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Geometric properties of an integral operator associated with Mittag-Leffler functions

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abstract: The main object of this paper is to introduce a new integral operator associated with Mittag-Leffler function. Further, we obtain some sufficient condition for this integral operator belonging to certain classes of starlike functions.

Key Words: Analytic function, univalent function, Mittag-Leffler function, convex function, integral operator.

Contents

1. Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f of the form (1.1) which are also univalent in Δ.

A function $f(z) \in \mathcal{A}$ is said to be starlike of order δ if it satisfies the following analytic criteria

$$\Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \delta, \quad z \in \Delta, \quad \text{for some } \delta(0 \le \delta < 1).$$

Also, a function $f(z) \in \mathcal{A}$ is said to be convex of order δ if it satisfies the following analytic criteria

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \delta, \quad z \in \Delta, \text{where} \quad 0 \le \delta < 1.$$

The classes of all starlike functions and convex functions of order δ are denoted by $\mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$ respectively, studied by Robertson [13] and Silverman [14].

The Mittag-Leffler function $E_\alpha(z)$ was introduced by Mittag-Leffler [8] and defined as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \ \Re(\alpha) > 0).$$

In 1905, Wiman ([17], [18]) generalized the Mittag-Leffler function in $E_{\alpha,\, \beta}(z)$ by the relation

$$E_{\alpha,\, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

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where $z, \alpha, \beta \in \mathbb{C}$, $\Re(\alpha) > 0$. It should be easy to see that the function $E_{\alpha,\,\beta}(z)$ defined by (1.2) is not in class $\mathcal{A}$.
Thus, first we normalize the Mittag-Leffler function as follows

$$\mathbb{E}_{\alpha,\,\beta}(z) = \Gamma(\beta) z\, E_{\alpha,\,\beta}(z)$$
$$\mathbb{E}_{\alpha,\,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\, z^n, \tag{1.3}$$

where $z, \alpha, \beta \in \mathbb{C}$, $\beta \neq 0, -1, -2, \cdots, \Re(\alpha) > 0$.
In the present paper we shall restrict our attention to the case for real-valued α, β and $z \in \Delta$. For specific values of α and β, the function $\mathbb{E}_{\alpha,\,\beta}(z)$ reduces to many well-known functions

$$\mathbb{E}_{2,\,1}(z) = z \cosh \sqrt{z}$$
$$\mathbb{E}_{2,\,2}(z) = \sqrt{z} \sinh \sqrt{z}$$
$$\mathbb{E}_{2,\,3}(z) = 2[\cosh \sqrt{z} - 1] \quad \text{and}$$
$$\mathbb{E}_{2,\,4}(z) = \frac{6[\sinh \sqrt{z} - \sqrt{z}]}{\sqrt{z}}.$$

In 2016, Bansal and Prajapati [3] studied geometric properties such as starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $\mathbb{E}_{\alpha,\,\beta}(z)$. The integral operator associated with Bessel function of the first kind have been introduced by several researchers and studied various interesting analytic and geometric properties. Note worthy contribution in this direction may be found in ([1], [2], [4]- [7], [9]- [12]).

Motivated with the above mentioned work, it is natural to think for introducing these integral operators involving other special functions. Recently, Srivastava *et al.* [16] investigated a new integral operator associated with Mittag-Leffler functions.

In the present work we introduce a new integral operator involving Mittag-Leffler function in the following way

$$F_{\alpha_i,\,\beta_i,\,\gamma_i}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(t)}{t} \right)^{\gamma_i} dt \tag{1.4}$$

where the functions $\mathbb{E}_{\alpha_i,\,\beta_i}(z)$ is normalized Mittag-Leffler functions defined by (1.3) and parameters γ_i are positive real numbers such that the integral in (1.4) exists. Some sufficient conditions for the integral operator defined by (1.4) is in the class $\mathcal{S}^*$ are obtained.

2. Preliminary Results

To prove our main results we shall require the following lemmas.

Lemma 2.1. *([16]) Let $\alpha \geq 1, \beta \geq 1$. Then*

$$\left| \frac{z\, \mathbb{E}'_{\alpha,\,\beta}(z)}{\mathbb{E}_{\alpha,\,\beta}(z)} - 1 \right| \leq \frac{2\beta + 1}{\beta^2 - \beta - 1}, \quad z \in \Delta.$$

Lemma 2.2. *([15]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} < \frac{\delta + 1}{2(\delta - 1)}, \quad z \in \Delta, \textit{ for some } \quad 2 \leq \delta < 3,$$
$$\textit{or} \quad \Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} < \frac{5\delta - 1}{2(\delta + 1)}, \quad z \in \Delta, \textit{ for some } \quad 1 < \delta \leq 2, \textit{ then } f \in \mathcal{S}^*.$$

Lemma 2.3. *([15]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > -\frac{\delta + 1}{2\delta(\delta - 1)}, \quad z \in \Delta, \textit{ for some } \quad \delta \leq -1,$$
$$\textit{or} \quad \Re\left\{1 + \frac{z\, f''(z)}{f'(z)}\right\} > \frac{3\delta + 1}{2\delta(\delta + 1)}, \quad z \in \Delta, \textit{ for some } \quad \delta > 1, \textit{ then } f \in \mathcal{S}^*\left(\frac{\delta + 1}{2\delta}\right).$$

3. Main Results

Theorem 3.1. *Let* $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \geq 1,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n \geq \frac{1}{2}(1+\sqrt{5})$ *and suppose that* $\beta = \min\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ *and* $\gamma_1,\ \gamma_2,\ \cdots\ \gamma_n$ *are positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i \leq \frac{3-\delta}{2(\delta-1)}$$

is satisfied. Then the function $F_{\alpha_i,\,\beta_i,\,\gamma_i}$ *defined by* (1.4) *is in the class* $\mathcal{S}^*$ *for some* $2 \leq \delta < 3$.

Proof. We observe that $\mathbb{E}_{\alpha_i,\,\beta_i} \in \mathcal{A}$ i.e., $\mathbb{E}_{\alpha_i,\,\beta_i}(0) = 0 = \mathbb{E}'_{\alpha_i,\,\beta_i}(0) - 1$ for all $i \in \{1, 2, \cdots, n\}$.

Differentiating equation (1.4) we have

$$F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z) = \prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(t)}{t}\right)^{\gamma_i}.$$

Taking logarithmic differentiation, we have

$$\frac{F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)} = \sum_{i=1}^{n}\ \gamma_i\left(\frac{\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)} - \frac{1}{z}\right)$$

or equivalently

$$1 + \frac{z\,F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)} = \sum_{i=1}^{n}\ \gamma_i\left(\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)} - 1\right) + 1. \tag{3.1}$$

Taking the real part of both side of (3.1) we have

$$\Re\left\{1 + \frac{z\,F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}\right\} = \sum_{i=1}^{n}\ \gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} + 1 - \sum_{i=1}^{n}\ \gamma_i$$

$$\leq 1 + \sum_{i=1}^{n}\ \gamma_i\left|\frac{z\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)} - 1\right| \tag{3.2}$$

$$\leq 1 + \sum_{i=1}^{n}\ \gamma_i\left(\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right). \tag{3.3}$$

For all $z \in \Delta$ and $(\beta_1,\ \beta_2,\ \cdots,\ \beta_n) \geq \frac{1}{2}(1+\sqrt{5})$. Since the function $\phi : \left(\frac{1}{2}(1+\sqrt{5}), \infty\right) \to \mathbb{R}$, defined by $\phi(x) = \dfrac{2x+1}{x^2-x-1}$ is decreasing. Therefore, for all $i \in \{1,\ 2,\ \cdots,\ n\}$, we obtain

$$\frac{2\beta_i+1}{\beta_i^2-\beta_i-1} \leq \frac{2\beta+1}{\beta^2-\beta-1}.$$

Using this result, inequality (3.4) can be written as

$$\Re\left\{1 + \frac{z\,F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}\right\} \leq 1 + \frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\ \gamma_i.$$

Since

$$1 + \frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\ \gamma_i < \frac{\delta+1}{2(\delta-1)}$$

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\ \gamma_i < \frac{\delta+1}{2(\delta-1)} - 1$$

$$= \frac{3-\delta}{2(\delta-1)}.$$

Therefore, from Lemma 2.2, $F_{\alpha_i,\,\beta_i,\,\gamma_i}(z) \in \mathcal{S}^*$ for some $2 \le \delta \le 3$.

Thus, the proof of Theorem 3.1 is complete. $\square$

Theorem 3.2. *Let* $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \ge 1,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n \ge \frac{1}{2}(1+\sqrt{5}),\ \beta = \max\{\beta_1, \beta_2, \cdots, \beta_n\}$ *and* $\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_n$ *are positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i \le \frac{2\delta^2-\delta+1}{2\delta(\delta-1)} \tag{3.4}$$

is satisfied, then $F_{\alpha_i,\,\beta_i,\,\gamma_i}(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta \le -1$.

Proof. It is easy to see that for all $i \in \{1,\ 2,\ \cdots,\ n\}$, we have $\mathbb{E}_{\alpha_i,\,\beta_i}(z) \in \mathcal{A}$. Differentiating equation (1.4) we have $F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z) = \prod_{i=1}^{n}\left(\frac{\mathbb{E}_{\alpha_i,\,\beta_i}(t)}{t}\right)^{\gamma_i}$.

Taking logarthmic differentiation, we have

$$\Re\left\{1+\frac{z\,F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}\right\} = \sum_{i=1}^{n}\ \gamma_i\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right\}+1. \tag{3.5}$$

Using the result of Lemma 2.1, we have

$$\left|\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}-1\right| \le \frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using the identity $\Re\{z\} \le |z|$

$$\Re\left\{1-\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} \le \frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

$$\Re\left\{\frac{z\,\mathbb{E}'_{\alpha_i,\,\beta_i}(z)}{\mathbb{E}_{\alpha_i,\,\beta_i}(z)}\right\} \ge 1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}.$$

Using this inequality in (3.5)

$$\begin{aligned}\Re\left\{1+\frac{z\,F''_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}{F'_{\alpha_i,\,\beta_i,\,\gamma_i}(z)}\right\} &\ge \sum_{i=1}^{n}\ \gamma_i\left(1-\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\right)+1-\sum_{i=1}^{n}\ \gamma_i\\ &= -\frac{2\beta_i+1}{\beta_i^2-\beta_i-1}\sum_{i=1}^{n}\ \gamma_i+1\\ &\ge -\frac{\delta+1}{2\delta(\delta-1)}.\end{aligned}$$

Hence from Lemma 2.3 and condition (3.4), we have $F_{\alpha_i,\,\beta_i,\,\gamma_i}(z) \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ for some $\delta \le -1$. This completes the proof of Theorem 3.2. $\square$

Theorem 3.3. *Let* $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \ge 1,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n \ge \frac{1}{2}(1+\sqrt{5}),\ \beta = \min\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ *and* $\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_n$ *are positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i \le \frac{3(\delta-1)}{2(\delta+1)}.$$

is satisfied. Then $F_{\alpha_i,\,\beta_i,\gamma_i}(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}^*$ *for some* $1 < \delta \le 2$.

Proof. The proof of above theorem is much akin to that of Theorem 3.1. Therefore, we omit the details. □

Theorem 3.4. *Let* $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n \geq 1,\ \beta_1,\ \beta_2,\ \cdots,\ \beta_n \geq \frac{1}{2}(1+\sqrt{5}),\ \beta = \max\{\beta_1,\ \beta_2,\ \cdots,\ \beta_n\}$ *and* $\gamma_1,\ \gamma_2,\ \cdots,\ \gamma_n$ *are positive real numbers. Moreover, suppose that these numbers satisfy the following inequality*

$$\frac{2\beta+1}{\beta^2-\beta-1}\sum_{i=1}^{n}\gamma_i \leq \frac{\delta+1-2\delta^2}{2\delta(\delta+1)}$$

is satisfied. Then the function $F_{\alpha_i,\,\beta_i,\,\gamma_i}(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta > 1$.

Proof. The proof of above theorem is similar to that of Theorem 3.2. Therefore, we omit the details. □

4. Some Consequences

Here we give some examples.

Example 4.1.

1. *If* $0 \leq 5\gamma \leq \frac{3-\delta}{2(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $2 \leq \delta < 3$.
2. *If* $0 \leq \frac{7}{5}\gamma < \frac{3-\delta}{2(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{2[\cosh\sqrt{t}-1]}{t}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $2 \leq \delta < 3$.
3. *If* $0 \leq \frac{9}{11}\gamma < \frac{3-\delta}{2(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{6[\sinh\sqrt{t}-\sqrt{t}]}{t^{3/2}}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $2 \leq \delta < 3$.

Example 4.2.

1. *If* $0 \leq 5\gamma \leq \frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta \leq -1$.
2. *If* $0 \leq \frac{7}{5}\gamma < \frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{2[\cosh\sqrt{t}-1]}{t}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta \leq -1$.
3. *If* $0 \leq \frac{9}{11}\gamma < \frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$, *then* $\int\limits_0^z \left(\frac{6[\sinh\sqrt{t}-\sqrt{t}]}{t^{3/2}}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta \leq -1$.

Example 4.3.

1. *If* $0 \leq 5\gamma \leq \frac{3(\delta-1)}{2(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $1 < \delta \leq 2$.
2. *For* $0 \leq \frac{7}{5}\gamma < \frac{3(\delta-1)}{2(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{2[\cosh\sqrt{t}-1]}{t}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $1 < \delta \leq 2$.
3. *If* $0 \leq \frac{9}{11}\gamma < \frac{3(\delta-1)}{2(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{6[\sinh\sqrt{t}-\sqrt{t}]}{t^{3/2}}\right)^{\gamma} dt \in \mathcal{S}^*$ *for some* $1 < \delta \leq 2$.

Example 4.4.

1. *If* $0 \leq 5\gamma \leq \frac{\delta+1-2\delta^2}{2\delta(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{\sinh\sqrt{t}}{\sqrt{t}}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta > 1$.
2. *For* $0 \leq \frac{7}{5}\gamma \leq \frac{\delta+1-2\delta^2}{2\delta(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{2[\cosh\sqrt{t}-1]}{t}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta > 1$.
3. *If* $0 \leq \frac{9}{11}\gamma < \frac{\delta+1-2\delta^2}{2\delta(\delta+1)}$, *then* $\int\limits_0^z \left(\frac{6[\sinh\sqrt{t}-\sqrt{t}]}{t^{3/2}}\right)^{\gamma} dt \in \mathcal{S}^*\left(\frac{\delta+1}{2\delta}\right)$ *for some* $\delta > 1$.

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Authors' contributions
All authors equally worked on the results and they read and approved the final manuscript.

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