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Nontrivial Solutions for a General $p(x)$-Laplacian Robin Problem

Fouad Kissi and Abdelrachid El Amrouss

ABSTRACT: We establish the existence of multiple nontrivial solutions for a class of $p(x)$-Laplacian Robin problem. Our approach relies on the variable exponent theory of generalized Lebesgue-Sobolev spaces, combined with adequate variational methods and a variant of the Mountain Pass lemma.

Key Words: $p(x)$-Laplacian, generalized Lebesgue-Sobolev spaces, Robin problem, variational method, critical point.

Contents

1. Introduction

The purpose of the present paper is to study the following Robin problem involving the $p(x)$-Laplacian

$$(\mathcal{P})\begin{cases} -\Delta_{p(x)}u = f(x,u)+g(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v}+\beta(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial v}$ is the outer unit normal derivative on $\partial\Omega$, p is Lipschitz continuous on $\bar{\Omega}$, with

$$1 < p^- = \inf_{\bar{\Omega}} p(x) \leq p(x) \leq \sup_{\bar{\Omega}} p(x) = p^+ < N,$$

$\beta \in L^\infty(\partial\Omega)$ with $\beta^- = \inf_{x\in\partial\Omega} > 0$, and f, g are continuous functions on $\bar{\Omega}\times\mathbb{R}^N$.

The study of differential equations and variational problems with $p(x)$-growth conditions was an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics. For axample, see [5,14,16] and references therein.

The operator $-\Delta_{p(x)}u = \mathrm{div}(\mid \nabla \mathrm{u} \mid^{\mathrm{p(x)-2}} \nabla \mathrm{u})$ with $p(x) > 1$ is called the $p(x)$-Laplacian which is natural generalization of the p-Laplacian (where $p > 1$ is a constant). When $p(x) \neq \mathit{constant}$, the $p(x)$-Laplacian possesses more complicated nonlinearity than the p-Laplacian, say, it is nonhomogeneous.

The $p(x)$-Laplacian Dirichlet, Neumann, Steklov, and Robin problems on a bounded domain have been investigated and some interesting results have been obtained (see [1,2,4,6,7,8,9,11,13,15] and references therein).

In a recent paper [2], the authors considered the above problem and using variational methods, by the assumptions on the function f, they established the existence of at least three solutions of the problem.

Inspired by the above references and the work of Allaoui, El amrouss and Ourraoui [3], we prove that there exist two nontrivial solutions for $(\mathcal{P})$.

In order to obtain this result, we suppose the following conditions :
(F_1) $|f(x,t)| \leq b(x)|t|^{\gamma(x)-1}$, such that

$$p^+ < \gamma^- = \inf_{x\in\bar{\Omega}} \gamma(x) \leq \gamma^+ = \sup_{x\in\bar{\Omega}} \gamma(x) \ll p^*(x),\ \forall x \in \bar{\Omega}$$

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with
$b \in L^{\infty}(\bar{\Omega}) \cap L^{r(x)}(\bar{\Omega})$,
$p^*(x) = \frac{Np(x)}{N-p(x)}$,
and
$r(x) > \frac{Np(x)}{Np(x)+p(x)\gamma(x)-N\gamma(x)}$,
for a.e.x in $\bar{\Omega}$, $r \in C(\bar{\Omega})$, which implies that
$p(x) < \gamma(x)r(x) < p^*(x)$.
We denote by $d(x) \ll j(x)$ that $\inf_{x\in\Omega}(j(x)-d(x)) > 0$.
(F_2) There exists $\theta > p^+$ such that
$0 < \theta F(x,t) \leq tf(x,t)$, for a.e. $x \in \mathbb{R}^N$, $t \in \mathbb{R}^N$,
where $F(x,t) = \int_0^t f(x,s)\,ds$.
(F_3) $|g(x,t)| \leq a(x)|t|^{\delta(x)-1}$, such that
$1 < \delta^- = \inf_{x\in\bar{\Omega}} \delta(x) \leq \delta^+ = \sup_{x\in\bar{\Omega}} \delta(x) < p^-$, $a \in L^{\infty}(\bar{\Omega}) \cap L^{q(x)}(\bar{\Omega})$,
$\frac{1}{q(x)} + \frac{\delta(x)}{s(x)} = 1$, $q \in C(\bar{\Omega})$ and $p(x) \leq s(x) \leq p^*(x)$.
(F_4) $G(x,t) \geq b_0(x)t^{\delta_0}$ as $t \to 0^+$, $0 < \delta_0 < p^-$, $b_0(x) \geq 0$,
where $G(x,t) = \int_0^t g(x,s)\,ds$.
The main result reads as follows.

Theorem 1.1. *Under the assumptions* (F_1)-(F_4)*, the problem* $(\mathcal{P})$ *has at least two nontrivial solutions.*

This paper is organized as follows. In Section 2, we give the necessary notations and preliminaries, we also include some useful results involving the variable exponenet Lebesgue and Sobolev spaces in order to facilitate the reading of the paper. Finally, in Section 3, we will give the proof of our main result.

2. Preliminary

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [10,12]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p(x) \in C_+(\bar{\Omega})$ where

$$C_+(\bar{\Omega}) = \{p;\ p \in C(\bar{\Omega}),\ p(x) > 1 \text{ for all } \ x \in \bar{\Omega}\}.$$

Denote by

$$p^+ = \sup_{x\in\bar{\Omega}} p(x) \text{ and } \mathrm{p}^- = \inf_{\mathrm{x}\in\bar{\Omega}} \mathrm{p(x)}.$$

Define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mesurable and } \int_{\Omega} |\ u(x)\ |^{p(x)}\ dx < \infty\},$$

with the norm

$$|\ u\ |_{p(x)} = \inf\{\mu > 0;\ \int_{\Omega} |\ \frac{u(x)}{\mu}\ |^{p(x)}\ dx \leq 1\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\ \nabla u\ | \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|\ u\ \| = \inf\{\mu > 0;\ \int_{\Omega}(|\ \frac{\nabla u(x)}{\mu}\ |^{p(x)} + |\ \frac{u(x)}{\mu}\ |^{p(x)})\,dx \leq 1\},$$

$$\|\ u\ \| = |\ \nabla u\ |_{p(x)} + |\ u\ |_{p(x)}\ .$$

We refer the reader to [10,12] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1. *Both* $(L^{p(x)}(\Omega), |\,.\,|_{p(x)})$ *and* $(W^{1,p(x)}(\Omega), \|\,.\,\|)$ *are separable and uniformly convex Banach spaces.*

Lemma 2.2. *Hölder inequality holds, namely*

$$|\int_\Omega uv\,dx\,| \leq 2\mid u\mid_{p(x)}\mid v\mid_{p'(x)} \quad \forall u \in L^{p(x)}(\Omega), \forall v \in L^{p'(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Lemma 2.3. *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\bar{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *for* $x \in \bar{\Omega}$, *then there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Now, we introduce a norm, which will be used later.
Let $\beta \in L^\infty(\partial\Omega)$ with $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$ and for $u \in W^{1,p(x)}(\Omega)$, define

$$\| u \|_\beta = \inf\{\mu > 0; \int_\Omega \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} dx + \int_{\partial\Omega} \beta(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} d\sigma \leq 1\}.$$

Then, by Theorem 2.1 in [8], $\|\,.\,\|_\beta$ is also a norm on $W^{1,p(x)}(\Omega)$ which is equivalent to $\|\,.\,\|$.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the the mapping defined by the following.

Lemma 2.4. *Let* $I(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx + \int_{\partial\Omega} \beta(x)\,|u(x)|^{p(x)}\,d\sigma$ *with* $\beta^- > 0$. *For* $u \in W^{1,p(x)}(\Omega)$ *we have*

$$\| u \|_\beta \geq 1 \Rightarrow \| u \|_\beta^{p^-} \leq I(u) \leq \| u \|_\beta^{p^+},$$

$$\| u \|_\beta \leq 1 \Rightarrow \| u \|_\beta^{p^+} \leq I(u) \leq \| u \|_\beta^{p^-},$$

$$\| u \|_\beta < 1 (resp. = 1; > 1) \Leftrightarrow I(u) < 1 (resp. = 1; > 1).$$

Let $X = W^{1,p(x)}(\Omega)$. The Euler-Lagrange functional associated with $(\mathcal{P})$ is defined as $\Phi : X \to \mathbb{R}$,

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)} |u|^{p(x)}\,d\sigma - \int_\Omega F(x,u)\,dx - \int_\Omega G(x,u)\,dx.$$

We say that $u \in X$ is a weak solution of $(\mathcal{P})$ if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2} uv\,d\sigma = \int_\Omega f(x,u)v\,dx + \int_\Omega g(x,u)v\,dx,$$

for all $v \in X$.

Standard arguments imply that $\Phi \in C^1(X, \mathbb{R})$ and

$$\langle \Phi'(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v\,dx + \int_{\partial\Omega} \beta(x)|u|^{p(x)-2} uv\,d\sigma - \int_\Omega f(x,u)v\,dx - \int_\Omega g(x,u)v\,dx,$$

for all u, $v \in X$. Thus the weak solution of $(\mathcal{P})$ coincide with the critical points of Φ.

3. Proof of main result

For the proof of our theorem, we will use Mountain Pass Lemma. We need to establish some lemmas.

Lemma 3.1. *Under the conditions of Theorem 1.1, the functional Φ satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a (PS) sequence for the functional Φ: $\Phi(u_n) \leq C$ and $\Phi'(u_n) \to 0$. We have, for $\| u_n \|_\beta > 1$

$$\begin{aligned}
\Phi(u_n) &= \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u_n|^{p(x)}\,d\sigma - \int_\Omega F(x,u_n)\,dx - \int_\Omega G(x,u_n)\,dx \\
&\geq (\frac{1}{p^+} - \frac{1}{\theta}) \| u_n \|_\beta^{p^-} + \frac{1}{\theta}\Phi'(u_n)u_n + \int_\Omega (\frac{1}{\theta} f(x,u_n)u_n - F(x,u_n))\,dx \\
&+ \int_\Omega (\frac{1}{\theta} g(x,u_n)u_n - G(x,u_n))\,dx \\
&\geq (\frac{1}{p^+} - \frac{1}{\theta}) \| u_n \|_\beta^{p^-} + \frac{1}{\theta}\Phi'(u_n)u_n - C(1+\frac{1}{\theta}) \mid a \mid_\infty \| u_n \|_\beta^{\delta^+}, \qquad (3.1)
\end{aligned}$$

which is a contradiction. So $(u_n)_n$ is bounded and the Palais-Smale conditions are satified. □

Now it remains to check the geometric condition of Mountain Pass Theorem.

Lemma 3.2. *There exist $\rho > 0$ and $\alpha > 0$ such that $\Phi(u) \geq \alpha$, for all $u \in X$ with $\| u \|_\beta = \rho$.*

Proof. For $\| u \|_\beta$ small enough, we have

$$\begin{aligned}
\Phi(u)|_{\|u\|_\beta=\rho} &\geq (\frac{1}{p^+} \| u \|_\beta^{p^+} - \frac{\mid b \mid_\infty}{\gamma^-} \int_\Omega \mid u \mid^{\gamma(x)}\, dx - \frac{\mid a \mid_\infty}{\delta^-} \int_\Omega \mid u \mid^{\delta(x)}\, dx)|_{\|u\|_\beta=\rho} \\
&\geq \frac{1}{p^+} \| u \|_\beta^{p^+} - C_1 \| u \|_\beta^{\gamma^+} - C_2 \| u \|_\beta^{\delta^+} \\
&= \| u \|_\beta^{p^+} [\frac{1}{p^+} - C_1 \| u \|_\beta^{\gamma^+ - p^+} - C_2 \| u \|_\beta^{\delta^+ - p^+}] \\
&= \rho^{p^+} H(\rho). \qquad (3.2)
\end{aligned}$$

It easy to see that $H(\rho)$ has an absolute maximum $\rho_0 = (\frac{C_2}{C_1})^{\frac{1}{\gamma^+ - \delta^+}}$. Afterwards, there exists $\alpha > 0$ such that $\Phi(u)|_{\|u\|_\beta=\rho} \geq \alpha$. □

Proof of theorem 1.1. In order to apply the Mountain Pass Theorem, we must prove that

$$\Phi(tu) \to -\infty \text{ as t} \to +\infty,$$

for a certain $u \in X$. From the condition (F_2), there exists $M > 0$ such that

$$F(x,t) \geq M \mid t \mid^\theta, \text{ for all (x, t)} \in \bar{\Omega} \times \mathbb{R}.$$

Let $u \in X$ and $t > 1$ we have,

$$\begin{aligned}
\Phi(tu) &= \int_\Omega \frac{t^{p(x)}}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} t^{p(x)}\frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma - \int_\Omega F(x,tu)\,dx - \int_\Omega G(x,tu)\,dx \\
&\leq t^{p^+}[\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|u|^{p(x)}\,d\sigma] - M \mid t \mid^\theta \int_\Omega \mid u \mid^\theta\, dx \\
&- \mid t \mid^{\delta_0} \int_\Omega b_0(x) \mid u \mid^{\delta_0}\, dx.
\end{aligned}$$

Hence, we infer that $\Phi(tu) \to -\infty$ as $t \to +\infty$, provided $\theta > p^+ > \delta_0$. Then, by Mountain Pass Theorem and the lemmas 3.1 and 3.2, we obtain the existence of a nontrivial solution u_1 to problem $(\mathcal{P})$.

On the other hand, we can see that $\Phi(u)|_{\|u\|_\beta=\rho_0} > 0$ and $\Phi(u)$ is bounded below for $\| u \|_\beta < \rho_0$, then $\inf_{\bar{B}_{\rho_0}(0)} \Phi > -\infty$.
Furthermore, Φ is weakly lower semi continuous, thus for u_n is a minimizing sequence in $B_{\rho_0}(0)$, we may extract a weakly convergent subsequence what we call also $(u_n)_n$, then we have

$u_n \rightharpoonup u$ and $\Phi(u_n) \to \inf_{\bar{B}_{\rho_0}(0)} \Phi$,

we observe that

$\Phi(u) \leq \liminf \Phi(u_n) = \Phi(u)$,

hence, Φ attains a local minimum at $u_2 \in B_{\rho_0}(0)$. We note that u_2 must be nontrivial since we have for some $v \in C_0^\infty(\Omega)$,

$$\Phi(sv) \leq s^{p^-}[\int_\Omega \frac{1}{p(x)}|\nabla v|^{p(x)}\,dx + \int_{\partial\Omega} \frac{\beta(x)}{p(x)}|v|^{p(x)}\,d\sigma] + \frac{s^{\gamma^-}}{\gamma^-} \mid b \mid_\infty \int_\Omega \mid v \mid^{\gamma(x)}\ dx$$

$$-s^{\delta_0}\int_\Omega b_0(x) \mid v \mid^{\delta_0}\ dx < 0,$$

with $s > 0$ small enough. It follows that Φ attains its local minimum at $u_2 \in B(0, \rho_0)$ which yields $\Phi'(u_2) = 0$. We point out that $u_2 \notin \partial B_{\rho_0}$ since $\Phi(u_2) < 0$ and $\Phi(u)|_{\|u\|_\beta=\rho_0} < 0$. The proof of theorem 1.1 is now complete

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References

1. M. Allaoui and A. R. El Amrouss,*Solutions for Steklov boundary value problems involving $p(x)$-Laplace operators*, Bol. Soc. Paran. Math., 32(2014), 163-173.
2. M. Allaoui, A. R. El Amrouss, F. Kissi and A. Ourraoui,*Existence and multiplicity of solutions for a Robin problem*, J. Math. Comput. Sci., 10 (2014), 163-172.
3. M. Allaoui, A. R. El Amrouss and A. Ourraoui,*Existence results for a class of $p(x)$-Laplacian problems in $\mathbb{R}^N$*, Comput. Math. Appl., 69 (2015) 582-591.
4. J. Chabrowski and Y. Fu,*Existence of solutions for $p(x)$-Laplacian problems on a bounded domain*, J. Math. Anal. Appl. 306 (2005), 604-618.
5. Y.M. Chen, S. Levine, M. Rao,*Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (2006) 1383-1406.
6. S. G. Dend, Q. Wang and S. J. Cheng,*On the $p(x)$-Laplacian Robin eigenvalue problem*, Appl. Math. Comput., 217 (2011), 5643-5649.
7. S. G. Dend,*A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations*, Appl. Math. Comput., 211 (2009), 234-241.
8. S. G. Dend,*Positive solutions for Robin problem involving the $p(x)$-Laplacian*, J. Math. Anal. Appl. 360 (2009) 548-560.
9. X. Ding and X. Shi,*Existence and multiplicity of solutions for a general $p(x)$-Laplacian Neumann problem*, Nonlinear. Anal., 70(2009), 3713-3720.
10. X. L. Fan, J. S. Shen, D. Zhao,*Sobolev embedding theorems for spaces $W^{k,p(x)}$*, J. Math. Anal. Appl. 262 (2001), 749-760.
11. N. Tsouli, M. Haddaoui and ELM Hssini,*Multiple Solution for a Critical $p(x)$-Kirchhoff Type Equations*, Bol. Soc. Paran. Mat.(3s.) v. 38 4 (2020): 197-211.
12. X. L. Fan, D. Zhao,*On the spaces $L^{p(x)}$ and $W^{m,p(x)}$*, J. Math. Anal. Appl. 263 (2001), 424-446.
13. X. L. Fan, Q. H. Zhang,*Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. 52 (2003) 1843-1852.
14. M. Rüzicka,*Electrorheological fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.
15. L. L. Wang, Y. H. Fan and W. G. Ge,*Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear. Anal., 71 (2009), 4259-4270.

16. V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv. 29 (1987) 33-66.

Fouad Kissi,
Department of Mathematics and computer,
Faculty of Sciences, University Mohammed I
Oujda, Morocco.
E-mail address: kissifouad@hotmail.com

and

Abdelrachid El Amrouss,
Department of Mathematics and computer,
Faculty of Sciences, University Mohammed I
Oujda, Morocco.
E-mail address: a.elamrouss@fso.ump.ma