xy\in E_3(LTG_k)} m_{4,8}x^4y^8 + \sum_{xy\in E_4(LTG_k)} m_{6,6}x^6y^6 \\
& + \sum_{xy\in LTG_k} m_{6,8}x^6y^8 + \sum_{xy\in E_6(LTG_k)} m_{8,8}x^8y^8 \\
& + \sum_{xy\in E_7(LTG_k)} m_{8,10}x^8y^{10} + \sum_{xy\in E_8(LTG_k)} m_{10,10}x^{10}y^{10} \\
= & |E_1(LTG_k)|x^4y^4 + |E_2(LTG_k)|x^4y^6 \\
& +|E_3(LTG_k)|x^4y^8 + |E_4(LTG_k)|x^6y^6 \\
& +|E_5(LTG_k)|x^6y^8 + |E_6(LTG_k)|x^8y^8 \\
& ++|E_7(LTG_k)|x^8y^{10} + |E_8(LTG_k)|x^{10}y^{10}
\end{aligned}
$$

$$
\begin{aligned}
= \;& 3x^4y^4\left(1+4y^2+2y^4+(k-2)x^2y^2+4(k-3)x^2y^4\right) \\
& +3x^8y^8\left((2k-3)+8(k-4)y^2+\frac{5k^2-43k+92}{2}x^2y^2\right)
\end{aligned}
$$

Therefore,

$$MLTG_3(x,y) = g(x,y)+h(x,y).$$

□

Proposition 4.1 *Let LTG_k be the line graph of the triangular graph with even degree vertices, then we have:*

$$M_1(LTG_k) = \begin{cases} 204, & \text{if} \quad k=3 \\ 150k^2-558k+528, & \text{if} \quad k>3. \end{cases}$$

$$M_2(LTG_k) = \begin{cases} 492, & \text{if} \quad k=3 \\ 750k^2-3462k+4128, & \text{if} \quad k>3. \end{cases}$$

$$F(LTG_k) = \begin{cases} 1032, & \text{if} \quad k=3 \\ 1500k^2-6780k-7872, & \text{if} \quad k>3. \end{cases}$$

$$ReZG_3(LTG_k) = \begin{cases} 4944, & \text{if} \quad k=3 \\ 15000k^2-78396k+107328, & \text{if} \quad k>3. \end{cases}$$

$${}^mM_2(LTG_k) = \begin{cases} \dfrac{23}{24}, & \text{if} \quad k=3 \\ \dfrac{3}{40}k^2+\dfrac{197}{2400}k-\dfrac{11}{4800}, & \text{if} \quad k>3. \end{cases}$$

$$R_\alpha(LTG_k) = \begin{cases} 3.2^{4\alpha+1} + 3^{\alpha+1}.2^{3\alpha+2} + 3^{2\alpha+1}.2^{2\alpha}, & \textit{if} \quad k = 3 \\ g(k) + h(k), & \textit{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = 3.2^{2\alpha}\left(2^{2\alpha} + 3^{\alpha}.2^{\alpha+2} + 2^{3\alpha+1} + 3^{2\alpha}(k-2) + 3^{\alpha}.2^{2(\alpha+1)}(k-3)\right),$$

$$h(k) = 3.2^{6\alpha}(2k-3) + 3.2^{4\alpha+3}.5^{\alpha}(k-4) + 3.2^{2\alpha-1}.5^{2\alpha}(5k-23)(k-4)$$

$$RR_\alpha(LTG_k) = \begin{cases} 3.2^{1-4\alpha} + 3^{1-\alpha}.2^{2-3\alpha} + 3^{1-2\alpha}.2^{-2\alpha}, & \textit{if} \quad k = 3 \\ g(k) + h(k), & \textit{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = \frac{3}{2^{2\alpha}}\left(\frac{1}{2^{2\alpha}} + \frac{1}{3^{\alpha}.2^{\alpha-2}} + \frac{1}{2^{3\alpha-1}} + \frac{(k-2)}{3^{2\alpha}} + \frac{(k-3)}{3^{\alpha}.2^{2(\alpha-1)}}\right)$$

$$h(k) = \frac{3}{2^{2\alpha}}\left(\frac{(2k-3)}{2^{4\alpha}} + \frac{(k-4)}{2^{2\alpha-3}.5^{\alpha}} + \frac{(5k-23)(k-4)}{2.5^{2\alpha}}\right)$$

$$SSD(LTG_k) = \begin{cases} 48, & \textit{if} \quad k = 3 \\ 15k^2 - \frac{184}{5}k + \frac{106}{5}, & \textit{if} \quad k > 3. \end{cases}$$

$$H(LTG_k) = \begin{cases} \frac{22}{5}, & \textit{if} \quad k = 3 \\ \frac{3}{4}k^2 - \frac{67}{210}k + \frac{1667}{840}, & \textit{if} \quad k > 3. \end{cases}$$

$$I(LTG_k) = \begin{cases} \frac{249}{5}, & \textit{if} \quad k = 3 \\ \frac{75}{2}k^2 - \frac{5951}{42}k + \frac{14354}{105}, & \textit{if} \quad k > 3. \end{cases}$$

$$A(LTG_k) = \begin{cases} \frac{22499}{1125}, & \textit{if} \quad k = 3 \\ g(k), & \textit{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = \frac{3125}{243}k^2 - \frac{674175761}{10418625}k - \frac{20972572717}{93767625}$$

Proof: By Theorem 4.1,

$$f(x,y) = M(LTG_k) = \begin{cases} 6x^4y^4 + 12x^4y^6 + 3x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = \; & 3x^4y^4 + 12x^4y^6 + 6x^4y^8 + 3(k-2)x^6y^6 + 12(k-3)x^6y^8 \\ & +3(2k-3)x^8y^8, \end{aligned}$$

$$h(x,y) = 24(k-4)x^8y^{10} + \frac{3}{2}(5k-23)(k-4)x^{10}y^{10}.$$

Then

$$D_x(f(x,y)) = \begin{cases} 24x^4y^4 + 48x^4y^6 + 18x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 12x^4y^4 + 48x^4y^6 + 24x^4y^8 + 18(k-2)x^6y^6 + 72(k-3)x^6y^8,$$
$$h(x,y) = 24(2k-3)x^8y^8 + 192(k-4)x^8y^{10} + 15(5k-23)(k-4)x^{10}y^{10}.$$

$$D_y(f(x,y)) = \begin{cases} 24x^4y^4 + 72x^4y^6 + 18x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 12x^4y^4 + 72x^4y^6 + 48x^4y^8 + 18(k-2)x^6y^6 + 96(k-3)x^6y^8,$$
$$h(x,y) = 24(2k-3)x^8y^8 + 240(k-4)x^8y^{10} + 15(5k-23)(k-4)x^{10}y^{10}.$$

$$(D_x + D_y)(f(x,y)) = \begin{cases} 48x^4y^4 + 120x^4y^6 + 36x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 24x^4y^4 + 120x^4y^6 + 72x^4y^8 + 36(k-2)x^6y^6 + 168(k-3)x^6y^8$$
$$h(x,y) = 48(2k-3)x^8y^8 + 432(k-4)x^8y^{10} + 30(5k-23)(k-4)x^{10}y^{10}.$$

$$D_x^2(f(x,y)) = \begin{cases} 96x^4y^4 + 192x^4y^6 + 108x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 48x^4y^4 + 192x^4y^6 + 96x^4y^8 + 108(k-2)x^6y^6 + 432(k-3)x^6y^8$$
$$h(x,y) = 192(2k-3)x^8y^8 + 1536(k-4)x^8y^{10} + 150(5k-23)(k-4)x^{10}y^{10}))$$

$$D_y^2(f(x,y)) = \begin{cases} 96x^4y^4 + 432x^4y^6 + 108x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 48x^4y^4 + 432x^4y^6 + 384x^4y^8 + 108(k-2)x^6y^6 + 768(k-3)x^6y^8,$$
$$h(x,y) = 192(2k-3)x^8y^8 + 2400(k-4)x^8y^{10} + 150(5k-23)(k-4)x^{10}y^{10}.$$

$$(D_x^2 + D_y^2)(f(x,y)) = \begin{cases} 192x^4y^4 + 624x^4y^6 + 216x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 96x^4y^4 + 624x^4y^6 + 480x^4y^8 + 216(k-2)x^6y^6 + 1200(k-3)x^6y^8,$$
$$h(x,y) = 384(2k-3)x^8y^8 + 3936(k-4)x^8y^{10} + 300(5k-23)(k-4)x^{10}y^{10}.$$

$$D_yD_x(f(x,y)) = \begin{cases} 96x^4y^4 + 288x^4y^6 + 108x^6y^6, & \text{if} \quad k = 3 \\ g(x,y) + h(x,y), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(x,y) = 48x^4y^4 + 288x^4y^6 + 192x^4y^8 + 108(k-2)x^6y^6 + 576(k-3)x^6y^8$$
$$h(x,y) = 192(2k-3)x^8y^8 + 1920(k-4)x^8y^{10} + 150(5k-23)(k-4)x^{10}y^{10}$$

$$D_xD_y(f(x,y)) = \begin{cases} 96x^4y^4 + 288x^4y^6 + 108x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = 48x^4y^4 + 288x^4y^6 + 192x^4y^8 + 108(k-2)x^6y^6 + 576(k-3)x^6y^8$$
$$h(x,y) = 192(2k-3)x^8y^8 + 1920(k-4)x^8y^{10} + 150(5k-23)(k-4)x^{10}y^{10}$$

$$D_xD_y(D_x+D_y)(f(x,y)) = \begin{cases} 768x^4y^4 + 2880x^4y^6 + 1296x^6y^6 & \text{if} \quad k=3 \\ g(x,y)+h(x,y) & \text{if} \quad k>3 \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = \ & 384x^4y^4 + 2880x^4y^6 + 2304x^4y^8 + 1296(k-2)x^6y^6 \\ & +8064(k-3)x^6y^8 \end{aligned}$$
$$\begin{aligned} h(x,y) = \ & 3072(2k-3)x^8y^8 + 34560(k-4)x^8y^{10} \\ & +3000(5k-23)(k-4)x^{10}y^{10} \end{aligned}$$

$$S_y(f(x,y)) = \begin{cases} \frac{3}{2}x^4y^4 + 2x^4y^6 + \frac{1}{2}x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = \frac{3}{4}x^4y^4 + 2x^4y^6 + \frac{3}{4}x^4y^8 + \frac{1}{2}(k-2)x^6y^6 + \frac{3}{2}(k-3)x^6y^8$$
$$h(x,y) = \frac{3}{8}(2k-3)x^8y^8 + \frac{12}{5}(k-4)x^8y^{10} + \frac{3}{20}(5k-23)(k-4)x^{10}y^{10}$$

$$S_xS_y(f(x,y)) = \begin{cases} \frac{3}{8}x^4y^4 + \frac{1}{2}x^4y^6 + \frac{1}{12}x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = \frac{3}{16}x^4y^4 + \frac{1}{2}x^4y^6 + \frac{3}{16}x^4y^8 + \frac{1}{12}(k-2)x^6y^6 + \frac{1}{4}(k-3)x^6y^8$$
$$h(x,y) = \frac{3}{64}(2k-3)x^8y^8 + \frac{3}{10}(k-4)x^8y^{10} + \frac{3}{200}(5k-23)(k-4)x^{10}y^{10}$$

$$D_x^{\alpha}(f(x,y)) = \begin{cases} 6.4^{\alpha}x^4y^4 + 12.4^{\alpha}x^4y^6 + 3.6^{\alpha}x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3 \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = \ & 3.4^{\alpha}x^4y^4 + 12.4^{\alpha}x^4y^6 + 6.4^{\alpha}x^4y^8 + 3.6^{\alpha}(k-2)x^6y^6 \\ & +12.6^{\alpha}(k-3)x^6y^8 \end{aligned}$$
$$\begin{aligned} h(x,y) = \ & 3.8^{\alpha}(2k-3)x^8y^8 + 24.8^{\alpha}(k-4)x^8y^{10} \\ & +\frac{3}{2}.10^{\alpha}(5k-23)(k-4)x^{10}y^{10}(k-3) \end{aligned}$$

$$D_y^\alpha(f(x,y)) = \begin{cases} 6.4^\alpha x^4y^4 + 12.6^\alpha x^4y^6 + 3.6^\alpha x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3 \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = {} & 3.4^\alpha x^4y^4 + 12.6^\alpha x^4y^6 + 6.8^\alpha x^4y^8 + 3.6^\alpha(k-2)x^6y^6 \\ & +12.8^\alpha(k-3)x^6y^8 \\ h(x,y) = {} & 3.8^\alpha(2k-3)x^8y^8 + 24.10^\alpha(k-4)x^8y^{10} \\ & +\tfrac{3}{2}.10^\alpha(5k-23)(k-4)x^{10}y^{10} \end{aligned}$$

$$D_x^\alpha D_y^\alpha(f(x,y)) = \begin{cases} 6.4^{2\alpha} x^4y^4 + 12.6^\alpha.4^\alpha x^4y^6 + 3.6^{2\alpha} x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3 \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = {} & 3.4^{2\alpha}x^4y^4 + 12.6^\alpha.4^\alpha x^4y^6 + 6.8^\alpha.4^\alpha x^4y^8 + 3.6^{2\alpha}(k-2)x^6y^6 \\ h(x,y) = {} & 12.8^\alpha.6^\alpha(k-3)x^6y^8 + 3.8^{2\alpha}(2k-3)x^8y^8 \\ & +24.8^\alpha.10^\alpha(k-4)x^8y^{10} + \frac{3}{2}.10^{2\alpha}(5k-23)(k-4)x^{10}y^{10} \end{aligned}$$

$$S_y^\alpha(f(x,y)) = \begin{cases} \dfrac{6}{4^\alpha}x^4y^4 + \dfrac{12}{6^\alpha}x^4y^6 + \dfrac{3}{6^\alpha}x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = {} & \frac{3}{4^\alpha}x^4y^4 + \frac{12}{6^\alpha}x^4y^6 + \frac{6}{8^\alpha}x^4y^8 + \frac{3(k-2)}{6^\alpha}x^6y^6 + \frac{12(k-3)}{8^\alpha}x^6y^8 \\ h(x,y) = {} & \frac{3}{8^\alpha}(2k-3)x^8y^8 + \frac{24}{10^\alpha}(k-4)x^8y^{10} \\ & +\frac{3}{2.10^\alpha}(5k-23)(k-4)x^{10}y^{10} \end{aligned}$$

$$S_x^\alpha S_y^\alpha(f(x,y)) = \begin{cases} \dfrac{6}{4^{2\alpha}}x^4y^4 + \dfrac{12}{6^\alpha.4^\alpha}x^4y^6 + \dfrac{3}{6^{2\alpha}}x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$\begin{aligned} g(x,y) = {} & \frac{3}{4^\alpha.4^\alpha}x^4y^4 + \frac{12}{6^\alpha.4^\alpha}x^4y^6 + \frac{6}{8^\alpha.4^\alpha}x^4y^8 + \frac{3}{6^\alpha.6^\alpha}(k-2)x^6y^6 \\ h(x,y) = {} & \frac{12(k-3)}{8^\alpha.6^\alpha}x^6y^8 + \frac{3(2k-3)}{8^{2\alpha}}x^8y^8 + \frac{24(k-4)}{10^\alpha.8^\alpha}x^8y^{10} \\ & +\frac{3(5k-23)(k-4)}{2.10^{2\alpha}}x^{10}y^{10} \end{aligned}$$

$$(S_yD_x)(f(x,y)) = \begin{cases} 6x^4y^4 + 8x^4y^6 + 3x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = 3x^4y^4 + 8x^4y^6 + 3x^4y^8 + 3(k-2)x^6y^6 + 9(k-3)x^6y^8$$

$$h(x,y) = 3(2k-3)x^8y^8 + \frac{96(k-4)}{5}x^8y^{10} + \frac{3(5k-23)}{2}(k-4)x^{10}y^{10}$$

$$(S_xD_y)(f(x,y)) = \begin{cases} 6x^4y^4 + 18x^4y^6 + 3x^6y^6, & \text{if} \quad k=3 \\ g(x,y)+h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = 3x^4y^4 + 18x^4y^6 + 12x^4y^8 + 3(k-2)x^6y^6 + 16(k-3)x^6y^8$$

$$h(x,y) = 3(2k-3)x^8y^8 + 30(k-4)x^8y^{10} + \frac{3}{2}(5k-23)(k-4)x^{10}y^{10}$$

$$(S_yD_x + S_xD_y)(f(x,y)) = \begin{cases} 12x^4y^4 + 26x^4y^6 + 6x^6y^6, & \text{if} \quad k=3 \\ g(x,y) + h(x,y), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x,y) = 6x^4y^4 + 26x^4y^6 + 15x^4y^8 + 6(k-2)x^6y^6 + 25(k-3)x^6y^8$$

$$h(x,y) = 6(2k-3)x^8y^8 + \frac{246(k-4)}{5}x^8y^{10} + 3(5k-23)(k-4)x^{10}y^{10}$$

$$J(f(x,y)) = \begin{cases} 6x^8 + 12x^{10} + 3x^{12}, & \text{if} \quad k=3 \\ g(x) + h(x), & \text{if} \quad k>3 \end{cases}$$

Where:

$$g(x) = 3x^8 + 12x^{10} + 6kx^{12} + 3(k-2)x^{12} + 12(k-3)x^{14}$$

$$h(x) = 3(2k-3)x^{16} + 24(k-4)x^{18} + \frac{3(5k-23)(k-4)}{2}x^{20}$$

$$(S_xJ)(f(x,y)) = \begin{cases} \frac{3}{4}x^8 + \frac{6}{5}x^{10} + \frac{1}{4}x^{12}, & \text{if} \quad k=3 \\ g(x) + h(x), & \text{if} \quad k>3 \end{cases}$$

Where:

$$g(x) = \frac{3}{8}x^8 + \frac{6}{5}x^{10} + \frac{1}{4}kx^{12} + \frac{6}{7}(k-3)x^{14}$$

$$h(x) = \frac{3}{16}(2k-3)x^{16} + \frac{4}{3}(k-4)x^{18} + \frac{3}{40}(5k-23)(k-4)x^{20}$$

$$(JD_xD_y)(f(x,y)) = \begin{cases} 96x^8 + 288x^{10} + 108x^{12}, & \text{if} \quad k=3 \\ g(x) + h(x), & \text{if} \quad k>3 \end{cases}$$

Where:

$$g(x) = 48x^8 + 288x^{10} + 192x^{12} + 108(k-2)x^{12} + 576(k-3)x^{14}$$

$$h(x) = 192(2k-3)x^{16} + 1920(k-4)x^{18} + 150(5k-23)(k-4)x^{20}$$

$$(S_xJD_xD_y)(f(x,y)) = \begin{cases} 12x^8 + \frac{144}{5}x^{10} + 9x^{12}, & \text{if} \quad k=3 \\ g(x) + h(x), & \text{if} \quad k>3. \end{cases}$$

Where:

$$g(x) = 6x^8 + \frac{144}{5}x^{10} + 16x^{12} + 9(k-2)x^{12} + \frac{288}{7}(k-3)x^{14}$$

$$h(x) = 12(2k-3)x^{16} + \frac{320(k-4)}{3}x^{18} + \frac{15}{2}(5k-23)(k-4)x^{20}$$

$$(D_x^3D_y)(f(x,y)) = \begin{cases} 24.4^3x^4y^4 + 72.4^3x^4y^6 + 18.6^3x^6y^6, & \text{if} \quad k=3 \\ g(x,y) + h(x,y), & \text{if} \quad k>3 \end{cases}$$

Where:

$$g(x,y) = 12.4^3x^4y^4 + 72.4^3x^4y^6 + 48.4^3x^4y^8 + 18.6^3(k-2)x^6y^6$$

$$
\begin{aligned}
h(x,y) = \quad & 96.6^3(k-3)x^6y^8 + 24.8^3(2k-3)x^8y^8 + 240.8^3(k-4)x^8y^{10} \\
& +15.10^3(5k-23)(k-4)x^{10}y^{10}
\end{aligned}
$$

$$
(JD_x^3D_y)(f(x,y)) = \begin{cases} 1536x^8 + 4608x^{10} + 3888x^{12}, & \text{if} \quad k=3 \\ g(x)+h(x), & \text{if} \quad k>3 \end{cases}
$$

Where:

$$
g(x) = 768x^8 + 4608x^{10} + 3072x^{12} + 3888x^{12} + 20736(k-3)x^{14}
$$

$$
h(x) = 12288(2k-3)x^{16} + 122880(k-4)x^{18} + 15000(5k-23)(k-4)x^{20}
$$

$$
(Q_{-2}JD_x^3D_y)(f(x,y)) = \begin{cases} 1536x^6 + 4608x^8 + 3888x^{10}, & \text{if} \quad k=3 \\ g(x)+h(x), & \text{if} \quad k>3 \end{cases}
$$

Where:

$$
g(x) = 768x^6 + 4608x^8 + 3072x^{10} + 3888(k-2)x^{10} + 20736(k-3)x^{12}
$$

$$
h(x) = 12288(2k-3)x^{14} + 122880(k-4)x^{16} + 15000(5k-23)(k-4)x^{18}
$$

$$
(S_x^3Q_{-2}JD_x^3D_y)(f(x,y)) = \begin{cases} \frac{64}{9}x^6 + 9x^8 + \frac{486}{125}x^{10}, & \text{if} \quad k=3 \\ g(x)+h(x), & \text{if} \quad k>3 \end{cases}
$$

Where:

$$
g(x) = \frac{32}{9}x^6 + 9x^8 + \frac{384}{125}x^{10} + \frac{486}{125}(k-2)x^{10} + 12(k-3)x^{12}
$$

$$
h(x) = \frac{1536}{343}(2k-3)x^{14} + 30(k-4)x^{16} + \frac{625}{243}(5k-23)(k-4)x^{18}
$$

1.- First Zagreb index

$$
M_1(LTG_k) = (D_x + D_y)(MLTG_k(x,y))|_{x=y=1}
$$

$$
M_1(LTG_k) = \begin{cases} 204, & \text{if} \quad k=3 \\ 150k^2 - 558k + 528, & \text{if} \quad k>3. \end{cases}
$$

2.- Second Zagreb index

$$
M_2(LTG_k) = (D_xD_y)(MLTG_k(x,y))|_{x=y=1}.
$$

$$
M_2(LTG_k) = \begin{cases} 492, & \text{if} \quad k=3 \\ 750k^2 - 3462k + 4128, & \text{if} \quad k>3. \end{cases}
$$

3.- Forgotten topological index

$$
F(LTG_k) = (D_x^2 + D_y^2)(MG(x,y))|_{x=y=1}.
$$

$$
F(LTG_k) = \begin{cases} 1032, & \text{if} \quad k=3 \\ 1500k^2 - 6780k + 7872, & \text{if} \quad k>3. \end{cases}
$$

4.- Redefined third Zagreb index

$$
ReZG_3 = D_xD_y(D_x + D_y)|_{x=y=1}.
$$

$$ReZG_3 = \begin{cases} 4944, & \text{if} \quad k = 3 \\ 15000k^2 - 78936k + 107328, & \text{if} \quad k > 3. \end{cases}$$

5.- Modified Second Zagreb index

$$^mM_2(LTG_k) = (S_xS_y)(MLTG_k(x,y))|_{x=y=1}$$

$$^mM_2(LTG_k) = \begin{cases} \frac{23}{24}, & \text{if} \quad k = 3 \\ \frac{3}{40}k^2 + \frac{197}{2400}k + \frac{11}{4800}, & \text{if} \quad k > 3. \end{cases}$$

6.- Generalized Randic Index

$$R_\alpha(LTG_k) = (Dx^\alpha Dy^\alpha)(MLTG_k(x,y))|_{x=y=1}$$

$$R_\alpha(LTG_k) = \begin{cases} 3.2^{4\alpha+1} + 3^{\alpha+1}.2^{3\alpha+2} + 3^{2\alpha+1}.2^{2\alpha}, & \text{if} \quad k = 3 \\ g(k) + h(k), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = 3.2^{4\alpha} + 3^{\alpha+1}.2^{3\alpha+2} + 3.2^{5\alpha+1} + 3^{2\alpha+1}.2^{2\alpha}(k-2) + 3^{\alpha+1}.2^{4\alpha+2}(k-3)$$

$$h(k) = 3.2^{6\alpha}(2k-3) + 3.2^{4\alpha+3}.5^\alpha(k-4) + (3.2^{2\alpha-1}.5^{2\alpha}(5k-23)(k-4)$$

7.- Inverse Randic Index

$$RR_\alpha(LTG_k) = (S_x^\alpha S_y^\alpha)(MLTG_k(x,y))|_{x=y=1}$$

$$RR_\alpha(LTG_k) = \begin{cases} 3.2^{1-4\alpha} + 3^{1-\alpha}.2^{2-3\alpha} + 3^{1-2\alpha}.2^{-2\alpha}, & \text{if} \quad k = 3 \\ g(k) + h(k), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = \frac{3}{2^{4\alpha}} + \frac{3^{1-\alpha}}{2^{3\alpha-2}} + \frac{3}{2^{5\alpha-1}} + \frac{3^{1-2\alpha}}{2^{2\alpha}}(k-2) + \frac{3^{1-\alpha}}{2^{4\alpha-2}}(k-3)$$

$$h(k) = \frac{3}{2^{6\alpha}}(2k-3) + \frac{3.2^{3-4\alpha}}{5^\alpha}(k-4) + \frac{3.2^{-1-2\alpha}}{5^{2\alpha}}(5k-23)(k-4)$$

8.- Symmetric Division Deg Index

$$SSD(LTG_k) = (S_yD_x + S_xD_y)(MLTG_k(x,y))|_{x=y=1}$$

$$SSD(LTG_k) = \begin{cases} 48, & \text{if} \quad k = 3 \\ 15k^2 - \frac{184}{5}k + \frac{106}{5}, & \text{if} \quad k > 3. \end{cases}$$

9.- Harmonic index

$$H(LTG_k) = 2S_xJ(MLTG_k(x,y))|_{x=y=1}$$

$$H(LTG_k) = \begin{cases} \frac{22}{5}, & \text{if} \quad k = 3 \\ \frac{3}{4}k^2 - \frac{67}{210}k + \frac{1667}{840}, & \text{if} \quad k > 3. \end{cases}$$

10.- Inverse Sum index

$$I(LTG_k) = (S_xJD_xD_y)(MLTG_k(x,y))\ |_{x=y=1}$$

$$I(LTG_k) = \begin{cases} \dfrac{249}{5}, & \text{if} \quad k = 3 \\ \dfrac{75}{2}k^2 - \dfrac{5951}{42}k + \dfrac{14354}{105}, & \text{if} \quad k > 3. \end{cases}$$

11.- Augmented Zagreb index

$$A(LTG_k) = S_x^3 Q_{-2} J D_x^3 Dy = (MLTGk(x,y))|_{x=y=1}$$

$$A(LTG_k) = \begin{cases} \dfrac{22499}{1125}, & \text{if} \quad k = 3 \\ g(k), & \text{if} \quad k > 3. \end{cases}$$

Where:

$$g(k) = \frac{3125}{243}k^2 - \frac{674175761}{10418625}k - \frac{20972572717}{93767625}$$

□

5. Application

Fullerenes have sp2 and sp3 hybrid carbon atoms as is shown in Figure 3. These molecules have an extremely high affinity for electrons and can be reversibly reduced to absorb 6 electrons. Although this molecule is made of conjugated carbon rings, the electrons here are not delocalized, and therefore, these molecules lack the property of superaromaticity [7]. These molecules have a very high tensile strength and recover their original shape after being subjected to more than 3,000 atmospheric pressures [4]. Due to its relative ease of synthesis, C60 fullerene remains widespread, and much research has been conducted for its potential applications [33]. C60 fullerene consists of 60 carbons at 60 vertices forming a spherical structure. It consists of 12 pentagonal rings and 20 hexagonal rings adjacent to each other. These rings are conjugated with double bonds. The C-C bond length for the hexagonal rings is 1.40 A° and 1.46 A° for the pentagonal rings, with an average bond length equal to 1.44 A°. Considering their unique chemistry in materials science, researchers have discovered several applications of fullerenes, including medical, [5] and nanotechnological applications such as superconductors and optical fibre [29] and [22]. Considering that the described compounds (fullerenes) contain even base, the application of the above-described topological indices on these compounds might be a fundamental step to understand the quantitative relationships of their structures with specific physicochemical properties; these could help the development of new derivatives with improved properties and new physical applications.

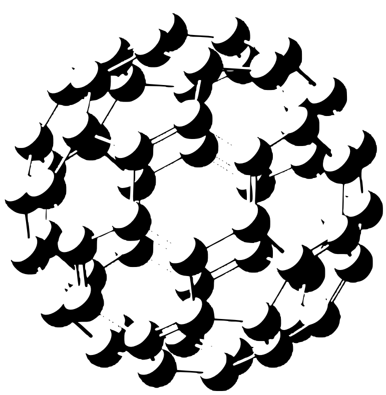

Figure 3: Fullerene

6. Conclusion

In this paper, we computed M-polynomials of the triangular graph with even degree vertices and its line graph. From these M-polynomials, we recover: First Zagreb index, Second Zagreb index, Forgotten topological index, Redefined third Zagreb index, Modified Second Zagreb index, Generalized Randic index, Inverse Randic index, Symmetric Division index, Harmonic index, Inverse Sum index and Augmented Zagreb index.

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