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On subharmonic solutions for second order difference equations with relativistic operator *

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ABSTRACT: The purpose of this paper is to investigate the existence of subharmonic solutions of second-order difference equations with relativistic operator. Our approach is variational and based on the use of the critical point theory for convex, lower semi-continuous perturbations of C^1-functionals due to A. Szulkin.

Key Words: Subharmonic solution, Relativistic operator, (PS)-condition, Critical point theory, Difference equation.

Contents

1. Introduction

In this paper, we study the existence of subharmonic solutions for the following problem :

$$\Delta[\phi(\Delta u(n-1))] - a(n)u(n) + g(n, u(n)) = 0, \ n \in \mathbb{Z}, \tag{1.1}$$

where $a : \mathbb{Z} \to \mathbb{R}$ is a positive and T-periodic function for some integer $T > 0$, $g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is T-periodic in the first variable and continuous in the second one and $\phi(x) = \dfrac{x}{\sqrt{1-|x|^2}}$ for all $x \in]-1, 1[$. Moreover, the forward difference operator Δ is defined as $\Delta u(n-1) = u(n) - u(n-1)$.

We said that $u : \mathbb{Z} \to \mathbb{R}$ is a subharmonic solution of (1.1) if it is a periodic solution whose period is multiple of T, i.e.

$$u(n + mT) = u(n), \ \forall n \in \mathbb{Z}.$$

Variational methods in the study of periodic and subharmonic solutions of Lagrangian and Hamiltonian systems of difference equations were first introduced in 2003 by Guo and Yu [7]. They proved the existence of subharmonic solutions for the equation

$$\Delta^2 u(n-1) + f(n, u(n)) = 0, \quad n \in \mathbb{Z},$$

through the use of Rabinowitz's saddle point theorem. Since then, a large literature has been devoted to this class of problems, we refer to the survey paper [1] for more references.

In recent years, problems involving the so-called relativistic operator : $u \longmapsto (\phi(u'))'$ and the corresponding discrete version $u(n) \mapsto \Delta[\phi(\Delta u(n-1))]$ received special attention by the researchers. Especially, in 2015, the authors of [8] established a multiplicity result of periodic solutions for the following pendulum-type differential system

$$-(\phi(u'))' = \nabla_u F(t, u) + h(t), \ u(0) - u(T) = 0 = u'(0) - u'(T), \tag{1.2}$$

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where $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is of class C^1 on $\mathbb{R}^N$, w_i-periodic $(w_i > 0)$ with respect to each u_i $(i = 1, ..., N)$ and $h \in L^1([0,T], \mathbb{R}^N)$.

Concerning the discrete case and inspired by the corresponding approach for differential problems studied in [8], J. Mawhin [11] investigated the existence of periodic solutions for the following difference equation

$$\Delta[\phi(\Delta u(n-1))] = \nabla_u F(n, u(n)) + h(n),\ u(n+T) = u(n),\ n \in \mathbb{Z}, \tag{1.3}$$

under the following conditions:

(H_F) F is continuous, $F(n, .)$ is differentiable on $\mathbb{R}^N$ for all $n \in \mathbb{Z}$, $\nabla_u F$ is continuous and $F(., u)$ is T-periodic for all $u \in \mathbb{R}^N$,

(H_p) there are positive real numbers $w_1, ..., w_N$ such that

$$F(n, u_1 + w_1, ..., u_N + w_N) = F(n, u_1, ..., u_N),$$

for all $n \in \mathbb{Z}$, $u = (u_1, ..., u_N) \in \mathbb{R}^N$,

(H_h) $h : \mathbb{Z} \to \mathbb{R}^N$, $h(n+T) = h(n)$ for all $n \in \mathbb{Z}$.

Notice that ϕ is only defined on $]-1, 1[$, so the classical critical point theory cannot be applied. To deal with this kind of problems, the main idea used in [8,11] consists to reduce the singular problems (1.2) and (1.3) to a modified non-singular problems to which classical variational methods can be applied. Precisely, they replaced ϕ by the following homeomorphism

$$\begin{aligned} \phi_0 :\ \mathbb{R}^N &\longrightarrow \mathbb{R}^N \\ x &\longmapsto \begin{cases} \dfrac{x}{\sqrt{1-|x|^2}}, & if\ |x| \leqslant R, \\ \dfrac{x}{\sqrt{1-R^2}}, & if\ |x| > R, \end{cases} \end{aligned}$$

where $R \in (0, 1)$ and it turns out that T-periodic solutions of the modified problems coincide with those of (1.2) and (1.3). Also, Coello et al. [5,6] used similar ideas to investigate positive solutions of some problems involving the curvature operator in Minkowski space.

Recently the authors of [9] chose to deal with similar problems using a variational approach which relies on a generalization of a result for smooth functionals to convex, lower semi-continuous perturbations of C^1-functionals presented by A.Szulkin in 1987 (for more details see [14]). They established a multiplicity result of periodic solutions for problems involving Fisher-Kolmogorov nonlinearities of the type

$$-(\phi(u')' = \lambda u(1 - |u|^q),\ u(0) - u(T) = 0 = u'(0) - u'(T),$$

and respectively

$$-\Delta[\phi(\Delta(n-1))] = \lambda u(n)(1 - |u(n)|^q),\ u(n+T) = u(n),\ n \in \mathbb{Z},$$

where $q > 0$ is fixed and $\lambda > 0$ is a real parameter.

Motivated by the above works, the aim of this paper is to investigate the existence of subharmonic solutions for (1.1) with various nonlinearities using the Szulkin's critical point theorems. To the best of our knowledge, no previous research has studied this problem.

Let the function $a : \mathbb{Z} \to \mathbb{R}$ satisfies:

(A) $a(n) > 0$ and $a(n+T) = a(n)$ for all $n \in \mathbb{Z}$.

Denote by a_0 and a_1 the minimum and the maximum of $\{a(n)\}$ respectively. Moreover, let the function $G : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ defined by

$$G(n, x) = \int_0^x g(n, s)ds, \quad \forall n \in \mathbb{Z},\ \forall x \in \mathbb{R}.$$

Our main results are the following:

Theorem 1.1 *Assume that* (A) *and the following assumptions hold:*

(G_1) *there exist* $1 \leqslant \beta < 2$ *and* $\alpha_0 > 0$ *such that*

$$|g(n,x)| \leqslant \alpha_0 |x|^{\beta-1}, \ \forall (n,x) \in \mathbb{Z} \times \mathbb{R},$$

(G_2) *there exist* $n_0 \in \mathbb{Z}$ *and constants* $\rho > 0$ *and* $1 \leq \xi < 2$ *with*

$$G(n_0, x) \geqslant \rho |x|^{\xi}, \ \forall |x| \leqslant 1.$$

Then, for all $m \in \mathbb{N}$*, the problem* (1.1) *has at least one nontrivial* mT*-periodic solution.*

In order to look for nonconstant solutions of problem (1.1), we have:

Corollary 1.1 *Assume that* $(A), (G_1)$ *and* (G_2) *hold. Moreover,*

(G_3) $\quad g(n,x) = a(n)x, \forall \ n \in \mathbb{Z}, \quad$ *if and only if* $\quad x = 0.$
Then, for all $m \in \mathbb{N}$*, problem* (1.1) *has at least one nonconstant* mT*-periodic solution.*

Note that any $T-$periodic solution is a fortiori $mT-$periodic one, thus an additional argument is required to show that any of these solutions is indeed distinct. In this direction we have :

Theorem 1.2 *Let* $m \in \mathbb{N}$ *and* $k \in \mathbb{N}$ *with* $1 \leqslant k \leqslant mT$*. Assume that* (A) *and* (G_1) *hold. If further*

(G_2') *there exist* $\overline{\rho} > 0$ *and* $1 \leq \xi < 2$ *with*

$$G(n,x) \geqslant \overline{\rho} |x|^{\xi}, \ \forall (n,x) \in \mathbb{Z} \times \mathbb{R}, \ |x| \leqslant 1,$$

(G_4) $g(n,-x) = -g(n,x), \forall x \in \mathbb{R}, \forall n \in \mathbb{Z}.$
Then problem (1.1) *has at least* k *distinct pairs of nontrivial* mT*-periodic solutions.*

Corollary 1.2 *Under the assumptions of Theorem 1.2* and (G_3)*, the problem* (1.1) *has at least* k *distinct pairs of nonconstant* mT*-periodic solutions.*

Example 1.1 *As an example, we define, for a natural number* $T \geq 1$ *given, the following functions:*

$$a(n) = 1, \quad g(n,x) = \big(2 + \cos\frac{\pi n}{T}\big)\frac{x}{\sqrt{|x|}}, \ \forall \ (n,x) \in \mathbb{Z} \times \mathbb{R}.$$

It is easy to see that all the assumptions $(A), (G_1) - (G_4)$ *and* (G_2') *are satisfied with* $\beta = \xi = \frac{3}{2}$*. Hence, the conclusions of Theorem 1.1, Theorem 1.2, Corollary 1.1* and *Corollary 1.2* hold true.

In what follows, we investigate another kind of nonlinearities including the super-quadratic and the asymptotically quadratic cases.

Theorem 1.3 *Assume that* (A) *and the following assumptions hold:*
(G_5) *there exist positive constants* $\eta \leqslant \frac{1}{2}$ *and* $\alpha_1 < a_0$ *such that*

$$|g(n,x)| \leqslant \alpha_1 |x|, \ \forall (n,x) \in \mathbb{Z} \times \mathbb{R}, \ |x| \leqslant \eta,$$

(G_6) *there exist constants* $\sigma > 0$ *and* $d > \frac{8+a_1}{2}$ *such that*

$$G(n,x) \geqslant d|x|^2, \ \forall (n,x) \in \mathbb{Z} \times \mathbb{R}, \ |x| \geqslant \sigma.$$

Then, for all $m \in \mathbb{N}$*, problem* (1.1) *has at least one nontrivial* mT*-periodic solution.*

Corollary 1.3 *Assume that* $(A), (G_3), (G_5)$ *and* (G_6) *hold. Then, for all* $m \in \mathbb{N}$*, problem* (1.1) *has at least one nonconstant* mT*-periodic solution.*

Corollary 1.4 *Under the assumptions* $(A), (G_3), (G_5)$ *and*

$$(G_6') \qquad \lim_{|x|\longrightarrow+\infty} \frac{G(n,x)}{|x|^2} = +\infty \text{ uniformly in } n \in \mathbb{Z},$$

the problem (1.1) *has at least one nonconstant* mT*-periodic solution for all* $m \in \mathbb{N}$.

Example 1.2 *To illustrate the results of the super-quadratic case, let, for* $T \geq 1$*, the following functions:*

$$a(n) = 1, \; g(n,x) = \big(2 + \cos\frac{\pi n}{T}\big)x^3, \; \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}.$$

It is easy to see that g *satisfies the assumptions* $(G_3), (G_5), (G_6)$ *and* (G_6'). *Hence, the conclusions of Theorem* 1.3*, Corollary* 1.3 *and Corollary* 1.4 *hold true.*

2. Preliminary results

Firstly, we recall some topics in the frame of Szulkin's critical point theory [14], which will be needed in the sequel.

Let $(Y, \|.\|)$ be a real Banach space and let I a function satisfying the following hypothesis
(H) $I : Y \to (-\infty, +\infty]$ be a functional of the type

$$I = F + \psi, \tag{2.1}$$

where $F \in C^1(Y, \mathbb{R})$ and $\psi : Y \to (-\infty, +\infty]$ is convex, lower semi-continuous and proper (i.e., $D(\psi) := \{u \in Y : \psi < +\infty\} \neq \emptyset$).

A point $u \in Y$ is said to be a critical point of I if $u \in D(\psi)$ and satisfies the inequality

$$\langle F'(u), v - u\rangle + \psi(v) - \psi(u) \geqslant 0 \;\; \forall v \in D(\psi). \tag{2.2}$$

A sequence $\{u_n\} \subset D(\psi)$ is called a PS-sequence if $I(u_n) \to c \in \mathbb{R}$ and

$$\langle F'(u_n), v - u_n\rangle + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n\|v - u_n\|, \;\; \forall\ v \in D(\psi),$$

where $\varepsilon_n \to 0$, as $n \longrightarrow \infty$.
The functional I is said to satisfy the PS condition if any PS-sequence has a convergent subsequence in Y.

Let $S = \big\{u = \{u(n)\} | u(n) \in \mathbb{R}, \, n \in \mathbb{Z}\big\}$ be a vector space with $au + bv = \{au(n) + bv(n)\}$ for $u, v \in S$ and $a, b \in \mathbb{R}$. In addition, for each fixed $m \in \mathbb{N}$ let H_m be the set of all mT-periodic sequence, i.e.

$$H_m = \{u \in S \,:\, u(mT + n) = u(n), \, \forall n \in \mathbb{Z}\}.$$

Clearly, H_m is isomorphic to $\mathbb{R}^{mT}$. Hence it can be equipped with the following norms for $\zeta \geq 1$,

$$\|u\|_{\zeta m} = \left(\sum_{n=1}^{mT} |u(n)|^{\zeta}\right)^{\frac{1}{\zeta}}, \; \|u\| = \left(\sum_{n=1}^{mT} a(n)|u(n)|^2\right)^{\frac{1}{2}}, \; \|u\|_{\infty m} = \max_{1\leqslant n\leqslant mT} |u(n)|, \; \forall u \in H_m.$$

Obviously, from assumption (A) we have

$$a_0^{\frac{1}{2}}\|u\|_{2m} \leqslant \|u\| \leqslant a_1^{\frac{1}{2}}\|u\|_{2m}, \; \forall u \in H_m. \tag{2.3}$$

So the norms $\|.\|$ and $\|.\|_{2m}$ are equivalent independent of m. Also for each $u \in H_m$, we set

$$\overline{u} := \frac{1}{mT}\sum_{n=1}^{mT} u(n), \;\; \tilde{u} := u - \overline{u}.$$

Now, let the function

$$\begin{array}{rcl}\Phi: & [-1,1] & \longrightarrow \mathbb{R} \\ & x & \longmapsto 1-\sqrt{1-x^2}.\end{array}$$

It is easy to see that $\Phi \in C^1(]-1,1[)$ with $\phi = \Phi' :]-1,1[\to \mathbb{R}$ is strictly increasing and $\phi(0) = 0$. Evidently, Φ is strictly convex and $\Phi(x) \geqslant 0$ for each $x \in [-1,1]$. Furthermore, an easy computation exhibits that

$$\frac{1}{2}|x|^2 \leqslant \Phi(x) \leqslant |x|^2, \ \ \forall x \in [-1,1]. \tag{2.4}$$

Define the closed convex subset

$$L := \{u \in H_m;\ \|\Delta u\|_{\infty m} \leqslant 1\}.$$

Also, let the even function $\Psi : H_m \to (-\infty, +\infty]$ be defined by

$$\Psi(u) = \begin{cases} \displaystyle\sum_{n=1}^{mT} \Phi(\Delta u(n)) & if\ u \in L, \\ \\ +\infty & otherwise. \end{cases}$$

From [9], we know that Ψ is proper, convex and lower semi-continuous on H_m.

Now, define $F : H_m \to \mathbb{R}$ by

$$\begin{aligned} F(u) &= \sum_{n=1}^{mT} [\frac{1}{2} a(n)|u(n)|^2 - G(n, u(n))] \\ &= \frac{1}{2}\|u\|^2 - \sum_{n=1}^{mT} G(n, u(n)). \end{aligned}$$

Notice that F is of class C^1 on H_m and its derivative is given by

$$\langle F'(u), v\rangle = \sum_{n=1}^{mT} a(n)u(n)v(n) - g(n, u(n))v(n), \ \ \forall u, v \in H_m.$$

The energy functional $J : H_m \to (-\infty, +\infty]$ associated to (1.1) will be defined by

$$J = \Psi + F,$$

and has the structure required by Szulkin's critical point theory. Moreover, any solution $u \in H_m$ of problem (1.1) is such that $|\Delta u(n)| < 1$ for all $n \in \mathbb{Z}$.

Proposition 2.1 *Assume that (A) holds. Then any critical point $u \in H_m$ of J is a solution of problem* (1.1).

Proof: From Lemmas 5 and 6 in [12] we know that for every $l \in H_m$, the problem

$$\Delta[\phi(\Delta u(n-1))] = \overline{u} + l(n),\ \ u(n) = u(n + mT),\ \ n \in \mathbb{Z} \tag{2.5}$$

has a unique solution u_l which is a solution of the following variational inequality

$$\sum_{i=1}^{mT} \Big[\Phi(\Delta v(i)) - \Phi(\Delta u(i)) + \overline{u}(\overline{v} - \overline{u}) + l(i)(v(i) - u(i))\Big] \geqslant 0,\ \ \forall v \in L. \tag{2.6}$$

Moreover, from Proposition 3.1 in [9] we know that u_l is also the unique solution for (2.6).

Now, let z be a critical point of J. By (2.2), for every $v \in L$, one obtains

$$\sum_{i=1}^{mT}\Big[\Phi(\Delta v(i)) - \Phi(\Delta z(i)) + (a(i)z(i) - g(i, z(i)))(v(i) - z(i))\Big] \geqslant 0.$$

So, it implies that for every $v \in L$

$$\sum_{i=1}^{mT}\Big[\Phi(\Delta v(i)) - \Phi(\Delta z(i)) + \overline{z}(v(i) - z(i))\Big] + \sum_{i=1}^{mT}\Big[a(i)z(i) - g(i, z(i)) - \overline{z}\Big](v(i) - z(i)) \geqslant 0.$$

Then, it yields that z is a solution of the variational inequality

$$\sum_{i=1}^{mT}\Big[\Phi(\Delta v(i)) - \Phi(\Delta z(i)) + \overline{z}(\overline{v} - \overline{z}) + l_z(i)(v(i) - z(i))\Big] \geqslant 0, \forall v \in L,$$

such that for $n \in \mathbb{Z}$, $l_z(n) = a(n)z(n) - g(n, z(n)) - \overline{z} \in H_m$. Consequently, this combined with the uniqueness of the solution of (2.6) together with (2.5) yield

$$\begin{aligned}\Delta[\phi(\Delta z(n-1))] &= \overline{z} + l_z(n)\\ &= a(n)z(n) - g(n, z(n)).\end{aligned}$$

Hence z solves the problem (1.1). □

The following theorems presented in [14] are the main tools in order to prove our fundamental results.

Theorem 2.1 *Let E be a real Banach space and $I : E \to (-\infty, +\infty]$ satisfying (H) and the Palais-Smale condition. Suppose that I is bounded from below, then*

$$c = \inf_{x \in E} I(x)$$

is a critical value of I.

Let Π be the collection of all symmetric subsets of $E\setminus\{0\}$ which are closed in E. We said that a nonempty set $A \in \Pi$ have genus k (denoted by $\gamma(A) = k$) if k is the smallest integer such that with there is an odd continuous mapping $\tau : A \to \mathbb{R}^k \setminus \{0\}$. If k does not exist, $\gamma(A) = +\infty$.

Lemma 2.1 *Let $A \in \Pi$. If $\tau : A \to S^{k-1}$ ($k-1$ dimension sphere in the Euclidean space $\mathbb{R}^k$) is an odd homeomorphism. Then $\gamma(A) = k$.*

Let $\Omega \subset 2^E$ be the sub-collection of Π consisting of all nonempty compact symmetric subsets of E, considered with the Hausdorff-Pompeiu distance and let

$$\Omega_i := cl\{A \in \Omega : 0 \notin A, \gamma(A) \geqslant i\},$$

where cl is the closure in Ω.

Theorem 2.2 *Let E be a real Banach space and $I : E \to (-\infty, +\infty]$ satisfying (H), the Palais-Smale condition, $I(0) = 0$ and F, ψ are even. Define*

$$c_i := \inf_{A \in \Omega_i} \sup_{u \in A} I(u).$$

If $-\infty < c_i < 0$ for $i = 1, ..., k$, then I has at least k distinct pairs of nontrivial critical points.

Theorem 2.3 *Let E be a real Banach space and $I : E \to (-\infty, +\infty]$ satisfying (H) and the Palais-Smale condition. Suppose that $I(0) = 0$ and*
(I_1) there are constants $\rho, \alpha > 0$ such that $I\,|_{\partial B_\rho} \geqslant \alpha$,
(I_2) there is an $e \in E \setminus B_\rho$ such that $I(e) \leqslant 0$.
Then I possesses a critical value $c \geqslant \alpha$ given by

$$c = \inf_{f \in \Gamma} \max_{s \in [0,1]} I(f(s)),$$

where

$$\Gamma = \{f \in C([0, 1], E);\ f(0) = 0,\ f(1) = e\}.$$

3. Sub-quadratic case

Proposition 3.1 *Assume that* (A) *and* (G_1) *hold. Then* J *satisfies the Palais-Smale condition.*

Proof: Let $\{u_p\}_{p\in\mathbb{N}} \subset L$ be a sequence with $\lim_{p\to+\infty} J(u_p) = l_0 \in \mathbb{R}$ and $\varepsilon_p \to 0$ as $p \to +\infty$ such that

$$\langle F'(u_p), v - u_p\rangle + \Psi(v) - \Psi(u_p) \geqslant -\varepsilon_p \|v - u_p\|, \ \ \forall v \in L. \tag{3.1}$$

As $\{J(u_p)\}_{p\in\mathbb{N}}$ is convergent, there is $M_0 > 0$ such that

$$|J(u_p)| \leqslant M_0, \ \ \forall\, p \in \mathbb{N}.$$

Moreover, using (G_1), we get

$$\begin{aligned} |G(n,x)| &= \Big|\int_0^x g(n,s)ds\Big|, \\ &\leqslant \frac{\alpha_0}{\beta}|x|^\beta, \ \forall x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Hence, one gets for every $u \in H_m$

$$\begin{aligned} \left|\sum_{n=1}^{mT} G(n,u(n))\right| &\leqslant \frac{mT\alpha_0}{\beta}\|u\|_{\infty m}^\beta \\ &\leqslant \frac{mT\alpha_0}{\beta\sqrt{a_0}^\beta}\|u\|^\beta. \end{aligned}$$

From the fact that $\Psi \geqslant 0$, we obtain

$$\begin{aligned} M_0 &\geqslant J(u_p), \\ &\geqslant \frac{1}{2}\|u_p\|^2 - \sum_{n=1}^{mT} G(n,u_p(n)), \\ &\geqslant \frac{1}{2}\|u_p\|^2 - \frac{mT\alpha_0}{\beta\sqrt{a_0}^\beta}\|u_p\|^\beta, \ \forall\, p \in \mathbb{N}. \end{aligned}$$

Since $\beta < 2$, we can easily see that $\{u_p\}_{p\in\mathbb{N}}$ is bounded in H_m and hence it admits a convergent subsequence. □

Proof of Theorem 1.1 Considering Proposition 3.1, the functional J satisfies (H) and the Palais-Smale condition. In what follows, we prove that J is bounded from below. By using (3.2) and (2.3) it is easy to check that

$$J(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{mT\alpha_0}{\beta\sqrt{a_0}^\beta}\|u\|^\beta, \ \forall\, u \in H_m. \tag{3.3}$$

Again, by $\beta < 2$, it is obvious that $J(u) \to +\infty$ as $\|u\| \to +\infty$ and hence J is bounded from below. Now, using Theorem 2.1, we have $c = \inf_{u\in H_m} J(u)$ is a critical value of J which yields a critical point $u^* \in H_m$ with $J(u^*) = c$.

Finally, it remains to show that $u^* \neq 0$. Let $u_0 \in H_m$ be such that

$$u_0(n) = \begin{cases} \vartheta, \ if\ n = n_0, \\ 0, \ otherwise. \end{cases}$$

By (G_2) and (2.4), we can find $0 < \vartheta < 1$ small enough such that

$$\begin{aligned} J(u_0) &\leqslant \Psi(u_0) + \frac{a(n_0)}{2}\vartheta^2 - \rho\vartheta^\xi \\ &\leqslant \Phi(\Delta u_0(n_0-1)) + \Phi(\Delta u_0(n_0)) + \frac{a_1}{2}\vartheta^2 - \rho\vartheta^\xi \\ &\leqslant |\Delta u_0(n_0-1)|^2 + |\Delta u_0(n_0)|^2 + \frac{a_1}{2}\vartheta^2 - \rho\vartheta^\xi \\ &\leqslant (2 + \frac{a_1}{2})\vartheta^2 - \rho\vartheta^\xi \\ &< 0. \end{aligned}$$

Hence

$$c = J(u^*) = \inf_{u \in H_m} J(u) \leqslant J(u_0) < 0.$$

From the definition of G and J, it is obvious that $J(0) = 0$, so, we conclude that u^* is a non-trivial critical point of J which yields, from Proposition 2.1, a non-trivial $mT-$periodic solution for problem (1.1).

Note that if the periodic solution u^* obtained above is constant then, from (1.1), we get

$$-a(n)u^* + g(n, u^*) = 0, \ \forall \ n \in \mathbb{Z}.$$

Hence, we have obviously Corollary 1.1 by (G_3).

Proof of Theorem 1.2 From (G_4) it is easy to see that J is even. Also the functional J satisfies (H) and the Palais-Smale condition. In what follows, we shall use Theorem 2.2 to prove that J has k distinct critical points for every $1 \leqslant k \leqslant mT$. For this purpose we have to prove that there is some $A_k \in \Omega_k \subset 2^{H_m}$ such that

$$\sup_{u \in A_k} J(u) < 0. \tag{3.4}$$

Let $e_1, e_2, ..., e_{mT}$ be an orthonormal basis in the space H_m equipped with the Euclidean norm $\|.\|_{2m}$. For $k \in \mathbb{N}$ such that $1 \leqslant k \leqslant mT$, set

$$A_k := \{u = \sum_{j=1}^{k} \alpha_j e_j : \alpha_1^2 + ... + \alpha_k^2 = \rho^2\},$$

where $0 < \rho \leqslant \dfrac{1}{2\sqrt{k}}$. As the mapping

$$\begin{aligned} D: \ A_k &\longrightarrow S^{k-1} \\ u &\longmapsto D(\sum_{j=1}^{k}\alpha_j e_j) = (\frac{\alpha_1}{\rho}, ..., \frac{\alpha_k}{\rho}) \end{aligned}$$

is an odd homeomorphism, we deduce, by Lemma 2.1, that $\gamma(A_k) = k$ and hence we get $A_k \in \Omega_k$. Choose $u = \sum_{j=1}^{k}\alpha_j e_j \in A_k$. For each $n = 1, ..., mT$, one has

$$\begin{aligned} |\Delta u(n)| &\leqslant \sum_{j=1}^{k}|\alpha_j e_j(n+1)| + \sum_{j=1}^{k}|\alpha_j e_j(n)| \\ &\leqslant 2\sum_{j=1}^{k}|\alpha_j| \\ &\leqslant 2\sqrt{k}(\sum_{j=1}^{k}|\alpha_j|^2)^{\frac{1}{2}} = 2\rho\sqrt{k}. \end{aligned} \tag{3.5}$$

As $\rho \leqslant \dfrac{1}{2\sqrt{k}}$, it is easy to check that $\|\Delta u\|_{\infty m} \leqslant 1$ and so $u \in L$. Also we have

$$\|u\|_{2m}^2 = \sum_{n=1}^{mT} u(n)^2 = \sum_{j=1}^{k} \alpha_j^2 = \rho^2, \tag{3.6}$$

which implies that $\|u\|_{\infty m} \leqslant \rho \leqslant 1$. This combined with (G_2') yield

$$-\sum_{n=1}^{mT} G(n, u(n)) \leqslant -\overline{\rho} \sum_{n=1}^{mT} |u(n)|^{\xi}. \tag{3.7}$$

Consequently, by (2.3), (2.4), (3.6) and (3.7) one gets

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 + \Psi(u) - \sum_{n=1}^{mT} G(n, u(n)) \\ &\leqslant \frac{a_1}{2}\|u\|_{2m}^2 + \sum_{n=1}^{mT} |\Delta u(n)|^2 - \overline{\rho}\|u\|_{\xi m}^{\xi}. \end{aligned}$$

Since the norms $\|.\|_{\xi m}$ and $\|.\|_{2m}$ are equivalent, there is $\delta > 0$ such that

$$\begin{aligned} J(u) &\leqslant \frac{a_1}{2}\|u\|_{2m}^2 + \|\Delta u\|_{2m}^2 - \delta\overline{\rho}\|u\|_{2m}^{\xi} \\ &\leqslant \frac{a_1}{2}\rho^2 + 4\rho^2 - \delta\overline{\rho}\rho^{\xi}. \end{aligned}$$

As $\xi < 2$, thus we can find $0 < \rho < 1$ small enough such that $J(u) < 0$. Hence

$$c_k := \inf_{A \in \Omega_k} \sup_{u \in A} J(u) < 0,$$

for every $1 \leqslant k \leqslant mT$ and the proof is complete.

4. Non-sub-quadratic nonlinearities

Proposition 4.1 *Assume that (A) and (G_6) hold. Then J satisfies the Palais-Smale condition.*

Proof: Let $\{u_p\}_{p\in\mathbb{N}} \subset L$ be a sequence with $\lim\limits_{p\to+\infty} J(u_p) = l_1 \in \mathbb{R}$ and $\varepsilon_p \to 0$ as $p \to +\infty$ such that (3.1) holds. Let

$$R_\sigma := \max_{1\leqslant n\leqslant mT} \{|G(n,x) - d|x|^2|;\ |x| \leqslant \sigma\}.$$

By assumption (G_6), it is easy to see that

$$G(n,x) \geqslant d|x|^2 - R_\sigma,\ \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}. \tag{4.1}$$

On the other hand, since $\{J(u_p)\}_{p\in\mathbb{N}}$ is convergent, there is $M_1 > 0$ such that

$$|J(u_p)| \leqslant M_1, \quad \forall\ p \in \mathbb{N}.$$

Hence, from (2.4), (2.3) and (4.1), we get

$$\begin{aligned} -M_1 &\leqslant J(u_p) \\ &\leqslant \Psi(u_p) + \frac{1}{2}\|u_p\|^2 - \sum_{n=1}^{mT} G(n, u_p(n)) \\ &\leqslant \|\Delta u_p\|_{2m}^2 + \frac{1}{2}\|u_p\|^2 - d\|u_p\|_{2m}^2 + mTR_\sigma \\ &\leqslant 4\|u_p\|_{2m}^2 + \frac{a_1}{2}\|u_p\|_{2m}^2 - d\|u_p\|_{2m}^2 + 2mTR_\sigma \\ &\leqslant (\frac{8+a_1}{2} - d)\|u_p\|_{2m}^2 + mTR_\sigma, \quad \forall\ p \in \mathbb{N}. \end{aligned}$$

Thus

$$\|u_p\|_{2m}^2 \leqslant \frac{2M_1 + 2mTR_\sigma}{2d - 8 - a_1}.$$

As a result, from (2.3) one obtains

$$\|u_p\|^2 \leqslant a_1 \left(\frac{2M_1 + 2mTR_\sigma}{2d - 8 - a_1} \right).$$

Hence the sequence $\{u_p\}_{p \in \mathbb{N}}$ is bounded in H_m and therefore it contains a convergent subsequence. So the functional J satisfies the Palais-Smale condition. □

Proof of Theorem 1.3 First of all, considering Proposition 4.1, the functional J satisfies (H) and the Palais-Smale condition. Next, we prove that J satisfies (I_1) of Theorem 2.3. From (G_5), we get

$$|G(n,x)| \leqslant \frac{\alpha_1}{2}|x|^2, \ \forall n \in \mathbb{Z}, \ |x| \leqslant \eta. \tag{4.2}$$

Let $u \in H_m$ be such that

$$\|u\| \leqslant \sqrt{a_0}\eta.$$

Thus from using (2.3) and $\|u\|_{\infty m} \leqslant \|u\|_{2m}$, one obtains

$$\|u\|_{\infty m} \leqslant \frac{1}{\sqrt{a_0}}\|u\|.$$

Then we get

$$|u(n)| \leqslant \eta, \ \forall n \in \mathbb{Z}. \tag{4.3}$$

Let $\rho = \eta\sqrt{a_0}$. Since $\eta \leqslant \frac{1}{2}$ and (4.3) holds true, we get $|\Delta u(n)| \leqslant 2\eta \leqslant 1$ and hence $u \in L$. Now, for every $u \in \partial B_\rho$, from (4.2) and (4.3), we have

$$\begin{aligned} J(u) &= \Psi(u) + \frac{1}{2}\|u\|^2 - \sum_{n=1}^{mT} G(n,u(n)) \\ &\geqslant \frac{1}{2}\|u\|^2 - \sum_{n=1}^{mT} G(n,u(n)) \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{\alpha_1}{2}\|u\|_{2m}^2 \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{\alpha_1}{2a_0}\|u\|^2 \\ &\geqslant \frac{a_0 - \alpha_1}{2a_0}\rho^2. \end{aligned}$$

Since $\alpha_1 < a_0$, then J satisfies (I_1) of Theorem 2.3.

Finally, we show that J satisfies (I_2) of Theorem 2.3. Let $\lambda \geqslant \sigma$ and $e \in H_m$ be such that

$$e(n) = \lambda, \ \forall n \in \mathbb{Z}.$$

From (G_6), one obtains

$$\begin{aligned} J(e) &= \Psi(e) + \frac{1}{2}\|e\|^2 - \sum_{n=1}^{mT} G(n,e(n)) \\ &\leqslant \frac{a_1}{2}\|e\|_{2m}^2 - d\|e\|_{2m}^2 \\ &\leqslant (\frac{a_1}{2} - d)mT\lambda^2 \leqslant 0. \end{aligned}$$

So all assumptions of Theorem 2.3 are satisfied. Thus J admits a non-trivial critical value

$$c_m := J(u_m) \geqslant \frac{(a_0 - \alpha_1)\rho^2}{2a_0} > 0$$

which yields, from Proposition 2.1, a non-trivial $mT-$periodic solution u_m of problem (1.1).

The proofs of Corollary 1.3 and Corollary 1.4 are trivial.

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