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A Numerical Calculation of Arc Length and Area Using Some Spline Quasi-interpolants

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ABSTRACT: In this paper, we propose two methods to approach numerically the length of curves and the area of surface of revolution created by rotating a curve around an axis. The first one is based on an approximation of functions by quadratic spline discrete quasi-interpolant and calculating its exact length. The second one consists to approximate the values of the first derivatives by those of cubic spline discrete quasi-interpolant. These values are used to provide a quadrature formula to calculate the integral giving the length. In both methods, we prove that the order of convergence is $O(h^4)$. The theoretical results given in this work are illustrated by some numerical examples.

Key Words: Arc length, area of surface of revolution, spline quasi-interpolants, Simpson rule.

Contents

1. Introduction

The approximation of arc length curves and area surfaces are treated by many authors due to its importance in different domains such as Computer Aided Design (CAD), computer graphics, computer vision and other areas of computer science. For approaching the arc length, in [3], the authors use numerical integration on the derivative of the curve. In [4], the author proposes a method based entirely on some point evaluations. This method used specifically the Bézier curves (see also [5]). Other technique published in [6] uses a Pythagorean hodograph quintic splines,...

In our work, we are interested to approximate the area of surface of revolution created by rotating a curve around an axis, where we extend the methods used in approximation of arc length.

Let $\mathbf{f} : [a, b] \to \mathbb{R}^d,\ d \geq 2$ a regular parametric curve, by which we mean a continuously differentiable function such that $\mathbf{f}'(t) \neq 0$ for all $t \in [a, b]$ where we denote by $|\ .\ |$ the euclidean norm in $\mathbb{R}^d$.
The arc length of the curve $\mathbf{f}$ is given by the following formula (see [9], chapter 9)

$$L(\mathbf{f}) := \int_a^b |\mathbf{f}'(t)|dt = \int_a^b \sqrt{\sum_{i=1}^{d} (\mathbf{f}_i'(t))^2}dt, \tag{1.1}$$

with $\mathbf{f}(t) = (f_1(t), \cdots, f_d(t))$.

Since $L(\mathbf{f})$ is the integral of the function $|\mathbf{f}'|$ on $[a, b]$, a natural approach is to apply to $|\mathbf{f}'|$ some quadrature rules.

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In [3], the authors apply a method adaptively, however it has the drawbeak that involves derivatives of $\mathbf{f}$, which might be more time-consuming to evaluate than point of $\mathbf{f}$ or might simply not be available. One alternative way is *chord length* rule, but it only has second order accuracy. This is motivated in [4] to find a higher order using only point evaluations. After [8] investigate a much more general point-based method, which turns out to include these two methods as special cases.

The paper is organized as follows. In section 2, we give a remainder of univariate spline quasi-interpolant on uniform partition. In section 3, we approaching length of curve by two methods, the first is based in the approach a curve by classical quadratic quasi-interpolant and calculating its exact length, the second consists to approximating values of the first derivatives of a curve by those of his specific cubic quasi-interpolant and using Simpson rule to calculating the integral giving the arc length. In Section 4, we extend the previous methods to approximate the area of surface of revolution. Some numerical examples are given in the section 5 to illustrate the theoretical results.

2. Univariate spline quasi-interpolants on an uniform partition

For any integer $k \geq 1$, let $\mathbb{S}_k = \mathbb{S}_k^{k-1}([a,b])$ be the space splines in $C^{k-1}([a,b])$ and degree less than k on the interval $[a,b]$ endowed with the uniform partition $X_n = \{x_i = a + ih;\ 0 \leq i \leq n\}$ with $h = \frac{b-a}{n}$ and multiple knots at the endpoints.

Let $\{B_{i,k}, i \in J\}$ be the set of B-splines of $\mathbb{S}_k$ with $J = \{0, \cdots, n+k-1\}$ and support $\text{Supp}(B_{i,k}) = [x_{i-k}, x_{i+1}]$.

We denote by T_n the set $T_n = \{t_i,\ 0 \leq i \leq n+1\}$ with $t_0 = a$; $t_i = (x_{i-1} + x_i)/2$ for $i = 1, \cdots, n$ and $t_{n+1} = b$.

A discrete spline quasi-interpolant of degree k given by

$$Q_k\mathbf{f} := \sum_{i=0}^{n+k-1} \lambda_i(\mathbf{f})\ B_{i,k}, \tag{2.1}$$

where $\lambda_i(\mathbf{f})$ are the linear functionals of function $\mathbf{f}$ on either X_n if k is odd or on T_n if k is even. Moreover, Q_k is construct to be exact on $\mathbb{P}_k$ the space of polynomials of degree less than k.

3. Arc length approximation

3.1. Method based on a classic quadratic quasi-interpolant

We consider the quadratic quasi-interpolant Q_2 such that

$$Q_2\boldsymbol{f} := \sum_{i=0}^{n+1} \lambda_i(\boldsymbol{f})\ B_{i,2}. \tag{3.1}$$

The functional coefficients $\lambda_i(\mathbf{f})$ (see [2]), are given by

$$\begin{aligned}
&\lambda_0(\mathbf{f}) = \mathbf{f}_0,\\
&\lambda_1(\mathbf{f}) = \frac{1}{6}\big(-2\mathbf{f}_0 + 9\mathbf{f}_1 - \mathbf{f}_2\big),\\
&\lambda_n(\mathbf{f}) = \frac{1}{6}\big(-\mathbf{f}_{n-1} + 9\mathbf{f}_n - 2\mathbf{f}_{n+1}\big),\\
&\lambda_{n+1}(\mathbf{f}) = \mathbf{f}_{n+1},\\
&\lambda_j(\mathbf{f}) = \frac{1}{8}\big(-\mathbf{f}_{j-1} + 10\mathbf{f}_j - \mathbf{f}_{j+1}\big),\ 2 \leq j \leq n-1,
\end{aligned}$$

with $\mathbf{f}_i = \mathbf{f}(t_i)$, for $i = 0, \cdots, n+1$.
Q_2 is exact on the space $\mathbb{P}_2$. Moreover, we can write Q_2 as

$$Q_2\boldsymbol{f}(x) = \sum_{i=0}^{n+1} \mathbf{f}_i\ B^*_{i,2}(x),$$

with $B^*_{i,2}(x) = \frac{1}{8}\big(-B_{i-1,2} + 10B_i - B_{i+1,2}\big)$, for $2 \leq i \leq n-1$.

Specific formulas are given for extreme indices.
In [1], is shown that $\|Q_2\|_\infty \approx 1.47$.

Proposition 3.1. *For* $\boldsymbol{f} \in C^3([a,b])$ *and for all* $x \in [a,b]$, *there holds:*

$$|f_i(x) - Q_2 f_i(x)| \leq \frac{h^3}{3} \|f_i^{(3)}\|_\infty$$

and

$$|f_i'(x) - (Q_2 f_i)'(x)| \leq 1.2\ h^2\ \|f_i^{(3)}\|_\infty.$$

For the proof see [2].
Using the notation

$$\|\mathbf{f}^{(i)}\|_{[a,b]} = \max_{t\in[a,b]} |\mathbf{f}^{(i)}(t)|,$$

we have the following theorem.

Theorem 3.2. *For* $\boldsymbol{f} \in C^3([a,b])$, *we have the global approximation errors,*

$$\|Q_2\boldsymbol{f} - \boldsymbol{f}\|_{[a,b]} \leq \sqrt{d}\frac{h^3}{3}\ \|\boldsymbol{f}^{(3)}\|_{[a,b]},$$

and

$$\|(Q_2\boldsymbol{f}' - \boldsymbol{f}'\|_{[a,b]} \leq 1.2\ \sqrt{d}\ h^2\ \|\boldsymbol{f}^{(3)}\|_{[a,b]}.$$

Proof. By the previous proposition, we have

$$|f_i(x) - Q_2 f_i(x)| \leq \frac{h^3}{3}\ \max_{t\in[a,b]} |f_i^{(3)}(t)|.$$

For each $i = 1, \cdots, d$ we have

$$|f_i^{(3)}(t)| \leq |\mathbf{f}^{(3)}(t)|,$$

then

$$|f_i(x) - Q_2 f_i(x)| \leq \frac{h^3}{3}\ \max_{t\in[a,b]} |\mathbf{f}^{(3)}(t)|,$$

so that

$$|f_i(x) - Q_2 f_i(x)| \leq \frac{h^3}{3}\ \|\mathbf{f}^{(3)}\|_{[a,b]}.$$

Moreover,

$$\|\mathbf{f} - Q_2\mathbf{f}\|_{[a,b]} \leq \sqrt{d} \max_{t\in[a,b]} |f_i(t) - Q_2 f_i(t)|.$$

Finally

$$\|\mathbf{f} - Q_2\mathbf{f}\|_{[a,b]} \leq \sqrt{d}\frac{h^3}{3}\ \|\mathbf{f}^{(3)}\|_{[a,b]}.$$

By the same way we can get

$$\|(Q_2\mathbf{f})' - \mathbf{f}'\|_{[a,b]} \leq 1.2\ \sqrt{d}\ h^2\ \|\mathbf{f}^{(3)}\|_{[a,b]}.$$

□

In the following, we approach the arc length of curve of $\mathbf{f}$ by the associated quadratic quasi-interpolant $Q_2\mathbf{f}$, i.e

$$\int_a^b |\mathbf{f}'(t)|dt \simeq \int_a^b |(Q_2\mathbf{f})'(t)|dt.$$

First, we give and prove two lemmas that we use to prove the theorem giving the order of convergence.

Lemma 3.3. *If $\boldsymbol{f} \in C^4([a,b])$ and $\boldsymbol{f}$ is regular, then $|\boldsymbol{f}'(t)|'$ is bounded in $[a,b]$.*

Proof. We have

$$(|\mathbf{f}'(t)|^2)' = (\mathbf{f}'(t).\mathbf{f}'(t))'$$

and by applying the Leibniz formula for the functions $|\mathbf{f}'|$ and $\mathbf{f}'$, we obtain

$$(|\mathbf{f}'(t)|^2)' = 2|\mathbf{f}'(t)|'|\mathbf{f}'(t)|$$

and

$$(\mathbf{f}'(t).\mathbf{f}'(t))' = 2\mathbf{f}'(t).\mathbf{f}''(t).$$

The function $\mathbf{f}$ is supposed regular in the closed interval $[a,b]$, i.e $|\mathbf{f}'(t)| > 0$, then

$$|\mathbf{f}'(t)|' = \frac{\mathbf{f}'(t).\mathbf{f}''(t)}{|\mathbf{f}'(t)|}.$$

$\mathbf{f}'(t)$, $|\mathbf{f}'(t)|$ and $\mathbf{f}''(t)$ are bounded, then $|\mathbf{f}'(t)|'$ is bounded too. □

Lemma 3.4. *If $\boldsymbol{f} \in C^4([a,b])$, and $\boldsymbol{f}$ is regular, then $|(Q_2\boldsymbol{f})'(t)|'$ is bounded in $[a,b]$ independently which h is small enough.*

Proof. First, we prove that $Q_2\mathbf{f}$ is regular for a sufficiently small values of h.
By the triangular inequality, and the previous lemma, we have

$$|(Q_2\mathbf{f})'(t)| \geq |\mathbf{f}'(t)| - |(Q_2\mathbf{f})'(t) - \mathbf{f}'(t)| \geq |\mathbf{f}'(t)| - \|(Q_2\mathbf{f})'(t) - \mathbf{f}'(t)\|_{[a,b]} \geq |\mathbf{f}'(t)| - \ 1.2\ \sqrt{d}\ h^2\ \|\mathbf{f}^{(3)}\|_{[a,b]}.$$

Since $|\mathbf{f}'(t)| > 0$ for all $t \in [a,b]$, for sufficiently small value of h, we have $|(Q_2\mathbf{f})'(t)| > 0$ for all $t \in [a,b]$, i.e $Q_2\mathbf{f}$ is regular.

On the other hand $Q_2\mathbf{f}$ is a polynomial on each subinterval. By applying Lemma 3.3, $|(Q_2\mathbf{f})'(t)|'$ is bounded.

□

Theorem 3.5. *If $\mathbf{f} \in C^4([a,b])$ is regular, we have the global approximation error :*

$$L(\mathbf{f}) - L(Q_2\mathbf{f}) = O(h^4).$$

Proof. Let $E_2(t) = \mathbf{f}(t) - Q_2\mathbf{f}(t)$. By using some calculations, we get

$$|(Q_2\mathbf{f})'(t)| - |\mathbf{f}'(t)| = \frac{\big(\mathbf{f}'(t) - (Q_2\mathbf{f})'(t)\big)^2 - 2\mathbf{f}'(t)\big(\mathbf{f}'(t) - Q'(t)\big)}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|}.$$

Therefore,

$$L(\mathbf{f}) - L(Q_2\mathbf{f}) = \int_a^b \frac{E_2'.E_2'}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|}dt - 2\int_a^b \frac{\mathbf{f}'(t)E_2'}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|}dt.$$

Since $|\mathbf{f}'(t)|$ and $|Q_2\mathbf{f}'(t)|$ are bounded, moreover $\mathbf{f}$ and $Q_2\mathbf{f}$ are regular, so that

$$\int_a^b \frac{E_2'.E_2'}{|\mathbf{f}'(t)| + |Q_2\mathbf{f}'(t)|}dt = O(h^5).$$

It remains to evaluate the order of

$$I = \int_a^b \frac{\mathbf{f}'(t)E_2'}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|} dt.$$

An integration by parts gives

$$I = \int_a^b \frac{\mathbf{f}'(t)E_2'}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|} dt = \big[G(t).E_2\big]_a^b - \int_a^b G'(t).E_2 \; dt,$$

with

$$G(t) = \frac{\mathbf{f}'(t)}{|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|} \; ,$$

then

$$G'(t) = \frac{\mathbf{f}''(t)(|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|) - \mathbf{f}'(t)(|\mathbf{f}'(t)|' + |(Q_2\mathbf{f})'(t)|')}{(|\mathbf{f}'(t)| + |(Q_2\mathbf{f})'(t)|)^2}.$$

Since $Q_2\mathbf{f}$ interpolates $\mathbf{f}$ at endpoints, we have $E_2(a) = E_2(b) = 0$, i.e,

$$I = -\int_a^b G'(t).E_2 dt.$$

Moreover $|\mathbf{f}'|$, $\mathbf{f}''$, $|\mathbf{f}'|'$, $|(Q_2\mathbf{f})'(t)|$, and $|(Q_2\mathbf{f})'(t)|'$ are bounded, then G' is also bounded, therefore $I = O(h^4)$.
Finally, we have $L(\mathbf{f}) - L(Q_2\mathbf{f}) = O(h^4)$. □

3.2. Method based on a cubic quasi-interpolant and Simpson rule

We consider a specific cubic quasi-interpolant Q_3 given by

$$Q_3\boldsymbol{f} := \sum_{i=0}^{n+2} \mu_i(\boldsymbol{f}) \; B_{i,3}.$$

The functional coefficients $\mu_i(f)$ are given (see [7]) by

$$\begin{aligned}
\mu_0(\boldsymbol{f}) &= \boldsymbol{f}_0, \;\; \mu_{n+2}(\boldsymbol{f}) = \boldsymbol{f}_n,\\
\mu_1(\boldsymbol{f}) &= \frac{1}{36}\big(11\boldsymbol{f}_0 + 48\boldsymbol{f}_1 - 36\boldsymbol{f}_2 + 16\boldsymbol{f}_3 - 3\boldsymbol{f}_4\big),\\
\mu_2(\boldsymbol{f}) &= \frac{1}{36}\big(- 5\boldsymbol{f}_0 + 44\boldsymbol{f}_1 - 4\boldsymbol{f}_3 + \boldsymbol{f}_4\big),\\
\mu_n(\boldsymbol{f}) &= \frac{1}{36}\big(- 5\boldsymbol{f}_n + 44\boldsymbol{f}_{n-1} - 4\boldsymbol{f}_{n-3} + \boldsymbol{f}_{n-4}\big),\\
\mu_{n+1}(\boldsymbol{f}) &= \frac{1}{36}\big(11\boldsymbol{f}_n + 48\boldsymbol{f}_{n-1} - 36\boldsymbol{f}_{n-2} - 16\boldsymbol{f}_{n-3} - 3\boldsymbol{f}_{n-4}\big),\\
\mu_j(\boldsymbol{f}) &= \frac{1}{6}\big(\boldsymbol{f}_{j-4} - 10\boldsymbol{f}_{j-3} + 54\boldsymbol{f}_{j-2} - 10\boldsymbol{f}_{j-1} + \boldsymbol{f}_j\big), \;\; 3 \le j \le n-1,
\end{aligned}$$

with $\boldsymbol{f}_i = \boldsymbol{f}(\,x_i)$, for $i = 0, \cdots, n$. Q_3 is constructed to be exact on $\mathbb{P}_3$ and to give a surepconvergence of order 1 at all points of set X_n, i.e for $\boldsymbol{f} \in C^5([a,b])$, we have the results $Q_3 f_j(x_i) - f_j(x_i) = O(h^5)$ for each $i \in \{0, \cdots, n\}$ and $j = 1, \cdots, d$.

The following theorem gives the approximation order associated to Q_3.

Theorem 3.6. *If $\boldsymbol{f} \in C^4([a,b])$ is regular, we have the global approximation error :*

$$\|\boldsymbol{f} - Q_3\boldsymbol{f}\|_{[a,b]} = O(h^4).$$

Proof. By using the fact that $|f_i(x) - Q_3 f_i(x)| = O(h^4)$ (see [7]), we can easily prove this theorem. □

In this subsection, we approximate the arc length of a curve $\boldsymbol{f}$ by approaching the first derivatives of $\boldsymbol{f}$ at points x_i by those of $Q_3\boldsymbol{f}$, after we use this values to approximate the arc length of $Q_3\boldsymbol{f}$ by the Simpson rule.

The composite Simpson rule at points $x_i,\ i = 0, ...n$ is given by the following formula:

$$\int_a^b \boldsymbol{f}(t)dt \approx S(\boldsymbol{f}) := \frac{h}{3}(\boldsymbol{f}(x_i) + 4\sum_{i=1}^{n/2} \boldsymbol{f}(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} \boldsymbol{f}(x_{2i}) + \boldsymbol{f}(b)). \tag{3.2}$$

Then we have the approximation

$$\begin{aligned} L(Q_3\boldsymbol{f}) &= \int_a^b |(Q_3\boldsymbol{f})'(t)|dt \\ &\approx S(|(Q_3\boldsymbol{f})'|) = \frac{h}{3}(|(Q_3\boldsymbol{f})'(x_i)| + 4\sum_{i=1}^{n/2} |(Q_3\boldsymbol{f})'(x_{2i-1})| + 2\sum_{i=1}^{n/2-1} |(Q_3\boldsymbol{f})'(x_{2i})| + |(Q_3\boldsymbol{f})'(b)|). \end{aligned}$$

The following theorem give the approximation order associated to Q_3.

Theorem 3.7. *If* $\boldsymbol{f} \in C^5([a, b])$*, we have the global approximation error*

$$L(\mathbf{f}) - L(Q_3\mathbf{f}) = O(h^4).$$

Proof.

$$L(\mathbf{f}) - L(Q_3\mathbf{f}) = L(\mathbf{f}) - S(|\mathbf{f}'|) + S(|\mathbf{f}'|) - S(|(Q_3\mathbf{f})'|) + S(|(Q_3\mathbf{f})'|) - L(Q_3\mathbf{f}).$$

Since $L(\mathbf{f}) - S(|\mathbf{f}'|) = O(h^4)$ and $S(|(Q_3\mathbf{f})'|) - L(Q_3\mathbf{f}) = O(h^4)$,(see [8])

It remains to evaluate the approximation order of $S(|\mathbf{f}'|) - S(|(Q_3\mathbf{f})'|)$.

We have

$$\begin{aligned} \Big|S(|\mathbf{f}'|) - S(|(Q_3\mathbf{f})'|)\Big| &= \Big|\sum_{i=0}^{n} w_i|\mathbf{f}'(x_i)| - \sum_{i=0}^{n} w_i|(Q_3\mathbf{f})'(x_i)|\Big| \\ &= \Big|\sum_{i=0}^{n} w_i \left(|\mathbf{f}'(x_i)| - |(Q_3\mathbf{f})'(x_i)|\right)\Big| \\ &\le C_1 \sum_{i=0}^{n} w_i\Big|\ |\mathbf{f}'(x_i)|^2 - |(Q_3\mathbf{f})'(x_i)|^2)\Big| \\ &\le C_1 \sum_{i=0}^{n} w_i \sum_{j=1}^{d} \Big|(f_j'(x_i))^2 - ((Q_3 f_j)'(x_i))^2)\Big| \\ &\le C_2 \sum_{j=1}^{d} \max_{i\in\{0,\dots,n\}} \Big|f_j'(x_i) - (Q_3 f_j)'(x_i)\Big|, \end{aligned}$$

with

$$C_1 = \max_{i\in\{0,\dots,n\}} \frac{1}{\Big|\mathbf{f}'(x_i)| + |(Q_3\mathbf{f})'(x_i)\Big|},$$

$$C_2 = C_1 \sum_{i=0}^{n} w_i \max_{i,j} \Big|f_j'(x_i) + (Q_3 f_j)'(x_i)\Big|.$$

Since $f_j(x_i) - (Q_3 f_j)(x_i) = O(h^5)$ for each $j = 1, \cdots, d$,

$$f_j'(x_i) - (Q_3 f_j)'(x_i) = O(h^4),\ j = 1, \cdots, d.$$

Then

$$S(|\boldsymbol{f}'|) - S(|(Q_3\boldsymbol{f})'|) = O(h^4).$$

Finally, we obtain $L(\boldsymbol{f}) - L(Q_3\boldsymbol{f}) = O(h^4)$. □

4. Approaching area of a surface of revolution

The methods used previously to approximate the arc length of a curve can be extended to approximate the area of a surface of revolution.

We consider in this section, $\boldsymbol{f} = (f_1, f_2)$ a parametric curve over the interval $[a, b]$ in $\mathbb{R}^2$. We approximate the area of a surface of revolution $A(\boldsymbol{f})$ created by revolving the curve $\boldsymbol{f}$ around the x-axis.

$A(\boldsymbol{f})$ is given by the following integral (see [10]).

$$A(\mathbf{f}) := 2\ \pi \int_a^b f_2(t)|\mathbf{f}'(t)|dt.$$

4.1. Approaching $A(\boldsymbol{f})$ by $Q_2\boldsymbol{f}$

By approximating $\boldsymbol{f}$ by the previous quadratic quasi-interpolant given in (3.1), we obtain the following approach of area $A(\boldsymbol{f})$,

$$A(\boldsymbol{f}) \approx A(Q_2\boldsymbol{f}) := 2\ \pi \int_a^b Q_2 f_2(t)|(Q_2\boldsymbol{f})'(t)|dt.$$

Theorem 4.1. *If $\boldsymbol{f} \in C^4([a, b])$, and $\boldsymbol{f}$ regular, we have the global approximation error*

$$A(\mathbf{f}) - A(Q_2\mathbf{f}) = O(h^4).$$

Proof.

$$f_2(t)|\boldsymbol{f}'(t)| - Q_2 f_2(t)|(Q_3\boldsymbol{f})'(t)| = (f_2(t) - Q_2 f_2(t))|\boldsymbol{f}'(t)| + Q_2 f_2(t)(|\boldsymbol{f}'(t)| - |(Q_2\boldsymbol{f})'(t)|)$$

which implies

$$\begin{aligned} |A(\boldsymbol{f}) - A(Q_2\boldsymbol{f})| &= 2\ \pi \left| \int_a^b \left[(f_2(t) - Q_2 f_2(t))|\boldsymbol{f}'(t)| + Q_2 f_2(t)(|\boldsymbol{f}'(t)| - |(Q_2\boldsymbol{f})'(t)|)\right] dt \right| \\ &\leq 2\ \pi \sup_{t\in[a,b]} |\boldsymbol{f}'(t)| \left| \int_a^b (f_2(t) - Q_2 f_2(t))dt \right| \\ &+ \sup_{t\in[a,b]} |Q_2 f_2(t)| \left| \int_a^b (|\boldsymbol{f}'(t)| - |(Q_2\boldsymbol{f})'(t)|)dt \right|. \end{aligned}$$

We have $\left| \int_a^b (f_2(t) - Q_2 f_2(t))dt \right| = O(h^4)$, so that $|\boldsymbol{f}'|$ and $|(Q_2 f_2)'|$ are bounded, and by applying Theorem 3.5, we get

$$\left| \int_a^b (|\boldsymbol{f}'(t)| - |(Q_2\boldsymbol{f})'(t)|)dt \right| = O(h^4).$$

Then

$$A(\boldsymbol{f}) - A(Q_2\boldsymbol{f}) = O(h^4).$$

□

4.2. Approaching $A(\boldsymbol{f})$ using $Q_3\boldsymbol{f}$ and Simpson rule

We approximate the area $A(\boldsymbol{f})$ using second method applied to $A(Q_3\boldsymbol{f})$ in the same way we used it to approximate arc length.

We consider f a parametric curve over the interval $[a, b]$ in $\mathbb{R}^2$.

We want to give the convergence order of method approximation $A(\boldsymbol{f}) \approx A(Q_3\boldsymbol{f})$ using the Simpson rule to calculate the integral $A(Q_3\boldsymbol{f})$.

Applying the method given in (3.2) to the function $F(t) = 2\,\pi\, Q_3 f_2(t)|(Q_3\boldsymbol{f})'(t)|$, we get

$$A(Q_3\boldsymbol{f}) \approx S(F) := \frac{h}{3}(F(x_i) + 4\sum_{i=1}^{n/2} F(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} F(x_{2i}) + F(b)).$$

Theorem 4.2. *If $\boldsymbol{f} \in C^4([a, b])$, we have the global approximation error*

$$A(\mathbf{f}) - A(Q_3\mathbf{f}) = O(h^4).$$

Proof. We denote $H(t) = 2\,\pi f_2(t)|\mathbf{f}'(t)|$,

$$A(\boldsymbol{f}) - A(Q_3\boldsymbol{f}) = A(\boldsymbol{f}) - S(H) + S(H) - S(F) + S(F) - A(Q_3\boldsymbol{f}).$$

Since $A(\boldsymbol{f}) - S(H) = O(h^4)$ and $S(F) - L(Q_3\boldsymbol{f}) = O(h^4)$, (see [8]).

It remains to evaluate the approximation order of $S(H) - S(F)$. We have

$$\begin{aligned}
|S(H) - S(F)| &= \Big|\sum_{i=0}^{n} w_i\ H(xi) - \sum_{i=0}^{n} w_i\ F(x_i)\Big| \\
&= 2\,\pi\ \Big|\sum_{i=0}^{n} w_i\left(f_2(x_i)|\mathbf{f}'(x_i)| - Q_3 f_2(x_i)|(Q_3\mathbf{f})'(x_i)|\right)\Big| \\
&= 2\,\pi\ \Big|\sum_{i=0}^{n} w_i(f_2(x_i) - Q_3 f_2(x_i))|\mathbf{f}'(x_i)\Big| + \sum_{i=0}^{n} w_i|Q_3 f_2(x_i)|\Big|\mathbf{f}'(x_i)| - |(Q_3\mathbf{f})'(x_i)|\Big| \\
&\le K_1 \max_{i\in\{0,\cdots,n\}} \Big|f_2(x_i) - Q_3 f_2(x_i)\Big| + K_2 \max_{i\in\{0,\cdots,n\}, j=1,2} \Big|f_j'(x_i) - (Q_3 f_j)'(x_i)\Big|,
\end{aligned}$$

with

$$K_1 = 2\,\pi \max_{i\in\{0,\cdots,n\}} |\mathbf{f}'(x_i)| \sum_{i=0}^{n} w_i$$

and

$$K_2 = 2\,\sqrt{2}\,\pi \max_{i\in\{0,\cdots,n\}} |Q_3 f_2(x_i)| \sum_{i=0}^{n} w_i.$$

$\Big|f_2(x_i) - Q_3 f_2(x_i)\Big| = O(h^5)$ implies $\Big|f_2'(x_i) - (Q_3 f_2)'(x_i)\Big| = O(h^4)$.

As a result, we get

$$S(H) - S(F) = O(h^4).$$

Finally,

$$A(\boldsymbol{f}) - A(Q_3\boldsymbol{f}) = O(h^4).$$

□

5. Numerical examples

We consider the parametric curves $\boldsymbol{f}$ and $\mathbf{g}$ defined by

$$\boldsymbol{f}(t) = \big(tcos(t), tsin(t)\big), \quad t \in [0, \pi]$$

and

$$\mathbf{g}(t) = \Big(\frac{3t}{1+t^2}, \frac{3t^2}{1+t^3}\Big), \quad t \in [0, 1].$$

We obtain the surfaces of revolution of each curve by revolving them around the x-axis.

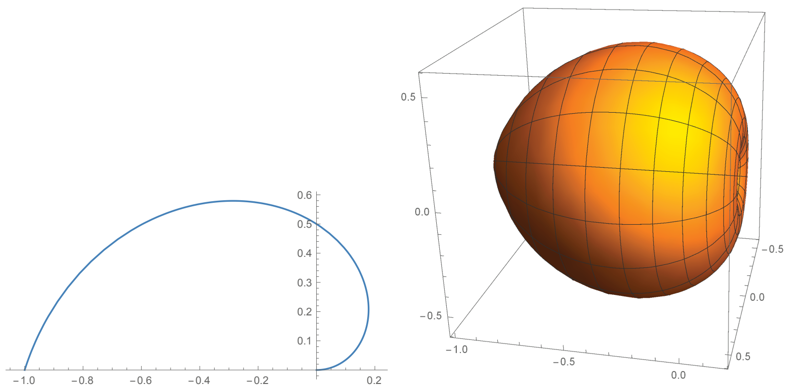

Figure 1: Curve $\boldsymbol{f}$ (left) and its surface of revolution (right)

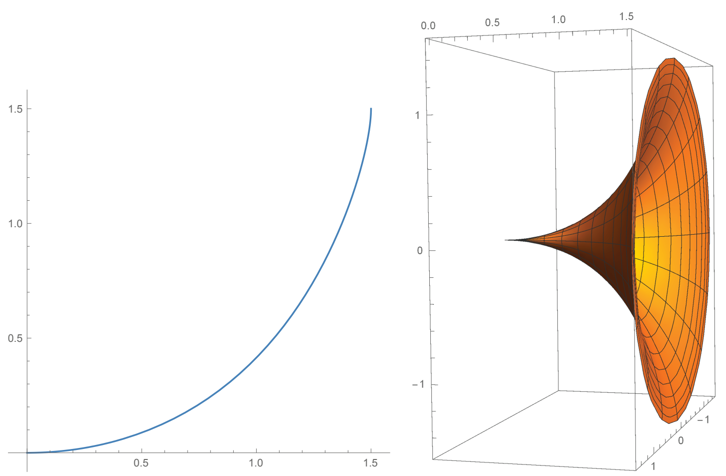

Figure 2: Curve $\mathbf{g}$ (left) and its surface of revolution (right)

In the following tables, we observe that by both methods, the errors

$$e_{L,2}(\boldsymbol{f}) = L(\boldsymbol{f}) - L(Q_2\boldsymbol{f}), \quad , e_{L,3}(\boldsymbol{f}) = L(\boldsymbol{f}) - L(Q_3\boldsymbol{f}),$$
$$e_{A,2}(\boldsymbol{f}) = A(\boldsymbol{f}) - A(Q_2\boldsymbol{f}), \quad e_{A,3}(\boldsymbol{f}) = A(\boldsymbol{f}) - A(Q_3\boldsymbol{f}),$$

obtained, with several values of n, are in $O(h^4)$. We notice by NCO the numerical convergence order

$$NCO = NCO(n_1 \to n_2) = \frac{\log\left(\dfrac{|e_{L,k,n_1}(\boldsymbol{f})|}{|e_{L,k,n_2}(\boldsymbol{f})|}\right)}{\log\left(\dfrac{n_1}{n_2}\right)}, \quad k = 2, 3.$$

We keep the same notation for the other errors defined above.

n	$\lvert e_{L,2}(\boldsymbol{f})\rvert$	NCO	$\lvert e_{L,2}(\boldsymbol{g})\rvert$	NCO
32	9.60899×10^{-6}	–	8.27564×10^{-7}	-
64	6.41148×10^{-7}	3.96	5.18711×10^{-8}	3.99
128	3.9338×10^{-8}	3.96	3.24189×10^{-9}	4
256	2.47413×10^{-9}	3.99	2.01967×10^{-10}	4
512	1.50092×10^{-10}	4.04	1.16665×10^{-11}	4.11

Table 1: Errors $e_{L,2}$ of the arc length of the curves $\boldsymbol{f}$ and $\boldsymbol{g}$.

n	$\lvert e_{L,3}(\boldsymbol{f})\rvert$	NCO	$\lvert e_{L,3}(\boldsymbol{g})\rvert$	NCO
32	2.16301×10^{-6}	–	2.38592×10^{-7}	–
64	1.38809×10^{-7}	3.96	1.25514×10^{-8}	4.25
128	8.85401×10^{-9}	3.97	7.47854×10^{-9}	4.07
256	6.9252×10^{-10}	3.67	4.64385×10^{-11}	4.01
512	7.03753×10^{-10}	3.85	3.16547×10^{-12}	3.87

Table 2: Errors $e_{L,3}$ of the arc length of the curves $\boldsymbol{f}$ and $\boldsymbol{g}$.

n	$\lvert e_{A,3}(\boldsymbol{f})\rvert$	NCO	$\lvert e_{A,3}(\boldsymbol{g})\rvert$	NCO
32	2.51624×10^{-6}	–	4.78159×10^{-6}	–
64	1.717441×10^{-7}	3.87	2.94637×10^{-7}	4.02
128	1.09835×10^{-8}	3.96	1.83574×10^{-8}	4
256	6.913071×10^{-10}	3.99	1.1488×10^{-9}	4
512	4.40394×10^{-11}	3.97	7.41309×10^{-11}	3.95

Table 3: Errors $e_{A,3}$ of the arc length of the curves $\boldsymbol{f}$ and $\boldsymbol{g}$.

n	$\lvert e_{A,2}(\boldsymbol{f})\rvert$	NCO	$\lvert e_{A,2}(\boldsymbol{g})\rvert$	NCO
32	3.18315×10^{-5}	–	8.10611×10^{-6}	–
64	1.97254×10^{-6}	4.01	5.20882×10^{-7}	3.96
128	1.22596×10^{-7}	4.01	3.30186×10^{-8}	3.98
256	7.63675×10^{-9}	4	2.07643×10^{-9}	3.99
512	4.75563×10^{-10}	4	1.26569×10^{-11}	4.03

Table 4: Errors $e_{A,2}$ of the arc length of the curves $\boldsymbol{f}$ and $\boldsymbol{g}$.

The results that we have obtained by the theoretical part and those obtained numerically justify that the different approximation methods that we propose in this paper prove their effectiveness and give good improvement on the approximation errors. These encourage us to do better in the future by looking for other new approaches by developing the current work.

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