Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–8.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.62764

Multiplicity of Solutions for a Nonlinear Nonlocal Problem with Variable Exponent

Abdelhak Bousgheiri and Anass Ourraoui

ABSTRACT: This work deals with a class of value problems involving the $p(x)$-biharmonic and $p(x)-$Laplacian

$$M_1(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx)\Delta^2_{p(x)}u - M_2(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\Delta_{p(x)}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega,$$
$$u = \Delta u = 0, \quad x \in \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$. with smooth boundary $\partial\Omega$. Our technical method is based on a theorem obtained by B. Ricceri.

Key Words: Biharmonic problem, Kirchhoff problem, elliptic equation.

Contents

1. Introduction

The development of Partial Differential Equations is based upon the famous weak solution impact. It plays a tremendous role in the area of partial differential equations. In other words, it has a direct efficient contribution in the mathematics and its different subfields. It also have a deep relationship with the famous Sobolov spaces. Experts in the field confirm ultimately that weak solutions represents the most advanced methods of analysis by the 20th century for sure. The differentials methods are clearly experimented in our real life. They have been exposed in many applications on the daily life. Actually, the experiments are taking place from the start of the 20th century. On the other hand, the electrorheological fluids are the first factors where experiments occurs. This is due to their viscosity and the powerful electric fluids in it. Actually, the electrorheological fluids are used in different varieties of applications such as robotics and aeronautics industry, for more detail we can refer to [4,12]. In fact, the following work is all concerning the nonlocal $p(x)-$biharmonic problem :

$$M_1(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx)\Delta^2_{p(x)u} - M_2(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\Delta_{p(x)}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \tag{1.1}$$
$$u = \Delta u = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic with $p \in C(\overline{\Omega})$,$p(x) > 1$ for every $x \in \overline{\Omega}$, and $\lambda, \mu \in \mathbb{R}_+$.We define $F(x,t) = \int_0^t f(x,s)ds$, $G(x,t) = \int_0^t g(x,s)ds$ and we denote by $p^+ := \sup_{x\in\overline{\Omega}} p(x)$ and $p^- := \inf_{x\in\overline{\Omega}} p(x)$.

Problems like (1.1) are usually called nonlocal problems because of the presence of the integral over the entire domain, and this implies that the first equation in (1.1). For more details see for example [3] and [9].

Throughout this paper, we suppose the following assumptions :

2010 *Mathematics Subject Classification:* 35P48, 35K92, 35B44.
Submitted March 03, 2022. Published September 27, 2022

There exist two positives constants C and δ, and a function $q \in C(\overline{\Omega})$ with

$$q^+ := \sup_{x\in\overline{\Omega}} q(x),\ \ q^- := \inf_{x\in\overline{\Omega}} q(x) \ and\ 1 < q^- \leq q^+ < p^-$$

such that
(F_1) $F(x,t) > 0$, for a.e $x \in \Omega$ and $t \in]0,\delta]$.
(F_2) there exists $q_1 \in \overline{\Omega}$ and $p^+ < q_1^- \leq q_1(x) < p_2^*$, Such as :

$$\limsup_{t\longrightarrow 0} \frac{F(x,t)}{|t|^{q_1(x)}} < +\infty,$$

uniformly a.e $x \in \Omega$, with

$$\begin{cases} p_2^*(x) = \frac{Np(x)}{N-2p(x)} & if\ p(x) < \frac{N}{2} \\ p_2^*(x) = +\infty & if\ p(x) \geq \frac{N}{2}. \end{cases}$$

(F_3) $|F(x,t)| \leq C(1+|t|^{q(x)})$, for $x \in \Omega$ and for $t \in \mathbb{R}$.
(F_4) $F(x,0) = 0$, for a.e $x \in \Omega$.
(G) $\sup_{(x,t)\in\Omega\times\mathbb{R}} \dfrac{G(x,t)}{1+|t|^{q_2(x)}} < +\infty$, with $q_2(x) \in C_+(\overline{\Omega})$ and $q_2(x) < p^*,\ \forall x \in \overline{\Omega}$.

We also assume that $M_i : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ is a continuous function for $i = 1,2$. For simplicity, we take $M(t) = M_1(t) = M_2(t)$ and suppose that there exist positives constants, a_0 *and* a_1 such that : (M) $0 < a_0 \leq M_1(t) \leq a_1$.

The goal of this paper is to prove the following result.

Theorem 1.1. *Under the conditions (F_1) to (F_4) and (G), There is an open interval $\wedge \subseteq [0,+\infty[$, and a positive real e, such that for every $\lambda \in \wedge$, there exists $\sigma > 0$, such as $\forall\mu \in [0,\sigma]$. Then, problem (1.1) admits at least three weak solutions whose norms in $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$ are smaller than e.*

Example 1.2. *For $N = 1$, and $\Omega =]0,1[$ we take $M_1 = M_2 = 1$ then the problem*

$$\begin{gathered} (|u''|^{p(x)-2}u'')'' - (|u'|^{p(x)-2}u')' = \lambda f(x,u) + \mu g(x,u) \quad in\ ]0,1[, \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{gathered}$$

has at least three weak solutions whose norms in $W^{2,p(x)}(]0,1[) \cap W_0^{1,p(x)}(]0,1[)$.

Many authors consider the existence of nontrivial solutions for some fourth order problems such as [1,2,5,6,7,8,9,12,14,15,16,17,19], which represent a generalization of the classical $p-$biharmonic operator obtained in the case when p is a positive constant.

Wang et al. [18] are first that considered the following fourth-order equation of Kirchhoff type,

$$\begin{gathered} \Delta^2 u - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u),\ in\ \Omega, \\ u = \Delta u = 0\ on\ \partial\Omega. \end{gathered} \tag{1.2}$$

Using the mountain pass theorem, the authors obtained at least one nontrivial solution for the previous problem.

Here we mention that the $p(x)$-biharmonic operator possesses more complicated non linearities than p-biharmonic, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero. This study is Inspired by the results of [1], [18], [10] and [11], we are to show the existence of three solutions of problem (1.1), which we use the three-critical-points theorems of Ricceri [15,16].

This manuscript is divided into three sections organized as follows: in section 2 we start with some preliminary basic results on the theory of Lesbegue-Sobolev spaces with variables exponent, then we recall the three-critical-points theorem of Ricceri with some required results. In section 3, we give the proof of the main result.

2. Preliminaries

In order to deal with the problem, We need some theory of variable exponent Sobolev Space. For convenience, We only recall some basic facts which will be used later, we refer to the book of Musielak [13] and [6,8]. Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Let $C_+(\overline{\Omega}) = \{\ p \in C(\overline{\Omega})\ and\ ess \inf_{x\in\overline{\Omega}} p(x) > 1\}$ for any $p(x) \in C_+(\overline{\Omega})$. Set $p^- = \min_{x\in\overline{\Omega}} p(x),\ p^+ = \max_{x\in\overline{\Omega}} p(x)$ and

$$p_k^*(x) = \frac{Np(x)}{N - kp(x)}\ if\ kp(x) < N\ and\ p_k^*(x) = +\infty\ if\ kp(x) \geq N.$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$,

$$L^{p(x)}(\Omega) = \{\ u : \Omega \longrightarrow \mathbb{R}\ mesurable\ :\ \int_\Omega |u|^{p(x)} dx < \infty\}\ .$$

Then $L^{p(x)}(\Omega)$ endowed with the norm :

$$||u||_{p(x)} = \inf\{\ \lambda > 0\ :\ \int_\Omega |\frac{u}{\lambda}|^{p(x)} dx \leq 1\}\ ,$$

becomes a Banach separable and reflexive space.

Proposition 2.1. *Set,* $\rho(u) = \int_\Omega |\nabla u|^{p(x)} + |\Delta u|^{p(x)} dx$ *for all* $u \in L^{p(x)}(\Omega)$*,*

- $||u||_{p(x)} \leq 1 \Longrightarrow ||u||_{p(x)}^{p^+} \leq \rho(u) \leq ||u||_{p(x)}^{p^-}$
- $||u||_{p(x)} \geq 1 \Longrightarrow ||u||_{p(x)}^{p^-} \leq \rho(u) \leq ||u||_{p(x)}^{p^+}$
- $||u_n||_{p(x)} \longrightarrow 0 \Longleftrightarrow \rho(u_n) \longrightarrow 0$
- $||u_n||_{p(x)} \longrightarrow +\infty \Longleftrightarrow \rho(u_n) \longrightarrow \infty.$

Define the variable exponent Sobolev space $W^{k,p}(\Omega)$:

$$W^{k,p(x)}(\Omega) = \{\ u \in L^{p(x)}(\Omega) :\ D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\}\ ,$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 ....\partial^{\alpha_N} x_N}$, with $\alpha = (\alpha_1, ...; \alpha_N)$ is a multi-index, and $|\alpha| = \sum_{i=1}^N \alpha_i$.

Then $W^{k,p(x)}(\Omega)$ endowed with the norm :

$$||u|| = \sum_{|\alpha|\leq k} ||D^\alpha u||_{p(x)},$$

becomes a Banach separable and reflexive space.
Define the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$, which is the closure of C^∞ functions compactly supported in Ω for the norm :
$||u||_{1,p(x)} = \int_\Omega |u(x)|^{p(x)} dx + \int_\Omega |\nabla u(x)|^{p(x)} dx$.

Proposition 2.2. *([8])*
For $p, r \in C_+(\overline{\Omega})$ *such that* $r(x) \leq p_k^*(x)$ *for all* $x \in \overline{\Omega}$*, there is a continuous and compact embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

We denote by $W_0^{k,p(x)}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p(x)}(\Omega)$.

Set $X = W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega)$, for the norm :

$$||u|| = |\Delta u|_{p(x)} + |\nabla u|_{p(x)} + |u|_{p(x)},\ \forall u \in X$$

Remarks 2.3. *([6])*
$(X, ||.||)$ is a separable and reflexive Banach space. By the above remark and Proposition 2.2 there is a continuous and compact embedding of X into $L^{r(x)}(\Omega)$ where $r(x) < p_2^*$ for all $x \in \overline{\Omega}$.

Proposition 2.4. *Set, $\rho(u) = \int_\Omega |\Delta u|^{p(x)} + |\nabla u|^{p(x)} dx,\ \forall u \in W^{2,p(x)}(\Omega)$,*

- $||u|| \leq 1 \Longrightarrow ||u||^{p^+} \leq \rho(u) \leq ||u||^{p^-}$
- $||u|| \geq 1 \Longrightarrow ||u||^{p^-} \leq \rho(u) \leq ||u||^{p^+}$
- $||u_n|| \longrightarrow 0 \Longleftrightarrow \rho(u_n) \longrightarrow 0$
- $||u_n|| \longrightarrow +\infty \Longleftrightarrow \rho(u_n) \longrightarrow \infty$

Proposition 2.5. *([8])*
For any $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{p^+})||u||_{p(x)}||v||_{q(x)},$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Definition 2.6. *Let $u \in X$, u is said to be a weak solution of the problem (P), if :*

$$M\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\, dx\Big)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta vdx + M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla vdx$$

$$-\lambda\int_\Omega f.vdx - \mu\int_\Omega g.vdx = 0,$$

for all $x \in X$.

Let define the following operators:

$$I(u) = \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) + \tilde{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big),$$

and

$$J(u) = -\int_\Omega F(x,u)dx \text{ } et \text{ } \Psi(u) = -\int_\Omega G(x,u)dx,$$

with $\tilde{M}_1(s) = \int_0^s M_1(t)dt$ and $\tilde{M}_2(s) = \int_0^s M_2(t)dt$. Set for all u, v in X,

$$\langle Lu, v\rangle = M_1(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx)\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta vdx + M_2(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla vdx.$$

Proposition 2.7. *(Theorem 1, [15]) Let X be a reflexive real Banach space; $K \subset \mathbb{R}$ an interval; $I: X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous C^1 functional whose derivative admits a continuous inverse on X^*; $J: X \to \mathbb{R}$ be a C^1 functional with compact derivative. In addition, I is bounded on each bounded subset of X. Assume that*

$$\lim_{\|x\|\to\infty} I(x) + \lambda J(x) = +\infty. \tag{2.1}$$

for $\lambda \in K$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in K} \inf_{x \in X} (I(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in K} (I(x) + \lambda(J(x) + \rho)). \tag{2.2}$$

Then, there exist a nonempty set $A \subseteq K$ and a positive number e with the following property: for every $\lambda \in A$ and every C^1 functional $\psi : X \to \mathbb{R}$ with compact derivative, there exists $\sigma > 0$ such that, for each $\mu \in [0, \sigma]$, the equation

$$I'(u) + \lambda J'(u) + \mu \psi'(u) = 0,$$

has at least three solutions in X whose norms are less than e.

Proposition 2.8. *[16] Let X be a nonempty set and I, J are two real functionals on X. Suppose there are $\gamma > 0,\ u_0, u_1 \in X$ such that*

$$I(u_0) = J(u_0) = 0,\ I(u_1) > \gamma, \sup_{u \in I^{-1}(]-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{I(u_1)}. \tag{2.3}$$

Then, for each ρ satisfying

$$I(u_0) = J(u_0) = 0,\ I(u_1) > \gamma, \sup_{u \in I^{-1}(]-\infty, \gamma])} J(u) < \rho < \gamma \frac{J(u_1)}{I(u_1)},$$

we have

$$\sup_{\lambda \geq 0} \inf_{u \in X} (I(u) + \lambda(\rho - J(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (I(u) + \lambda(\rho - J(u))).$$

3. Proof of the main result

Let verify the conditions of proposition 2.7. But firt, we start by the following lemma.

Lemma 3.1. *Under the condition (M), $L : X \longrightarrow X^*$ is continuous and admits a continuous inverse on X^*.*

Proof. Since L is the Fréchet derivative of I, it follows that L is continuous and bounded. Using the elementary inequalities :

$$|x - y|^\alpha \leq 2^\alpha (|x|^\alpha x - |y|^\alpha y)(x - y)\ if\ \alpha \geq 2.$$

$$|x - y|^\alpha \leq \frac{1}{(\alpha - 1)} (|x| + |y|)^{2-\alpha} (|x|^\alpha x - |y|^\alpha y)(x - y)\ if\ 1 < \alpha < 2,$$

for all $(x, y) \in (\mathbb{R}^N)^2$, where $x.y$ denotes the usual inner product in $\mathbb{R}^N$, we obtain for all $u, v \in X$ such that $u \neq v$,

So we obtain $\langle L(u) - L(v), u - v \rangle > 0$, for all $u, v \in X$ with $u \neq v$, which means that L is strictly monotone. Furthermore, for $\|u\| > 1$ we have that

$$L(u).u \geq m_0 \|u\|^{p^-},$$

then L is coercive.
Note that the strict monotonicity of L implies that L is injectivie.

Consequently, thanks to a Minty-Browder theorem [19], the operator L is a surjection and admits an inverse mapping. A standard argument guarantees that L^{-1} is continuous. □

Now we are ready to prove Theorem 1.1. In view of Lemma 3.1, I is continuously Gâteaux differentiable, whose Gâteaux derivative admits a continuous inverse on X^*. On the other side, since the functions $\hat{M}(t)$ is increasing and the convex functionals $I_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}$ and $I_2(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)}$ are both sequentially weakly lower semi-continuous, we can see that the functional $I : X \to \mathbb{R}$ is sequentially weakly lower semi-continuous. In addition ψ and J are continuously Gâteaux differentiable functions and its Gâteaux derivatives are compact. By a similar analysis to that in Fan and Zhang [7], from (F_3) and (G) we know that $J, \psi \in C^1(X, \mathbb{R})$, such as :

$$\langle J^{'}(u), v\rangle = -\int_\Omega f(x, u(x))v dx;\ \langle \psi^{'}(u), v\rangle = -\int_\Omega g(x, u(x))v dx$$

for $u, v \in X$. As $X \hookrightarrow L^{q(x)}(\Omega)$ is compact, $J^{'}$ and $\psi^{'} : X \longrightarrow X^*$ are also compact.

For $||u|| < 1$, we have :

$$\frac{a_0}{p^+}[\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx] \leq I(u) \leq \frac{a_1}{p^-}[\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx],$$

then

$$\frac{a_0}{p^+}||u||^{p^+} \leq I(u) \leq \frac{a_1}{p^+}||u||^{p^-}.$$

Let choose $c_0 > 0$ in order to get $c_0 \geq \frac{a_1}{p^+}||u||^{p^-} - \frac{a_0}{p^+}||u||^{p^+}$, that is,

$$I(u) \geq \frac{a_1}{p^+}||u||^{p^-} - c_0.$$

When $||u|| \geq 1$, one has $I(u) \geq \frac{a_0}{p^+}||u||^{p^-}$ and then $\forall u \in X$, we have

$$\begin{aligned}
\lambda J(u) &= -\lambda \int_\Omega F(x, u)dx \\
&\geq -\lambda \int_\Omega C(1 + |u|^{\alpha(x)})dx \\
&\geq -A_1(1 + ||u||^{q^+}_{q(x)}) \\
&\geq -A_2(1 + ||u||^{q^+}),
\end{aligned}$$

with $A_1 > 0,\ A_2 > 0$, hence,

$$I(u) + \lambda J(u) \geq \frac{a_1}{p^+}||u||^{p^-} - A_2(1 + ||u||^{q^+}) - c_0.$$

So $\lim_{||u|| \longrightarrow +\infty}(I(u) + \lambda J(u)) = +\infty$, the assumption (2.1) is satisfied. To check the assertion (2.2), it suffices to verify the conditions of proposition (2.7). Put $u_0 = 0$ then $I(u_0) = -J(u_0) = 0$, and take $x_0 \in \Omega$ since $(\Omega \neq \emptyset)$, and $r_2 > r_1 > 0$.

Let $w(x) \in C_0^\infty(\overline{\Omega})$, with $w(x) = 0$ for $x \in \overline{\Omega} - B(x^0, r_2)$, $w(x) = \frac{\delta}{r_2 - r_1}(r_2 - \|x_i - x_i^0\|_2)$, when $x \in B(x^0, r_2) - B(x^0, r_1)$, and $w(x) = \delta$ if $x \in B(x^0, r_1)$ with $\|x\|_2 = (\sum_{i=1}^N (x_i)^2)^{\frac{1}{2}}$; then

$$\begin{aligned}
-J(u_1) &= \int_\Omega F(x, w)dx \\
&= \int_{B(x^0, r_1)} F(x, w)dx + \int_{B(x^0, r_2) - B(x^0, r_1)} F(x, w)dx + \int_{\overline{\Omega}/B(x^0, r_2)} F(x, w)dx \\
&= \int_{B(x^0, r_1)} F(x, w)dx + \int_{B(x^0, r_2) - B(x^0, r_1)} F(x, w)dx \\
&= \int_{B(x^0, r_1)} F(x, \delta)dx + \int_{B(x^0, r_2) - B(x^0, r_1)} F(x, \frac{\delta}{r_2 - r_1}[r_2 - ||x_i - x_i^0||_2])dx
\end{aligned} \tag{3.1}$$

Since $x \notin B(x^0, r_2) - B(x^0, r_1)$, so $\frac{r_2 - ||x_i - x_i^0||_2}{r_2 - r_1} > 1$, and $\int_{B(x^0, r_2) - B(x^0, r_1)} F(x, w)dx > 0$.
By (F_2) there exists $\theta \in [0, 1], c_1 > 0$ such that:

$$F(x, t) \leq c_1|t|^{q(x)},\ \forall\ |t| < \theta,$$

for a.e $x \in \Omega$.

Put $P_1 = \sup_{|t|<\theta} \frac{c(1+|t|^{q^+})}{|t|^{q_1^-}}$, $P_2 = \sup_{|t|>\theta} \frac{c[1+|t|^{q^+}]}{|t|^{q^-}}$, $P_3 = \sup_{|t|<1} \frac{c[1+|t|^{q^+}]}{|t|^{q_1^-}}$, $P_3 = \sup_{|t|>1} \frac{c[1+|t|^{q^+}]}{|t|^{q_1^-}}$ and $M^* = \max\{\ c_1, P_i, i = 1...4\}$.
It follows that : $F(x,t) < M^*|t|^{q_1^-}$ for $t \in \mathbb{R}$, $a.e\ x \in \Omega$. Afterward, fix γ such that $0 < \gamma < 1$. When $\frac{1}{p^+}||u||^{p^+} \leq \gamma < 1$, by Sobolev embeddings theorem, there are positive constants c_2, c_3 such that:

$$-J(u) = \int_\Omega F(x,w)dx < M^* \int_{\Omega)} |u|^{q_1^-} dx \leq c_2||u||^{q_1^-} \leq c_3\gamma^{\frac{q_1^-}{p^+}}.$$

In view of $q_1^- > p^+$, it yields

$$\lim_{\gamma \longrightarrow 0^+} \frac{\sup_{\frac{1}{p^+}\|u\|^{p^+} \leq \gamma} -J(u)}{\gamma} = 0. \tag{3.2}$$

Let $w \in X$, with $-J(w) > 0$. Take γ_0 such that $\gamma < \gamma_0 < \frac{a_0}{p^+}\min\{\ ||w||^{p^+}, \|w\|^{p^-}, 1\} \leq a_0$. Two cases appear. If $||w|| < 1$, we have

$$\begin{aligned} I(u_1) &= I(w) = \tilde{M}(\int_\Omega \frac{1}{p(x)}|\Delta w|^{p(x)}dx) + \tilde{M}(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx) \\ &\geq \frac{a_0}{p^+}||w||^{p^+} \geq \gamma_0 > \gamma. \end{aligned}$$

From the last inequality we obtain

$$\sup_{\frac{1}{p^+}||u||^{p^+} \leq \gamma} -J(u) \leq \frac{\gamma}{2} \frac{-J(u)}{\frac{a_0}{p^+}||w||^{p^-}} \leq \frac{\gamma}{2} \frac{-J(u_1)}{I(u_1)} \leq \gamma \frac{-J(u_1)}{I(u_1)}. \tag{3.3}$$

If $||w|| \geq 1$, it follows that

$$I(u_1) = I(w) \geq \frac{a_0}{p^+}||w||^{p^-} \geq \gamma_0 > \gamma.$$

From (3.2) and since $\gamma \geq 0$, we get

$$\sup_{\frac{1}{p^+}||u||^{p^+} \leq \gamma} -J(u) \leq \frac{\gamma}{2} \frac{-J(u)}{\frac{a_0}{p^+}||w||^{p^-}} \leq \frac{\gamma}{2} \frac{-J(u_1)}{I(u_1)} < \gamma \frac{-J(u_1)}{I(u_1)}.$$

For all $u \in I^{-1}(]-\infty, \gamma])$, we have

$$\tilde{M}(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}) + \tilde{M}(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}) \leq \gamma$$

then $a_0\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) \leq \gamma$.
Therefore ,

$$\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega |\nabla u|^{p(x)}dx \leq \frac{\gamma p^+}{a_0} < \frac{\gamma_0 p^+}{a_0} < 1.$$

It means that $||u|| < 1$ and

$$\frac{a_0}{p^+}||u||^{p^+} \leq \hat{M}\Big(\int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\Big) + \hat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) \leq \gamma.$$

It follows that :

$$I^{-1}(]-\infty,\gamma]) \subset \{u \in X : \frac{1}{p^+}\|u\|^{p^+} < \gamma\} .$$

So by virtue of (3.3),

$$\sup_{u \in (]-\infty,\gamma])} -J(u) < \gamma \frac{-J(u_1)}{I(u_1)},$$

we can find ρ such that

$$\sup_{u\in(]-\infty,\gamma])} -J(u) < \rho < \gamma\frac{-J(u_1)}{I(u_1)}.$$

Taking $K = [0, +\infty[$, the assumptions of proposition 2.7 are satisfied. So the proof is complete.

References

1. M. Allaoui, A. Elamrouss, A. Ourraoui, Three Solutions For A Quasi-Linear Elliptic Problem, Applied Mathematics E-Notes, 13(2013), 51-59.
2. A. Ayoujil, A. El Amrouss, Multiple solutions for a Robin problem involving the $p(x)-$biharmonic operator, Annals of the University of Craiova, Mathematics, Volume 44(1), 2017, 87-93.
3. N. T. Chung, H. Q. Toan, Multiple solutions for a $p(x)-$Kirchhoff type problem with Neumann boundary condition, An.Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.) Tomul LXII, 2016, f.2.
4. J. I. Diaz, Nonlinear Partial Differential Equations and Free Boundaries Vol 1. Elliptic Equations, In: Pitman Res. Notes Ser 106, Boston, MA, Pitman, 1985.
5. A.R. El Amrouss, F. Moradi and M. Moussaoui, Existence of solutionss for fourth order PDES with variable exponent, Electronic journal of differential equations, 13(153)(2009).
6. A. R. El Amrouss and A. Ourraoui, Existence of solutions for a boundary problem involving $p(x)-$biharmonic operator, Bol. Soc. Paran. Mat.(3s.), 1(31)(2013), 179-192.
7. X. L. Fan and Q. H. Zhang, Existence of solutions for p(x)-Laplacian Dirichlet problems, Nonlinear Anal., 52(2003), 1843-1852.
8. X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m;p(x)}(\Omega)$ J. Math. Anal. Appl., 263(2001), 424-446.
9. S. Heidarkhani, L. A. De Araujo, G.A. Afrouzi, S. Moradi, Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions, Math Nachrichten. 2017,00:1-17.
10. C. Li and C. L. Tang, Three solutions for a Navier boundary value problem in volving the p-biharmonic, Nonlinear Anal. 72(2010), 1339-1347.
11. J. Liu and X. Shi, Existence of three solutions for a class of quasi-linear elliptic systems involving the (p(x); q(x))-Laplacian, Nonlinear Anal., 71(2009), 550-557.
12. L. K. Martinson and K. B. Pavlov, Unsteady shear flows of a conducting áuid with a rheological power flow, Magnit Gidrodinamika, (1970), 5869-5875.
13. J. Musielak, Orlicz Spaces and Modular Spaces, in Lecture Notes in Mathematics, Springer, Berlin, Vol. 1043, 1983.
14. A. Ourraoui, On a class of a boundary value problems involving the $p(x)-$Biharmonic operator, Proyecciones, 2019, 38(5), pp.955-967.
15. B. Ricceri, A three critical points theorem revisited, Nonlinear Anal., 70(2009), 3084-3089.
16. B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, Math. Comp. Model., 32(2000), 1485-1494.
17. N. Tsouli, O. Chakrone, M. Rahmani and O. Darhouche, Strict monotonicity and unique continuation of the biharmonic operator, Bol. Soc. Paran. Mat, (3s.) v. 30 2 (2012): 79-83.
18. F. Wang, Y. An, Existence and multiplicity of solutions for a fourth-order elliptic equation, Bound. Value. Probl., 2012, 6 (2012).
19. E. Zeidler; Nonlinear Function Analysis and its Applications, vol. II/B: NonlinearMonotone Operators, Springer, New york, 1990.

Abdelhak Bousgheiri, Anass Ourraoui,
University Mohamed I, FSO, Department of Mathematics, Oujda, Morocco.
E-mail address: abdelhak1993bousgheiri@gmail.com, a.ourraoui@gmail.com