Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–13.
ISSN-0037-8712 in press
doi:10.5269/bspm.62725

On a Class of Fractional Langevin Inclusion with Multi-point Boundary Conditions

Hamid Lmou*, Khalid Hilal and Ahmed Kajouni

ABSTRACT: The aim of this paper deals with the existence results for a class of fractional Langevin inclusion with multi-point boundary conditions. To prove the main results, we use the fixed point theorem for condensing multivalued maps, which is applicable to completely continuous operators. Our results extend and generalize several corresponding results from the existing literature.

Key Words: Caputo derivative, Langevin equation, Fractional Langevin inclusion, Boundary value problems, Fixed point theorem.

Contents

1 Introduction **1**

2 Preliminaries **2**
2.1 Fractional Calculus . . . 2
2.2 Multi-valued Analysis . . . 3

3 Existence Results **5**

4 Conclusion **12**

1. Introduction

Fractional derivatives give an excellent description of memory and hereditary properties of different processes. Properties of the fractional derivatives make the fractional order models more useful and practical than the classical integral-order models. Several researchers in the recent years have employed the fractional calculus as a way of describing natural phenomena in different fields such as physics, biology, finance, economics, and bioengineering (for more details see [3,8,9,13,14,18,19,22,24,25] and many other references).

With the recent outstanding development in fractional differential equations, the Langevin equation has been considered a part of fractional calculus, and thus, important results have been elaborated (see [1,2,7,20,23])

An equation of the form $m\frac{d^2x}{dt^2} = \lambda\frac{dx}{dt} + \eta(t)$ is called Langevin equation, introduced by Paul Langevin in 1908. The Langevin equation is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [15]. For some new developments on the fractional Langevin equation, see, for example, [4,5,17].

In this paper we prove the existence results for a fractional Langevin inclusion with multi-point boundary conditions of the form :

$$\begin{cases} {}^cD^{\alpha}({}^cD^{\beta}+\lambda)x(t) \in \mathcal{G}(t,x(t)), \quad 0<t<1 \\ x(0)=0, \quad D^{\beta}x(0)=0, \quad x(1)=\sum\limits_{i=1}^{n}\mu_i x(\eta_i)+\sum\limits_{i=1}^{n}\gamma_i\int_0^{\eta_i} x(s)ds, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}$ and ${}^cD^{\beta}$ are the Caputo fractional derivative of order $\alpha \in (1,2], \beta \in (0,1]$, $\lambda \in \mathbb{R}$ is the dissipative parameter, $\mu_i, \gamma_i \in \mathbb{R}$ and $\eta_i \in (0,1), i=1,...,n$ such that $n \in \mathbb{N}$ with $\omega = \sum\limits_{i=1}^{n}\mu_i\eta_i^{\beta+1} \neq 1$ and

2010 *Mathematics Subject Classification:* 26A33, 34A34.
Submitted March 01, 2022. Published May 11, 2022

$\mathcal{G} : [0,1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a bounded, closed, convex, multivalued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

The paper is organized as follows. In the second section, we will recall briefly some basic definitions and preliminary facts which will be used in the following section. In the third section, we discuss the existence results for our problem by making use of the fixed point theorem for condensing multivalued maps.

2. Preliminaries

2.1. Fractional Calculus

Definition 2.1. *[13] The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

Definition 2.2. *[13] The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function f can be defined as*

$$^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \tag{2.2}$$

where $n-1 < \alpha < n, n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *[13] Let α, β, non-negative then, the following relations hold:*

$$I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}, \tag{2.3}$$

$$^{c}D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \tag{2.4}$$

Lemma 2.4. *[13] Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$, if f is a continuous function, then, we have*

$$I^{\alpha c}D^{\alpha} f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + ... + a_{n-1} t^{n-1}. \tag{2.5}$$

For some $a_i \in \mathbb{R}, i = 1, ..., n-1, (n = [\alpha]+1)$.

In the following, $\mathcal{AC}^1([0,T], \mathbb{R})$ will denote the space of functions $x : [0,T] \longrightarrow \mathbb{R}$ that are absolutely continuous and whose first derivative is absolutely continuous.

Definition 2.5. *A function $x \in \mathcal{AC}^1([0,1], \mathbb{R})$ is called a solution of problem (1.1) if there exists a function $g \in \mathbb{L}^1([0,T], \mathbb{R})$ with $g(t) \in \mathcal{G}(t, x(t))$ for a.e $t \in [0,T]$ such that $^{c}D^{\alpha}(^{c}D^{\beta} + \lambda)x(t) = g(t)$ and $x(0) = 0, \quad D^{\beta}x(0) = 0, \quad x(1) = \sum_{i=1}^{n} \mu_i x(\eta_i) + \sum_{i=1}^{n} \gamma_i \int_0^{\eta_i} x(s) ds.$*

Lemma 2.6. *Let $g \in \mathcal{C}([0,1], \mathbb{R})$ Then, a unique solution of the boundary value problem*

$$\begin{cases} ^{c}D^{\alpha}(^{c}D^{\beta} + \lambda)x(t) = g(t), \quad 0 < t < 1 \\ x(0) = 0, \quad D^{\beta}x(0) = 0, \quad x(1) = \sum_{i=1}^{n} \mu_i x(\eta_i) + \sum_{i=1}^{n} \gamma_i \int_0^{\eta_i} x(s) ds, \end{cases} \tag{2.6}$$

is given by

$$
\begin{aligned}
x(t) = & \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}g(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds \\
& + \frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g(s)ds \\
& + \sum_{i=1}^{n}\frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g(s)ds - \sum_{i=1}^{n}\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x(s)ds \\
& + \sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}x(s)ds\Bigg].
\end{aligned}
$$

Proof. Applying Lemma 2.4, we get

$$
{}^cD^{\beta}x(t) = I^{\alpha}g(t) - \lambda x(t) + a_0 + a_1 t. \tag{2.7}
$$

Furthermore, we apply Lemma 2.4 and Lemma 2.3 the equation (2.7) becomes :

$$
\begin{aligned}
x(t) = & \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}g(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds \\
& + \frac{t^{\beta}}{\Gamma(\beta+1)}a_0 + \frac{t^{\beta+1}}{\Gamma(\beta+2)}a_1 + a_2.
\end{aligned}
$$

By using the boundary conditions $x(0) = 0$ and ${}^cD^{\alpha}x(0) = 0$ we get respectively $a_0 = 0$ and $a_2 = 0$, and by using $x(1) = \sum_{i=1}^{n}\mu_i x(\eta_i) + \sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}x(s)ds$ and $\omega = \sum_{i=1}^{n}\mu_i\eta_i^{\beta+1}$ we get :

$$
\begin{aligned}
a_1 = & \frac{\Gamma(\beta+2)}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g(s)ds \\
& + \sum_{i=1}^{n}\frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g(s)ds - \sum_{i=1}^{n}\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x(s)ds \\
& + \sum_{i=1}^{n}\gamma_i\int_0^{\eta_i}x(s)ds\Bigg].
\end{aligned}
$$

Substituting the value of a_0, a_1, and a_2, we obtain the desired results. Furthermore, it is not difficult to see that this solution verifies the boundary condition of the problem (1.1). This completes the proof □

2.2. Multi-valued Analysis

Now we recall some basic definitions on multivalued maps. [11,12,21]

Let $\mathcal{C}([0,1],\mathbb{R})$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = sup_{t\in[0,1]}|x|$. Let $\mathbb{L}^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x : [0,1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{\mathbb{L}^1} = \int_0^1 |x(t)|dt$.

For a normed space $(X, \|.\|)$ let :

$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \quad is \quad closed\}.$$

$$\mathcal{P}_{b}(X) = \{Y \in \mathcal{P}(X) : Y \quad is \quad bounded\}.$$

$$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \quad is \quad compact\}.$$

$$\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \quad is \quad compact \quad and \quad convex\}.$$

$$\mathcal{P}_{b,cl,cv}(X) = \{Y \in \mathcal{P}(X) : Y \quad is \quad bounded, \quad closed \quad and \quad convex\}.$$

Lemma 2.7. *Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \longrightarrow \mathcal{P}(X)$ is convex closed, if $G(x)$ is convex closed, for all $x \in X$; and G is bounded on bounded sets, if $G(B) = \underset{x\in B}{\cup} G(x)$ is bounded in X, for any bounded set B of X.*

Theorem 2.8. *G is called upper semi-continuous (u.s.c) on X, if for each $x \in X$, the set $G(x)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x)$, there exists an open neighborhood V of x such that $G(V) \in B$.*

Theorem 2.9. *G is said to be completely continuous if $G(B)$ is relatively compact, for every bounded subset $B \subset X$.*

Lemma 2.10. *If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph $\Big($i.e $x_n \longrightarrow x, y_n \longrightarrow y; y_n \in G(x_n)$ imply $y \in G(x)\Big)$.*

Definition 2.11. *an upper semi-continuous multivalued map $G : X \longrightarrow X$ is said to be condensing if for any subset $B \subset X$ with $\alpha(B) \neq 0$, we have $\alpha\big(G(B)\big) < \alpha\big(B\big)$, where α denotes the Kuratowski measure of noncompactness [16].*

Lemma 2.12. *:*
A completely continuous multivalued map is a condensing map

Theorem 2.13. *:(Arzelà-Ascoli's theorem)*
Let K be a compact space and (E, d) a metric space. $A \subset \mathcal{C}(K, E)$ is relatively compact (i.e. included in a compact) if and only if, for any x of K:

- *A is equicontinuous in x, i.e. for everything $\varepsilon > 0$, there exist a neighborhood V of x such that : $\forall f \in A, \forall y \in V \quad d(f(x), f(y)) < \varepsilon$*
- *The set $A(x) = \{f(x); f \in A\}$ is relatively compact.*

Definition 2.14. *A multivalued map $G : [0, 1] \longrightarrow \mathcal{P}(X)$ is said to be Carathéodory if*

(i)- $t \longrightarrow G(t, x)$ is measurable for each $x \in \mathbb{R}$.

(ii)- $x \longrightarrow G(t, x)$ is upper semicontinuous for almost all $t \in [0, 1]$.

Further a Carathéodory function G is called $\mathbb{L}^1$-Carathéodory if

(iii)- for each $\theta > 0$ there exist $\varphi_\theta \in \mathbb{L}^1([0, 1], \mathbb{R}^+)$ such that

$$\|G(t, x)\| = sup\{|g|, g \in G(t, x)\} \leq \varphi_\theta(t).$$

For all $\|x\|_\infty \leq \theta$ and for all a.e $t \in [0, 1]$
For each $x \in \mathcal{C}([0, 1], \mathbb{R})$, define the set of selections of G by

$$\mathcal{S}_{G,x} = \{g \in L^1(J, X) : g(t) \in G(t, x(t)); for \quad a.e \quad t \in [0, 1]\}.$$

Lemma 2.15. *[6]*
Let X be a Banach space. Let $G : [0, 1] \times \mathbb{R} \longrightarrow \mathcal{P}_{b,cl,cv}$ be a $\mathbb{L}^1$-Carathéodory multivalued. And let Υ be a linear continuous mapping from $L^1([0, 1], X)$ to $\mathcal{C}([0, 1], X)$. Then the operator :

$$\Upsilon \circ \mathcal{S}_G : \mathcal{C}([0, 1], X) \longrightarrow \mathcal{P}_{b,cl,cv}\big(\mathcal{C}([0, 1], X)\big); \quad x \longrightarrow \big(\Upsilon \circ \mathcal{S}_G\big)(x) = \Upsilon\big(\mathcal{S}_{G,x}\big),$$

is a closed graph operator in $\mathcal{C}([0, 1], X) \times \mathcal{C}([0, 1], X)$.

Theorem 2.16. *(Leray-Schauder's fixed point) [10]*
Let X be a Banach space and $\mathcal{N} : X \longrightarrow \mathcal{P}_{b,cl,cv}$ an u.s.c condensing map. If the set :

$$\Lambda := \{x \in X : \rho x \in \mathcal{N}x \quad for \quad \rho > 1\}$$

is bounded, then $\mathcal{N}$ has a fixed point.

3. Existence Results

Definition 3.1. *A function* $x \in \mathcal{C}([0,1],\mathbb{R})$ *is said to be a solution of* (1.1), *if there exists a function* $g \in \mathbb{L}^1([0,1],\mathbb{R})$ *with* $g(t) \in \mathcal{G}(t,x(t))$ *a.e* $t \in [0,1]$ *and*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}g(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds \\ &+ \frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g(s)ds \\ &+ \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g(s)ds - \sum_{i=1}^n \frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x(s)ds \\ &+ \sum_{i=1}^n \gamma_i\int_0^{\eta_i} x(s)ds\Bigg]. \end{aligned}$$

Assume that :

(H1)- $\mathcal{G} : [0,1]\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values, and for each fixed $x \in \mathcal{C}([0,1],\mathbb{R})$ the set :

$$\mathcal{S}_{G,x} = \{g \in \mathbb{L}^1([0,1],X) : g(t) \in G(t,x(t)); \quad t \in [0,1]\},$$

is nonempty.

(H2)- $\|G(t,x)\| := sup\{|g| : g \in G(t,x)\} \leq p(t)\Psi(\|x\|)$ for all $t \in J$ and all $x \in \mathcal{C}([0,1],X)$, where $p \in L^1([0,1],\mathbb{R}^+)$ and $\Psi : \mathbb{R}^+ \longrightarrow [0,+\infty)$ is continuous and nondecreasing function.

(H3)- there exists a number $M > 0$ such that :

$$\frac{\|p\|_\infty \Psi(\|x\|_\infty)\Phi}{1-\Omega} < \frac{\|p\|_\infty \Psi(M)\Phi}{1-\Omega}.$$

Where

$$\Phi = \frac{|1-\omega| + 1 + \sum\limits_{i=1}^n |\mu_i|\eta_i^{\beta+\alpha}}{|1-\omega|\Gamma(\beta+\alpha+1)}.$$

And

$$\Omega = \frac{|1-\omega||\lambda| + |\lambda|\big(1+\sum\limits_{i=1}^n |\mu_i|\eta_i^{\beta}\big) + \Gamma(\beta+1)\sum\limits_{i=1}^n |\gamma_i|\eta_i}{|1-\omega|\Gamma(\beta+1)}.$$

With $\Omega < 1$.

Our main result may be presented as the following theorem.

Theorem 3.2. *Assume that hypotheses* $(H1)-(H3)$ *hold, then the problem* (1.1) *has at least one solution.*

Proof. Let us introduce the operator $\mathcal{N} : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathcal{P}([0,1],\mathbb{R})$ as

$$\mathcal{N}(x) := \left\{ h \in \mathcal{C}([0,1],\mathbb{R}) : h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\beta+\alpha)} \displaystyle\int_0^t (t-s)^{\beta+\alpha-1} g(s)ds \\ -\dfrac{\lambda}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} x(s)ds \\ +\dfrac{t^{\beta+1}}{1-\omega} \Bigg[\dfrac{\lambda}{\Gamma(\beta)} \displaystyle\int_0^1 (1-s)^{\beta-1} x(s)ds \\ -\dfrac{1}{\Gamma(\beta+\alpha)} \displaystyle\int_0^1 (1-s)^{\beta+\alpha-1} g(s)ds \\ +\displaystyle\sum_{i=1}^{n} \dfrac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} g(s)ds \\ -\displaystyle\sum_{i=1}^{n} \dfrac{\lambda\mu_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} x(s)ds \\ +\displaystyle\sum_{i=1}^{n} \gamma_i \int_0^{\eta_i} x(s)ds \Bigg] \end{array} \right. \right\}.$$

For $g \in \mathcal{S}_{G,x}$.

Now we shall prove that $\mathcal{N}$ is a completely continuous multivalued map, u.s.c, with convex and closed values. The proof will be given in several steps.

Step 1 : $\mathcal{N}(x)$ is convex for each $x \in \mathcal{C}([0,1],\mathbb{R})$.

Indeed, if h_1, h_2 belong to $\mathcal{N}(x)$, then there exist $g_1, g_2 \in \mathcal{S}_{G,x}$ such that for each $t \in [0,1]$ we have:

$$\begin{aligned} h_j(t) &= \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} g_j(s)ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds \\ &+ \frac{t^{\beta+1}}{1-\omega} \Bigg[\frac{\lambda}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} x(s)ds - \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} g_j(s)ds \\ &+ \sum_{i=1}^{n} \frac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} g_j(s)ds - \sum_{i=1}^{n} \frac{\lambda\mu_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} x(s)ds \\ &+ \sum_{i=1}^{n} \gamma_i \int_0^{\eta_i} x(s)ds \Bigg]. \end{aligned}$$

For $j = 1, 2$. Let $0 \le k \le 1$ then, for each $t \in [0,1]$, we have

$$\begin{aligned} kh_1(t) + (1-k)h_2(t) &= \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} \Big[kg_1(s) + (1-k)g_2(s) \Big] ds \\ &- \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x(s)ds + \frac{t^{\beta+1}}{1-\omega} \Bigg[\frac{\lambda}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} x(s)ds \\ &- \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} \Big[kg_1(s) + (1-k)g_2(s) \Big] ds \\ &+ \sum_{i=1}^{n} \frac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} \Big[kg_1(s) + (1-k)g_2(s) \Big] ds \\ &- \sum_{i=1}^{n} \frac{\lambda\mu_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} x(s)ds + \sum_{i=1}^{n} \gamma_i \int_0^{\eta_i} x(s)ds \Bigg]. \end{aligned}$$

Thus $kg_1 + (1-k)g_2 \in \mathcal{N}(x)$ $\big($ because $\mathcal{S}_{G,x}$ is convex $\big)$, then $\mathcal{N}(x)$ is convex for each $x \in \mathcal{C}([0,1],\mathbb{R})$

Step 2 : $\mathcal{N}(x)$ maps bounded set into bounded set in $\mathcal{C}([0,1],\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in \mathcal{N}(x)$; $x \in \mathbf{B}_q = \{x \in \mathcal{C}([0,1],\mathbb{R} : \|x\|_\infty \leq q\}$ we have $\|h\|_\infty \leq l$.
If $h \in \mathcal{N}(x)$ then there exist $g \in \mathcal{S}_{G,x}$, such that for every $t \in [0,1]$ we have :

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1} g(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds \\ &+ \frac{t^{\beta+1}}{1-\omega}\Bigg[h_1(t)+h_2(t)\Bigg]. \end{aligned}$$

Where :

$$\begin{aligned} h_1(x) &= \frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x(s)ds - \sum_{i=1}^n \frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i - s)^{\beta-1}x(s)ds \\ &+ \sum_{i=1}^n \gamma_i \int_0^{\eta_i} x(s)ds, \end{aligned}$$

$$h_2(g) = \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i - s)^{\beta+\alpha-1}g(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g(s)ds, \text{ then :}$$

$$\begin{aligned} |h(t)| &\leq \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}|g(s)|ds + \frac{|\lambda|}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|x(s)|ds \\ &+ \frac{t^{\beta+1}}{|1-\omega|}\Bigg[|h_1(t)|+|h_2(t)|\Bigg], \end{aligned}$$

and

$$\begin{aligned} |h_1(x)| &\leq \frac{|\lambda|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\|x\|_\infty ds + \sum_{i=1}^n \frac{|\lambda||\mu_i|}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i - s)^{\beta-1}\|x\|_\infty ds \\ &+ \sum_{i=1}^n |\gamma_i| \int_0^{\eta_i} \|x\|_\infty ds \\ &\leq \|x\|_\infty \left(\frac{|\lambda|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}ds + \sum_{i=1}^n \frac{|\lambda||\mu_i|}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i - s)^{\beta-1}ds + \sum_{i=1}^n |\gamma_i| \int_0^{\eta_i} ds \right) \\ &\leq \frac{\|x\|_\infty}{\Gamma(\beta+1)}\left(|\lambda| + |\lambda|\sum_{i=1}^n |\mu_i|\eta_i^\beta + \Gamma(\beta+1)\sum_{i=1}^n |\gamma_i|\eta_i \right) \\ &\leq \frac{q}{\Gamma(\beta+1)}\left(|\lambda| + |\lambda|\sum_{i=1}^n |\mu_i|\eta_i^\beta + \Gamma(\beta+1)\sum_{i=1}^n |\gamma_i|\eta_i \right) \\ &\leq \frac{q}{\Gamma(\beta+1)}\left(|\lambda| + |\lambda|\sum_{i=1}^n |\mu_i|\eta_i^\beta + \Gamma(\beta+1)\sum_{i=1}^n |\gamma_i|\eta_i \right) \end{aligned}$$

$$
\begin{aligned}
|h_2(g)| &\leq \sum_{i=1}^{n} \frac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} \|g(s)\| ds + \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} \|g(s)\| ds \\
&\leq \frac{\|g(s)\|}{\Gamma(\beta+\alpha)} \left(\sum_{i=1}^{n} |\mu_i| \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} ds + \int_0^1 (1-s)^{\beta+\alpha-1} ds \right) \\
&\leq \frac{\|g(s)\|}{\Gamma(\beta+\alpha+1)} \left(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha} \right) \\
&\leq \frac{\|p\|_\infty \Psi(\|x\|_\infty)}{\Gamma(\beta+\alpha+1)} \left(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha} \right) \\
&\leq \frac{\|p\|_\infty \Psi(q)}{\Gamma(\beta+\alpha+1)} \left(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha} \right).
\end{aligned}
$$

Then

$$
\begin{aligned}
|h(t)| &\leq \frac{\|g(s)\|}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} ds + \frac{|\lambda| \|x\|_\infty}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \\
&+ \frac{t^{\beta+1}}{|1-\omega|} \Big[|h_1(t)| + |h_2(t)| \Big] \\
&\leq \frac{\|p\|_\infty \Psi(\|x\|_\infty) t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{|\lambda| \|x\|_\infty t^\beta}{\Gamma(\beta+1)} + \frac{t^{\beta+1}}{|1-\omega|} \Big[|h_1(x)| + |h_2(g)| \Big] \\
&\leq \frac{\|p\|_\infty \Psi(q) t^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{|\lambda| q t^\beta}{\Gamma(\beta+1)} \\
&+ \frac{t^{\beta+1}}{|1-\omega|} \left[\frac{q}{\Gamma(\beta+1)} \left(|\lambda| + |\lambda| \sum_{i=1}^{n} |\mu_i| \eta_i^\beta + \Gamma(\beta+1) \sum_{i=1}^{n} |\gamma_i| \eta_i \right) \right. \\
&+ \left. \frac{\|p\|_\infty \Psi(q)}{\Gamma(\beta+\alpha+1)} \left(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha} \right) \right] \\
&\leq \|p\|_\infty \Psi(q) \frac{|1-\omega| + 1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha}}{|1-\omega| \Gamma(\beta+\alpha+1)} \\
&+ q \frac{|1-\omega||\lambda| + |\lambda| \big(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^\beta \big) + \Gamma(\beta+1) \sum_{i=1}^{n} |\gamma_i| \eta_i}{|1-\omega| \Gamma(\beta+1)}.
\end{aligned}
$$

Then

$$\|h\|_\infty \leq \|p\|_\infty \Psi(q) \Phi + q\Omega := l.$$

where

$$\Phi = \frac{|1-\omega| + 1 + \sum_{i=1}^{n} |\mu_i| \eta_i^{\beta+\alpha}}{|1-\omega| \Gamma(\beta+\alpha+1)}.$$

And

$$\Omega = \frac{|1-\omega||\lambda| + |\lambda| \big(1 + \sum_{i=1}^{n} |\mu_i| \eta_i^\beta \big) + \Gamma(\beta+1) \sum_{i=1}^{n} |\gamma_i| \eta_i}{|1-\omega| \Gamma(\beta+1)}.$$

Step 3 : $\mathcal{N}$ maps bounded set into equicontinuous sets of $\mathcal{C}([0,1],\mathbb{R})$:

Let $t_1, t_2 \in [0,1]; t_1 < t_2$, and $x \in \mathbf{B}_q$ where $\mathbf{B}_q$, as above, is a bounded set of $\mathcal{C}([0,1],\mathbb{R})$; for each $x \in \mathbf{B}_q$ and $h \in \mathcal{N}(x)$; there exist $g \in \mathcal{S}_{G,x}$ such that :

$$\begin{aligned}
|h(t_2) - h(t_1)| \leq & \Big| \frac{1}{\Gamma(\beta+\alpha)} \int_0^{t_2} (t_2 - s)^{\beta+\alpha-1} g(s) ds - \frac{\lambda}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} x(s) ds \\
& - \frac{1}{\Gamma(\beta+\alpha)} \int_0^{t_1} (t_1 - s)^{\beta+\alpha-1} g(s) ds + \frac{\lambda}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} x(s) ds \Big| \\
& + \Big| \frac{t_2^{\beta+1} - t_1^{\beta+1}}{1-\omega} \Big[h_1(x) + h_2(g) \Big] \Big|. \\
& \leq \Big| \frac{1}{\Gamma(\beta+\alpha)} \int_0^{t_2} \Big[(t_2 - s)^{\beta+\alpha-1} - (t_1 - s)^{\beta+\alpha-1} \Big] g(s) ds \\
& + \frac{1}{\Gamma(\beta+\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\beta+\alpha-1} g(s) ds \Big| + \Big| \frac{\lambda}{\Gamma(\beta)} \int_0^{t_2} \Big[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1} \Big] x(s) ds \\
& + \frac{\lambda}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2 - s)^{\beta-1} x(s) ds \Big| + \frac{t_2^{\beta+1} - t_1^{\beta+1}}{|1-\omega|} \Big[|h_1(x)| + |h_2(g)| \Big].
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero, implies that $\mathcal{N}(x)$ is equicontinuous. Therefore it follows by Arzela-Ascoli theorem that $\mathcal{N} : \mathcal{C}([0,1],\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{C}([0,1],\mathbb{R}))$ is relatively compact then $\mathcal{N}$ is completely continuous.

Step 4 : $\mathcal{N}$ has a closed graph.

Let $x_n \longrightarrow x_*$, $h_n \in \mathcal{N}(x_n)$ and $h_n \longrightarrow h_*$, we shall prove that $h_* \in \mathcal{N}(x_*)$.
$h_n \in \mathcal{N}(x_n)$ then there exists $g_n \in \mathcal{S}_{G,x_n}$ such that for each $t \in [0,1]$

$$\begin{aligned}
h_n(t) = & \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} g_n(s) ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x_n(s) ds \\
& + \frac{t^{\beta+1}}{1-\omega} \Big[\frac{\lambda}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} x_n(s) ds - \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} g_n(s) ds \\
& + \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} g_n(s) ds - \sum_{i=1}^n \frac{\lambda \mu_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} x_n(s) ds \\
& + \sum_{i=1}^n \gamma_i \int_0^{\eta_i} x_n(s) ds \Big].
\end{aligned}$$

We should prove that $g_* \in \mathcal{S}_{G,x_*}$ such that for each $t \in [0,1]$:

$$\begin{aligned}
h_*(t) = & \frac{1}{\Gamma(\beta+\alpha)} \int_0^t (t-s)^{\beta+\alpha-1} g_*(s) ds - \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} x_*(s) ds \\
& + \frac{t^{\beta+1}}{1-\omega} \Big[\frac{\lambda}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} x_*(s) ds - \frac{1}{\Gamma(\beta+\alpha)} \int_0^1 (1-s)^{\beta+\alpha-1} g_*(s) ds \\
& + \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)} \int_0^{\eta_i} (\eta_i - s)^{\beta+\alpha-1} g_*(s) ds - \sum_{i=1}^n \frac{\lambda \mu_i}{\Gamma(\beta)} \int_0^{\eta_i} (\eta_i - s)^{\beta-1} x_*(s) ds \\
& + \sum_{i=1}^n \gamma_i \int_0^{\eta_i} x_*(s) ds \Big].
\end{aligned}$$

We have that :

$$\|\Bigg(h_n(t)+\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}x_n(s)ds+\frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}x_n(s)ds$$
$$-\sum_{i=1}^n\gamma_i\int_0^{\eta_i}x_n(s)ds+\sum_{i=1}^n\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x_n(s)ds\Bigg]\Bigg)$$
$$-\Bigg(h_*(t)+\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}x_*(s)ds+\frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}x_*(s)ds$$
$$-\sum_{i=1}^n\gamma_i\int_0^{\eta_i}x_*(s)ds+\sum_{i=1}^n\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x_*(s)ds\Bigg]\Bigg)\|\longrightarrow 0.$$

As $n\longrightarrow\infty$

Consider the linear operator :

$$\Upsilon:\mathbf{L}^1([0,1],\mathbb{R})\longrightarrow\mathcal{C}([0,1],\mathbb{R})$$

$$g\longrightarrow\Upsilon(g)(t).$$

With

$$\Upsilon(g)(t)=\frac{1}{\Gamma(\beta+\alpha)}\int_0^t(t-s)^{\beta+\alpha-1}g(s)ds$$
$$-\frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{1}{\Gamma(\beta+\alpha)}\int_0^1(1-s)^{\beta+\alpha-1}g(s)ds-\sum_{i=1}^n\frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g(s)ds\Bigg].$$

From Lemma 2.15; $\Upsilon\circ\mathcal{S}_G$ is a closed graph operator then we have that :

$$h_n(t)+\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}x_n(s)ds+\frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}x_n(s)ds$$
$$-\sum_{i=1}^n\gamma_i\int_0^{\eta_i}x_n(s)ds+\sum_{i=1}^n\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x_n(s)ds\Bigg]\in\Upsilon\Big(\mathcal{S}_{G,x_n}\Big).$$

Since $x_n\longrightarrow x_*$, and $h_n\longrightarrow h_*$ then :

$$h_*(t)+\frac{\lambda}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}x_*(s)ds+\frac{t^{\beta+1}}{1-\omega}\Bigg[\frac{\lambda}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}x_*(s)ds$$
$$-\sum_{i=1}^n\gamma_i\int_0^{\eta_i}x_*(s)ds+\sum_{i=1}^n\frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x_*(s)ds\Bigg]\in\Upsilon\Big(\mathcal{S}_{G,x_*}\Big).$$

It follows that $g_*\in\mathcal{S}_{G,x}$ such that

$$\begin{aligned}h_*(t) &= \frac{1}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}g_*(s)ds - \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x_*(s)ds \\ &+ \frac{t^{\beta+1}}{1-\omega}\left[\frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x_*(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g_*(s)ds\right. \\ &+ \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g_*(s)ds - \sum_{i=1}^n \frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x_*(s)ds \\ &\left.+ \sum_{i=1}^n \gamma_i \int_0^{\eta_i} x_*(s)ds\right].\end{aligned}$$

From the Step 1, step 2, step 3 and step 4 we deduce that $\mathcal{N}$ is u.s.c, completely continuous then by Lemma 2.12, $\mathcal{N}$ is a condensing, bounded, closed and convex operator. In order to prove that $\mathcal{N}$ has a fixed point, we need one more step.

Step 5 : The set $\Lambda := \{x \in X : \rho x \in \mathcal{N}x \quad \text{for} \quad \rho > 1\}$ is bounded.

Let $x \in \Lambda$. Then $\rho x \in \mathcal{N}(x)$, thus there exists $g \in \mathcal{S}_{G,x}$ such that :

$$\begin{aligned}x(t) &= \frac{\rho^{-1}}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1}g(s)ds - \frac{\rho^{-1}\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}x(s)ds \\ &+ \frac{\rho^{-1}t^{\beta+1}}{1-\omega}\left[\frac{\lambda}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}x(s)ds - \frac{1}{\Gamma(\beta+\alpha)}\int_0^1 (1-s)^{\beta+\alpha-1}g(s)ds\right. \\ &+ \sum_{i=1}^n \frac{\mu_i}{\Gamma(\beta+\alpha)}\int_0^{\eta_i}(\eta_i-s)^{\beta+\alpha-1}g(s)ds - \sum_{i=1}^n \frac{\lambda\mu_i}{\Gamma(\beta)}\int_0^{\eta_i}(\eta_i-s)^{\beta-1}x(s)ds \\ &\left.+ \sum_{i=1}^n \gamma_i \int_0^{\eta_i} x(s)ds\right].\end{aligned}$$

By $(H_1)-(H_3)$and (H_5) we have :

$$\|x\|_\infty \leq \|p\|_\infty \Psi(\|x\|_\infty)\Phi + \|x\|_\infty \Omega.$$

where

$$\Phi = \frac{|1-\omega| + 1 + \sum_{i=1}^n |\mu_i|\eta_i^{\beta+\alpha}}{|1-\omega|\Gamma(\beta+\alpha+1)}.$$

And

$$\Omega = \frac{|1-\omega||\lambda| + |\lambda|\big(1+\sum_{i=1}^n |\mu_i|\eta_i^{\beta}\big) + \Gamma(\beta+1)\sum_{i=1}^n |\gamma_i|\eta_i}{|1-\omega|\Gamma(\beta+1)}.$$

With $\Omega < 1$.
Then

$$\begin{aligned}\|x\|_\infty - \|x\|_\infty\Omega &\leq \|p\|_\infty \Psi(\|x\|_\infty)\Phi \\ &\leq \frac{\|p\|_\infty \Psi(\|x\|_\infty)\Phi}{1-\Omega} \\ &\leq \frac{\|p\|_\infty \Psi(M)\Phi}{1-\Omega} := K \\ &< K,\end{aligned}$$

where K depends only on the functions p and Ψ. This shows that Λ is bounded.
As a consequence of Theorem 2.16 (Leray-Schauders's fixed point) we deduce that $\mathcal{N}$ has a fixed point which is a solution of our problem (1.1). □

4. Conclusion

In the current article, we investigated the existence of solution for a fractional Langevin inclusion with multi-point boundary conditions. We established the existence resut by using Leray-Schauder's fixed point theorem.

References

1. J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Dekker, NewYork, (1980).
2. A. Cernea, *On a multi point boundary value problem for a fractional order differential inclusion*, Arab Journal of Mathematical Sciences 19, 73–83, (2013).
3. K. Deimling, *Multivalued Differential Equations*, Gruyter Series in Nonlinear Analysis and Applications vol. 1, Walter de Gruyter, Berlin, Germany, 1992.
4. H. Fazli and J. J. Nieto, *Fractional Langevin equation with anti-periodic boundary conditions*, Chaos, Solitons and Fractals 114, 332–337, (2018).
5. H. Fazli, H. G. Sun, and J. J. Nieto, *Fractional Langevin equation involving two fractional orders: existence and uniqueness*, Mathematics 8, 743, (2020).
6. A. Granas, J.Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, (2005).
7. K. Hilal, K. Guida, L. Ibnelazyz, and M. Oukessou, *Existence results for an impulsive fractional integro-differential equations with a non-compact semigroup*, in Recent Advances in Intuitionistic Fuzzy Logic Systems, S. Melliani and O. Castillo, Eds., Studies in Fuzziness and Soft Computing pp. 191–211, (2019).
8. K. Hilal, L. Ibnelazyz, K. Guida, and S. Melliani, *Existence of mild solutions for an impulsive fractional integro-differential equations with non-local condition*, in Recent Advances in Intuitionistic Fuzzy Logic Systems, S. Melliani and O. Castillo, Eds., Studies in Fuzziness and Soft Computing 251-271, (2019).
9. R. Hilfer, *Applications of Fractional Calculs in Physics*, World Scientific, Singapore, (2000).
10. S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis*, Mathematics and its Applications vol. 1, Kluwer Academic Publishers, Dordrecht, The Netherlands, (1997).
11. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, NorthHolland Mathematics Studies, vol. 204, Elsevier, Amsterdam, (2006)
12. V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Analysis: Theory, Methods and Applications 69, 3337–3343, (2008).
13. V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Analysis: Theory, Methods and Applications 69, 2677–2682, (2008).
14. A. Lasota, Z. Opial *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull Acad Polon Sci Ser Sci Math Astronom Phys. 13, 781–786, (1965).
15. C. Li, X. Luo, and Y. Zhou, *Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations*, Computers and Mathematcs with Applications 59, 1363–1375, (2010).
16. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, (1993).
17. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, (1993).
18. M. Rehman, R. Khan, and N. Asif, *Three point boundary value problems for nonlinear fractional differential equations*, Acta Mathematica Scientia 31, 1337–1346, (2011).
19. A. Salem and B. Alghamdi, *Multi-strip and multi-point boundary conditions for fractional Langevin equation*. Fractal and Fractional 4, 18, (2020).
20. A. Salem and B. Alghamdi, *Multi-point and anti-periodic conditions for generalized Langevin equation with two fractional orders*. Fractal and Fractional 3, 51,(2019) .
21. G. V. Smirnov, *Introduction to the Theory of Differential Inclusions*. Graduate Studies in Mathematics, American Mathematical Society vol 41, Providence, RI, USA, (2002).
22. J. Yang, J. C. Ma, S. Zhao, and Y. Ge, *Fractional multi-point boundary value problem of fractional differential equations*. Mathematics in Practice and Theory 41, 188–194, (2011) .
23. Y. Zhou, *Basic Theory of Fractional Differential Equations*. Xiangtan University, China, 2014.

24. Z. Zhou, Y. Qiao, *Solutions for a class of fractional Langevin equations with integral and anti-periodic boundary conditions.* Boundary Value Problems 2018 (2018).

25. A. Lachouri, M. S. Abdo, A. Ardjouni, B. Abdalla and T. Abdeljawad. *On a class of differential inclusions in the frame of generalized Hilfer fractional derivative.* AIMS Mathematics, 7(3), 3477-3493, (2022).

Hamid Lmou,
Khalid Hilal,
Ahmed Kajouni,
Laboratory of Applied Mathematics and Scientific Competing,
Faculty of Sciences and Technics, Sultan Moulay Slimane University,
Beni Mellal, Morocco.
E-mail address: hamid.lmou@usms.ma, hilalkhalid2005@yahoo.fr and kajjouni@gmail.com