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Semi-Delta-Open Sets in Topological Space

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abstract: The purpose of this paper is to introduce a new class of open sets, namely semi-delta-open sets (briefly δ_s-open sets). Further, some basic topological concepts such as neighbourhood axioms, border, exterior, and frontier of a set are defined and their properties have been investigated. In addition, in terms of these open sets, semi-delta-closed functions (briefly δ_s-closed functions) and semi-delta-continuous functions (briefly δ_s-continuous functions) are also defined and their properties have been discussed.

Key Words: δ_s-closed sets, δ_s-open sets, δ_s-closed functions, δ_s-continuous functions.

Contents

1. Introduction

The notion of an open set is very fundamental in topology. Many topologists have extensively studied open sets and their new versions for so long. Amongst them, Levine [7] was the first who made known the notion of semi-open sets. His work was not confined to this concept; he also introduced and studied the term semi-closed set and the concept of semi-continuity of a function. A subset G_1 of a topological space (G, τ) (briefly G) is termed as semi-open set if $G_1 \subseteq Cl[Int(G_1)]$. The complement of a semi-open set is termed as semi-closed set. For a subset G_1 of a space G, a point g in G is a semi-closure point of G_1 if for each semi-open set G_2 in G containing g , $G_2 \cap G_1 \neq \emptyset$. Levine's work opened up a new window for many researchers. Many topologists used his notion of semi-open sets as a substitute to open sets and proved various results. Veličko [10] purposed the notion of $\delta-$closure and $\theta-$closure of a set. $\delta-$closure of a subset G_1 of space G is defined as the set of all such g in G such that $Int[Cl(G_2)] \cap G_1 \neq \emptyset$, for each open set G_2 in G containing g, and $\delta-$interior of a subset G_1 of space G is the set of all such $g \in G$ such that $Int[Cl(G_2)] \subseteq G_1$ for some open set G_2 in G containing g. It is a well-established result that the collection of all $\delta-$open sets forms a topology on G, referred to as a semi-regularization topology on G. Andrijević [1] generalized open sets by introducing b-open sets. Dutta and Tripathi [3] proposed fuzzy b-θ open sets, and in 2019, Sarma and Tripathi [9] investigated several aspects of a fuzzy semi-pre quasi-neighbourhood of a fuzzy point. In 2020, Latif [6] introduced and studied $\theta-$irresolute, $\theta-$closed, pre-θ-open, and pre-θ-closed mappings and investigated their properties. Moreover, properties of $\theta-$continuous and $\theta-$open mappings are further investigated. Latif [5] also proposed and explored the various properties of $\delta-$derived, $\delta-$border, $\delta-$frontier of a set and concepts of δ-D-sets. Recently, Hassan and Labendia [4] introduced a new version of open sets called θ_s-open sets and explored various terms, namely θ_s-continuous, θ_s-open, and θ_s-closed function. In addition, some forms of separation axioms are introduced and characterized. The present paper gives an insight into semi-delta-open sets (briefly δ_s-open sets), semi-delta-neighbourhood axioms (briefly δ_s-neighbourhood axioms), and various other topological concepts using semi-delta-open sets. Moreover, the concepts of semi-delta-closed (briefly δ_s-closed) and semi-delta-continuous functions (briefly δ_s-continuous functions) are introduced and investigated.

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2. Preliminaries

In this paper, (G, τ) and (K, σ) represent topological spaces (briefly G and K) unless otherwise mentioned. $Cl(G_1)$ and $Int(G_1)$ symbolize the closure and the interior of the subset G_1 of space G, respectively.

Definition 2.1. *[7] Let G be a topological space. A subset G_1 of G is termed as semi-open set if $G_1 \subseteq Cl(Int(G_1))$ and semi-closed set if $Int(Cl(G_1)) \subseteq G_1$*

Definition 2.2. *[2] The intersection of all semi-closed supersets of subset G_1 of space G is called semi-closure of G_1 and is represented by $sCl(G_1)$. Also $sCl(G_1) = G_1 \cup Int(Cl(G_1))$.*

For the following Lemma, one may refer to Navalagi and Gurushantanavar [8].

Lemma 2.3. *For subsets G_1 and G_2 of G, the following hold for the semi-closure operator.*

(1) $G_1 \subset sCl(G_1) \subset Cl(G_1)$;

(2) $sCl(G_1) \subset sCl(G_2)$ *if* $G_1 \subset G_2$;

(3) $sCl(sCl(G_1)) = sCl(G_1)$;

(4) $sCl(G_1 \cap G_2) \subset sCl(G_1) \cap sCl(G_2)$;

(5) $sCl(G_1) \cup sCl(G_2) \subset sCl(G_1 \cup G_2)$;

(6) G_1 *is semi-closed if and only if* $sCl(G_1) = G_1$.

3. δ_s-Open Sets and Neighbourhood Axioms

The term δ_s-open sets, a new class of open sets, is defined in this section. Furthermore, the concept of δ_s-neighbourhood axioms is proposed and investigated.

Definition 3.1. *Let G be a topological space and $G_1 \subseteq G$. Then G_1 is said to be semi-delta-open (briefly δ_s-open) if for every $g \in G_1$, there exists an open set G_2(say) containing g such that $Int[sCl(G_2)] \subseteq G_1$.*

Definition 3.2. *Let G be a topological space. Let $g \in G$ and $G_1 \subseteq G$. We say that G_1 is a semi-delta-neighbourhood (briefly δ_s-neighbourhood) of g if there is a δ_s- open set G_2 of G such that $g \in G_2 \subseteq G_1$.*

Definition 3.3. *Let G be a topological space and $G_1 \subseteq G$. Then the semi-delta-closure (briefly δ_s-closure) of G_1 is denoted and defined by $Cl_{\delta_s}(G_1) = \cap\{G_2 : G_2$ is δ_s-closed and $G_1 \subseteq G_2\}$.*

Definition 3.4. *A point $g \in G$ is called the semi-delta-cluster point (briefly δ_s-cluster point) of $G_1 \subseteq G$ if $G_1 \cap Int[sCl(G_2)] \neq \emptyset$ for every open set G_2(say) of G containing g. Sometimes we define the δ_s-closure of the set G_1 as the set of all δ_s-cluster points of G_1.*

Definition 3.5. *Let G be topological space and $G_1 \subseteq G$. Then the semi-delta-interior (briefly δ_s-interior) of G_1 is denoted and defined by $Int_{\delta_s}(G_1) = \cup\{G_2 : G_2$ is δ_s-open and $G_2 \subseteq G_1\}$. Moreover, a point $g \in G$ is said to be a δ_s-interior point of G_1 if there exist a δ_s-open set G_2 containing g such that $G_2 \subseteq G_1$.*

Definition 3.6. *A subset $G_1 \subseteq G$ is called semi-delta-closed (briefly δ_s-closed) if $G_1 = Cl_{\delta_s}(G_1)$. Moreover, the complement of a semi-delta-closed set is a semi-delta-open set.*

Remark 3.7. *The arbitrary union of semi-delta-open sets is semi-delta-open.*

Remark 3.8. $Cl_{\delta_s}(G_1 \cap G_2) \subseteq Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G_2)$*, for any subsets G_1, G_2 of space G.*

Theorem 3.9. *Let G be a topological space. Then the following conditions hold:*

(1) *Empty set and space G are δ_s- closed.*

(2) *Arbitrary intersections of* δ_s- *closed sets are* δ_s-*closed.*

(3) *Finite union of* δ_s- *closed sets are* δ_s-*closed.*

Proof. (1) $\emptyset$ and G are δ_s- closed because they are the complement of δ_s- open sets G and $\emptyset$, respectively.

(2) Given a collection of δ_s-closed sets $\{F_\alpha\}_{\alpha\in I}$, we apply DeMorgan's law, $G - \bigcap\limits_{\alpha\in I} F_\alpha = \bigcup\limits_{\alpha\in I}(G - F_\alpha)$. Since the sets $G - F_\alpha$ are δ_s-open by definition and arbitrary union of δ_s- open sets is δ_s- open. Thus $\bigcap\limits_{\alpha\in I} F_\alpha$ is δ_s-closed.

(3) Similarly, if F_i is δ_s- closed for $i = 1, ...n$, consider the equality $G - \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n}(G - F_i)$. Since finite intersection of δ_s- open set is δ_s- open. Hence $\bigcup_{i=1}^{n} F_i$ is δ_s- closed. □

Theorem 3.10. *Let* G *be a topological space. Then the intersection of two* δ_s-*neighbourhoods of* $g \in G$ *is also a* δ_s- *neighbourhood of* g.

Proof. Let N_1 and N_2 be two δ_s-neighbourhoods of $g \in G$. Then there exists δ_s-open sets G_1 and G_2 such that $g \in G_1 \subseteq N_1$ and $g \in G_2 \subseteq N_2$. Therefore, $g \in G_1 \cap G_2 \subseteq N_1 \cap N_2$. Thus $G_1 \cap G_2$ is an δ_s- open set containing g and is contained in $N_1 \cap N_2$. This implies that $N_1 \cap N_2$ is also a δ_s-neighbourhood of g. □

Theorem 3.11. *Let* G *be a topological space. If* N *is a* δ_s-*neighbourhood of* $g \in G$ *then there exists a* δ_s-*neighbourhood* M *of* g *which is subset of* N *i.e* $M \subseteq N$ *such that* M *is a* δ_s-*neighbourhood of each of its points.*

Proof. Let N be a δ_s-neighbourhood of $g \in G$. Then there exists δ_s-open set M such that $g \in M \subseteq N$. Now M being a δ_s-open set, it is a δ_s-neighbourhood of each of its points. Hence the result follows. □

Theorem 3.12. *A subset of topological space is* δ_s-*open iff it is* δ_s-*neighbourhood of each of its points.*

Proof. Let G be a topological space. Let G_1 be a subset of G. Let N_g be δ_s-neighbourhood of $g \in G$. Then there exists δ_s-open set G_g(say) in G such that $g \in G_g \subseteq N_g \subseteq G_1$. Now $\bigcup\limits_{g\in G_1} G_g = G_1$. As arbitrary union of δ_s-open sets is also δ_s-open. Hence G_1 is δ_s-open set. Conversely, if G_1 is δ_s-open set, we can take $N_g = G_1$ for all $g \in G_1$. Hence for all $g \in G_1$, we have $N_g \in G_1$ such that $N_g \subseteq G_1$. □

4. Basic Properties of δ_s-Open Sets

In this section, the notions of semi-delta-limit point (briefly δ_s-limit point), semi-delta-border (briefly δ_s-border), semi-delta-frontier (briefly δ_s-frontier) and semi-delta-exterior (briefly δ_s-exterior) of a subset G_1 of space G have been introduced and investigated.

Definition 4.1. *Let* G_1 *be a subset of a space* G. *A point* $g \in G$ *is said to be* δ_s-*limit point of* G_1 *if for each* δ_s-*open set* G_2 *containing* $g, G_2 \cap (G_1 - \{g\}) \neq \emptyset$.
The set of all δ_s-*limit points of* G_1 *is called semi-delta-derived set (briefly* δ_s-*derived set) of* G_1 *and is denoted by* $D_{\delta_s}(G_1)$.

Remark 4.2. *For a subset* G_1 *of the space* G, *the following results hold.*

(1) $[G - Int_{\delta_s}(G_1)] = Cl_{\delta_s}(G - G_1)$.

(2) $Cl(G_1) \subseteq Cl_{\delta_s}(G_1)$.

(3) G_1 *is* δ_s-*open if and only if* $G_1 = Int_{\delta_s}(G_1)$.

(4) $Int_{\delta_s}[Int_{\delta_s}(G_1)] = Int_{\delta_s}(G_1)$.

(5) $Int_{\delta_s}(G_1) = [G_1 - D_{\delta_s}(G - G_1)]$.

(6) $Cl_{\delta_s}(G_1) = G_1 \cup D_{\delta_s}(G_1)$.

(7) $Int_{\delta_s}(G_1) \cup Int_{\delta_s}(G_2) \subseteq Int_{\delta_s}(G_1 \cup G_2)$.

Definition 4.3. *δ_s-border of a subset G_1 of space G is defined and denoted by $Bd_{\delta_s}(G_1) = G_1 - Int_{\delta_s}(G_1)$.*

Theorem 4.4. *For a subset G_1 of space G, the following statements hold:*

(1) $Bd(G_1) \subseteq Bd_{\delta_s}(G_1)$, *where $Bd(G_1)$ denotes the border of G_1.*

(2) $G_1 = Int_{\delta_s}(G_1) \cup Bd_{\delta_s}(G_1)$.

(3) $Int_{\delta_s}(G_1) \cap Bd_{\delta_s}(G_1) = \emptyset$.

(4) *G_1 is δ_s-open set if and only if $Bd_{\delta_s}(G_1) = \emptyset$.*

(5) $Bd_{\delta_s}[Int_{\delta_s}(G_1)] = \emptyset$.

(6) $Int_{\delta_s}[Bd_{\delta_s}(G_1)] = \emptyset$.

(7) $Bd_{\delta_s}[Bd_{\delta_s}(G_1)] = Bd_{\delta_s}(G_1)$.

(8) $Bd_{\delta_s}(G_1) = G_1 \cap [Cl_{\delta_s}(G - G_1)]$.

(9) $Bd_{\delta_s}(G_1) = D_{\delta_s}(G - G_1)$.

Proof. (1) $Bd(G_1) = G_1 \cap (Int(G_1))^c = G_1 \cap Cl(G_1^c)$. Since $Cl(G_1) \subseteq Cl_{\delta_s}(G_1)$, therefore $Bd(G_1) \subseteq G_1 \cap Cl_{\delta_s}(G_1)^c = G_1 \cap (Int_{\delta_s}(G_1))^c = Bd_{\delta_s}(G_1)$.

(2) $Int_{\delta_s}(G_1) \cup Bd_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cup [G_1 - Int_{\delta_s}(G_1)] = [Int_{\delta_s}(G_1) \cup G_1] \cap [Int_{\delta_s}(G_1) \cup (Int_{\delta_s}(G_1))^c] = G_1$.

(3) $Int_{\delta_s}(G_1) \cap Bd_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cap (G_1 - Int_{\delta_s}(G_1)) = [Int_{\delta_s}(G_1) \cap (Int_{\delta_s}(G_1))^c)] \cap G_1 = \emptyset$.

(4) If G_1 is δ_s-open, then using Remark 4.2, $Int_{\delta_s}(G_1) = G_1$. Therefore, $Bd_{\delta_s}(G_1) = \emptyset$. Conversely, if $Bd_{\delta_s}(G_1) = \emptyset \implies G_1 - Int_{\delta_s}(G_1) = \emptyset$, which implies $G_1 = Int_{\delta_s}(G_1)$. Hence G_1 is δ_s-open.

(5) $Bd_{\delta_s}[Int_{\delta_s}(G_1)] = Int_{\delta_s}(G_1) - Int_{\delta_s}(Int_{\delta_s}(G_1))= \emptyset$. Using Remark 4.2.

(6) If $g \in Int_{\delta_s}[Bd_{\delta_s}(G_1)]$, then $g \in Bd_{\delta_s}(G_1)$. On the other hand, since $Bd_{\delta_s}(G_1) \subseteq G_1$, $g \in Int_{\delta_s}[Bd_{\delta_s}(G_1)] \subseteq Int_{\delta_s}(G_1)$. Hence, $g \in Int_{\delta_s}(G_1) \cap Bd_{\delta_s}(G_1)$ which contradicts (3). Thus, $Int_{\delta_s}[Bd_{\delta_s}(G_1)] = \emptyset$.

(7) $Bd_{\delta_s}[Bd_{\delta_s}(G_1)] = Bd_{\delta_s}(G_1) - Int_{\delta_s}[Bd_{\delta_s}(G_1)] = \emptyset$. Now, using result proved in (6) we get the desired result.

(8) $Bd_{\delta_s}(G_1) = G_1 - Int_{\delta_s}(G_1)= G_1 - [G - Cl_{\delta_s}(G - G_1)] = G_1 \cap Cl_{\delta_s}(G - G_1)$.

(9) $Bd_{\delta_s}(G_1) = G_1 - Int_{\delta_s}(G_1) = G_1 - [G_1 - D_{\delta_s}(G - G_1)] = D_{\delta_s}(G - G_1)$.

□

Definition 4.5. *δ_s-frontier of a subset G_1 of space G is defined and denoted by $Fr_{\delta_s}(G_1) = Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1)$.*

Theorem 4.6. *For a subset G_1 of space G, the following statements hold:*

(1) $Fr(G_1) \subseteq Fr_{\delta_s}(G_1)$, *where $Fr(G_1)$ denotes the frontier of G_1.*

(2) $Cl_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cup Fr_{\delta_s}(G_1)$.

(3) $Int_{\delta_s}(G_1) \cap Fr_{\delta_s}(G_1) = \emptyset$.

(4) $Bd_{\delta_s}(G_1) \subseteq Fr_{\delta_s}(G_1)$.

(5) $Fr_{\delta_s}(G_1) = Bd_{\delta_s}(G_1) \cup D_{\delta_s}(G_1)$.

(6) *G_1 is a δ_s-open set if and only if* $Fr_{\delta_s}(G_1) = D_{\delta_s}(G_1)$.

(7) $Fr_{\delta_s}(G_1) = Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G - G_1)$.

(8) $Fr_{\delta_s}(G_1) = Fr_{\delta_s}(G - G_1)$.

(9) *$Fr_{\delta_s}(G_1)$ is δ_s- closed.*

(10) $Fr_{\delta_s}[Fr_{\delta_s}(G_1)] \subseteq Fr_{\delta_s}(G_1)$.

(11) $Fr_{\delta_s}[Int_{\delta_s}(G_1)] \subseteq Fr_{\delta_s}(G_1)$.

(12) $Fr_{\delta_s}[Cl_{\delta_s}(G_1)] \subseteq Fr_{\delta_s}(G_1)$.

(13) $Int_{\delta_s}(G_1) = G_1 - Fr_{\delta_s}(G_1)$.

Proof. (1) $Fr(G_1) = Cl(G_1) \cap [Int(G_1)]^c = Cl(G_1) \cap Cl(G_1)^c$. Since, $Cl(G_1)^c \subseteq Cl_{\delta_s}(G_1)^c$, therefore, $Fr(G_1) \subseteq Cl(G_1) \cap Cl_{\delta_s}(G_1)^c = Cl(G_1) - Int_{\delta_s}(G_1) = Fr_{\delta_s}(G_1)$.

(2) $Int_{\delta_s}(G_1) \cup Fr_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cup [Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1)]$
$=[Int_{\delta_s}(G_1) \cup Cl_{\delta_s}(G_1)] \cap [Int_{\delta_s}(G_1) \cup (G - Int_{\delta_s}(G_1))] = Cl_{\delta_s}(G_1)$.

(3) $Int_{\delta_s}(G_1) \cap Fr_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cap [Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1)]$
$= [Int_{\delta_s}(G_1) \cap Cl_{\delta_s}(G_1)] \cap [Int_{\delta_s}(G_1) \cap (G - Int_{\delta_s}(G_1))] = \emptyset$.

(4) $Bd_{\delta_s}(G_1) = G_1 - Int_{\delta_s}(G_1) = G_1 \cap [Int_{\delta_s}(G_1)]^c$. Since $G_1 \subseteq Cl_{\delta_s}(G_1)$, therefore $Bd_{\delta_s}(G_1) \subseteq Cl_{\delta_s}(G_1) \cap [Int_{\delta_s}(G_1)]^c = Fr_{\delta_s}(G_1)$.

(5) Since $Int_{\delta_s}(G_1) \cup Fr_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cup Bd_{\delta_s}(G_1) \cup D_{\delta_s}(G_1)$. Using Remark 4.2, result proved in (2) and Theorem 4.4. We have, $Fr_{\delta_s}(G_1) = Bd_{\delta_s}(G_1) \cup D_{\delta_s}(G_1)$.

(6) If G_1 is δ_s-open, this implies $Bd_{\delta_s}(G_1) = \emptyset \implies Fr_{\delta_s}(G_1) = D_{\delta_s}(G_1)$, using result proved in (5). Conversely, if $Fr_{\delta_s}(G_1) = D_{\delta_s}(G_1)$ then using result proved in (2) and Remark 4.2 $\implies G_1$ is δ_s-open.

(7) $Fr_{\delta_s}(G_1) = Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1) = Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G - G_1)$. By using Remark 4.2.

(8) From (7), $Fr_{\delta_s}(G_1) = Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G - G_1)$. Replacing G_1 by $G - G_1$ we have,
$Fr_{\delta_s}(G_1) = Fr_{\delta_s}(G - G_1)$.

(9) $Cl_{\delta_s}[Fr_{\delta_s}(G_1)] = Cl_{\delta_s}[Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G-G_1)] \subseteq Cl_{\delta_s}[Cl_{\delta_s}(G_1)] \cap Cl_{\delta_s}[Cl_{\delta_s}(G-G_1)] = Cl_{\delta_s}(G_1) \cap Cl_{\delta_s}(G - G_1) = Fr_{\delta_s}(G_1)$. Hence, $Fr_{\delta_s}(G_1)$ is δ_s- closed.

(10) $Fr_{\delta_s}[Fr_{\delta_s}(G_1)] = Cl_{\delta_s}[Fr_{\delta_s}(G_1)] \cap Cl_{\delta_s}[G - Fr_{\delta_s}(G_1)] \subseteq Cl_{\delta_s}[Fr_{\delta_s}(G_1)] = Fr_{\delta_s}(G_1)$.

(11) $Fr_{\delta_s}[Int_{\delta_s} G_1] = Cl_{\delta_s}[Int_{\delta_s}(G_1)] \cap Cl_{\delta_s}[Int_{\delta_s}(G_1)]^c \subseteq Cl_{\delta_s}[Fr_{\delta_s}(G_1)] = Fr_{\delta_s}(G_1)$. Using result proved in (3).

(12) $Fr_{\delta_s}[Cl_{\delta_s}(G_1)] = Cl_{\delta_s}[Cl_{\delta_s}(G_1)] - Int_{\delta_s}[Cl_{\delta_s}(G_1)] = Cl_{\delta_s}(G_1) - Int_{\delta_s}(Cl_{\delta_s}(G_1)) \subseteq [Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1)] = Fr_{\delta_s}(G_1)$.

(13) $G_1 - Fr_{\delta_s}(G_1) = G_1 - [Cl_{\delta_s}(G_1) - Int_{\delta_s}(G_1)] = Int_{\delta_s}(G_1)$.

□

Definition 4.7. *δ_s-exterior of a subset G_1 of space G is defined and denoted by $Ext_{\delta_s}(G_1) = Int_{\delta_s}(G - G_1)$.*

Theorem 4.8. *For the subset G_1 of space G, the following statements hold:*

(1) $Ext_{\delta_s}(G_1) \subseteq Ext(G_1)$, *where $Ext(G_1)$ denotes the exterior of G_1.*

(2) $Ext_{\delta_s}(G_1)$ *is δ_s- open.*

(3) $Ext_{\delta_s}(G_1) = Int_{\delta_s}(G - G_1) = G - Cl_{\delta_s}(G_1)$.

(4) $Ext_{\delta_s}[Ext_{\delta_s}(G_1)] = Int_{\delta_s}[Cl_{\delta_s}(G_1)]$.

(5) *If $G_1 \subseteq G_2$, then $Ext_{\delta_s}(G_2) \subseteq Ext_{\delta_s}(G_1)$.*

(6) $Ext_{\delta_s}(G_1) \cap Ext_{\delta_s}(G_2) \subseteq Ext_{\delta_s}(G_1 \cap G_2)$.

(7) $Ext_{\delta_s}(G) = \emptyset$.

(8) $Ext_{\delta_s}(\emptyset) = G$.

(9) $Ext_{\delta_s}(G_1) = Ext_{\delta_s}[G - Ext_{\delta_s}(G_1)]$.

(10) $Int_{\delta_s}(G_1) \subseteq Ext_{\delta_s}[Ext_{\delta_s}(G_1)]$.

(11) $G = Int_{\delta_s}(G_1) \cup Ext_{\delta_s}(G_1) \cup Fr_{\delta_s}(G_1)$.

(12) $Ext_{\delta_s}(G_1) \cup Ext_{\delta_s}(G_2) \subseteq Ext_{\delta_s}(G_1 \cap G_2)$.

Proof. (1) Since, $Ext_{\delta_s}(G_1) = Int_{\delta_s}(G - G_1)$, therefore, $Int_{\delta_s}(G - G_1) = G - Cl_{\delta_s}(G_1) \subseteq G - Cl(G_1) = Int(G - G_1) = Ext(G_1)$

(2) Since $Int_{\delta_s}(G_1)$ is δ_s- open for any subset G_1 of space G, this implies $Ext_{\delta_s}(G_1)$ is δ_s- open.

(3) Using result, $Int_{\delta_s}(G - G_1) = G - Cl_{\delta_s}(G_1)$.

(4) $Ext_{\delta_s}[Ext_{\delta_s}(G_1)] = Ext_{\delta_s}[G - Cl_{\delta_s}(G_1)] = Int_{\delta_s}[G - (G - Cl_{\delta_s}(G_1))] = Int_{\delta_s}[Cl_{\delta_s}(G_1)]$.

(5) As $G_1 \subseteq G_2 \implies G - G_2 \subseteq G - G_1$. Therefore, $Ext_{\delta_s}(G_2) = Int_{\delta_s}(G - G_2) \subseteq Int_{\delta_s}(G - G_1) = Ext_{\delta_s}(G_1)$.

(6) Using the fact, $G_1 \cap G_2 \subseteq G_1$, $G_1 \cap G_2 \subseteq G_2$ and result proved in (5).

(7) $Ext_{\delta_s}(G) = Int_{\delta_s}(\emptyset) = \emptyset$.

(8) $Ext_{\delta_s}(\emptyset) = Int_{\delta_s}(G)$.

(9) $Ext_{\delta_s}[G - Ext_{\delta_s}(G_1)] = Ext_{\delta_s}[G - Int_{\delta_s}(G - G_1)] = Int_{\delta_s}[Int_{\delta_s}(G - G_1)] = Int_{\delta_s}(G - G_1) = Ext_{\delta_s}(G_1)$.

(10) $Int_{\delta_s}(G_1) \subseteq Int_{\delta_s}[Cl_{\delta_s}(G_1)] = Int_{\delta_s}[G - Int_{\delta_s}(G - G_1)] = Int_{\delta_s}[G - Ext_{\delta_s}(G_1)] = Ext_{\delta_s}[Ext_{\delta_s}(G_1)]$.

(11) $Int_{\delta_s}(G_1) \cup Ext_{\delta_s}(G_1) \cup Fr_{\delta_s}(G_1) = Int_{\delta_s}(G_1) \cup Int_{\delta_s}(G - G_1) \cup Bd_{\delta_s}(G_1) \cup D_{\delta_s}(G_1) = G$.

(12) $Ext_{\delta_s}(G_1) \cup Ext_{\delta_s}(G_2) = Int_{\delta_s}(G - G_1) \cup Int_{\delta_s}(G - G_2) \subseteq Int_{\delta_s}[(G - G_1) \cup (G - G_2)] = Int_{\delta_s}[G - (G_1 \cap G_2)] = Ext_{\delta_s}(G_1 \cap G_2)$.

□

5. δ_s-Open Functions, δ_s-Closed Functions and δ_s-Continuous Functions

In this section, we introduce the concepts of δ_s-open, δ_s-closed, and δ_s-continuous functions and further study their properties.

Definition 5.1. *Let G and K be topological spaces. A function $g : G \to K$ is δ_s-open if $g(G_1)$ is δ_s-open in K for each open set G_1 in G.*

Definition 5.2. *Let G and K be topological spaces. A function $g : G \to K$ is δ_s-closed if $g(G_1)$ is δ_s-closed in K for every closed set G_1 in G.*

Definition 5.3. *A function $g : (G, \tau) \to (K, \sigma)$ is said to be δ_s-continuous function if $g^{-1}(K_1)$ is δ_s-open for every open set K_1 of K.*

Theorem 5.4. *Let G and K be topological spaces and $g : G \to K$ be a function. Then the following statements are equivalent:*

(1) *g is δ_s- closed on G.*

(2) *$Cl_{\delta_s}(g(G_1)) \subseteq g(Cl(G_1))$ for every $G_1 \subseteq G$.*

Proof. (1) $\Longrightarrow$ (2) Let $G_1 \subseteq G$. Note that $g(G_1) \subseteq g[Cl(G_1)]$ and $g[Cl(G_1)]$ is δ_s- closed. As δ_s-closure of G_1 is the smallest δ_s- closed set containing G_1. Therefore, $Cl_{\delta_s}[g(G_1)] \subseteq g[Cl(G_1)]$.

(2) $\Longrightarrow$ (1) Let G_1 be closed set in G. By assumption, $g(G_1) \subseteq Cl_{\delta_s}[g(G_1)] \subseteq g[Cl(G_1)] = g(G_1)$. Thus, $g(G_1)$ is δ_s-closed. Therefore, g is δ_s-closed in G. □

Theorem 5.5. *Let $g : (G, \tau) \to (K, \sigma)$ be δ_s-closed. If $K_1 \subseteq K$ and $G_1 \subseteq G$ is an open set containing $g^{-1}(K_1)$, then there exists a δ_s-open set $K_2 \subseteq K$ containing K_1 such that $g^{-1}(K_2) \subseteq G_1$.*

Proof. Let $K_2 = K - g(G - G_1)$. Since $g^{-1}(K_1) \subseteq G_1$, we have $g(G - G_1) \subseteq (K - K_1)$. Since g is δ_s-closed, then K_2 is a δ_s- open set and $g^{-1}(K_2) = G - g^{-1}[g(G - G_1)] \subseteq G - (G - G_1) = G_1$. □

Theorem 5.6. *Suppose that $g : (G, \tau) \to (K, \sigma)$ is a δ_s-closed function. Then $Int_{\delta_s}[Cl_{\delta_s}(g(G_1))] \subseteq g(Cl(G_1))$ for every subset G_1 of G.*

Proof. Suppose g is a δ_s-closed function and G_1 is an arbitrary subset of G. Then $g[Cl(G_1)]$ is δ_s-closed set in K. Then $Int_{\delta_s}[Cl_{\delta_s}(g(Cl(G_1)))] \subseteq g[Cl(G_1)]$. But also $Int_{\delta_s}[Cl_{\delta_s}(g(G_1))] \subseteq Int_{\delta_s}[Cl_{\delta_s}(g(Cl(G_1))]$. Hence $Int_{\delta_s}[Cl_{\delta_s}(g(G_1))] \subseteq g(Cl(G_1))$. □

Theorem 5.7. *Let $g : (G, \tau) \to (K, \sigma)$ be a δ_s- closed function, and $K_1, K_2 \subseteq K$. Then the following statements hold:*

(1) *If U is an open neighbourhood of $g^{-1}(K_1)$, then there exists a δ_s-open neighbourhood V of K_1 such that $g^{-1}(K_1) \subseteq g^{-1}(V) \subseteq U$.*

(2) *If g is also onto, then if $g^{-1}(K_1)$ and $g^{-1}(K_2)$ have disjoint open neighbourhoods, so have K_1 and K_2.*

Proof. (1) Let $V = K - g(G - U)$. Then $K - V = g(G - U)$. Since g is δ_s-closed, so V is a δ_s-open set. Since $g^{-1}(K_1) \subseteq U$, we have $K - V = g(G - U) \subseteq g[g^{-1}(K - K_1)] \subseteq (K - K_1)$. Hence, $K_1 \subseteq V$, thus V is a δ_s-neighbourhood of K_1. Further $G - U \subseteq g^{-1}[g(G - U)] = g^{-1}(K - V) = G - g^{-1}(V)$. This proves that $g^{-1}(V) \subseteq U$.

(2) If $g^{-1}(K_1)$ and $g^{-1}(K_2)$ have disjoint open neighbourhoods M and N, then by (1), we have δ_s-open neighbourhoods U and V of K_1 and K_2 respectively such that $g^{-1}(K_1) \subseteq g^{-1}(U) \subseteq Int_{\delta_s}(M)$ and $g^{-1}(K_2) \subseteq g^{-1}(V) \subseteq Int_{\delta_s}(N)$. Since M and N are disjoint, so are $Int_{\delta_s}(M)$ and $Int_{\delta_s}(N)$, hence so $g^{-1}(U)$ and $g^{-1}(V)$ are disjoint as well. It follows that U and V are disjoint too, as g is onto. □

Theorem 5.8. *Prove that a surjective mapping $g : (G, \tau) \to (K, \sigma)$ is δ_s-closed, if and only if for each subset K_1 of K and each open set G_1 in G containing $g^{-1}(K_1)$, there exists a δ_s-open set V in K containing K_1 such that $g^{-1}(V) \subseteq G_1$.*

Proof. **Necessity**. Follows from (1) of Theorem 5.7.

Sufficiency. Suppose F is an arbitrary closed set in G. Let k be an arbitrary point in $K - g(F)$. Then $g^{-1}(k) \subseteq G - g^{-1}[g(F)] \subseteq (G - F)$ and $(G - F)$ is open in G. By using assumption, there exists a δ_s-open set V_k containing k such that $g^{-1}(V_k) \subseteq (G - F)$. This implies that $k \in V_k \subseteq [K - g(F)]$. Thus $K - g(F) = \cup\{V_k : k \in K - g(F)\}$. Hence $K - g(F)$, being a union of δ_s- open sets, is δ_s-open. Thus its complement $g(F)$ is δ_s-closed. Which proves that g is δ_s-closed. □

Theorem 5.9. *Let G and K be topological spaces and $g : G \to K$ be a function.Then the following statements are equivalent:*

(1) *g is δ_s-continuous on G.*

(2) *$g^{-1}(F)$ is δ_s-closed in G for each closed subset F of K.*

(3) *$g^{-1}(K_1)$ is δ_s-open for each basic open set K_1 in K.*

(4) *For every $p \in G$ and every open set V of K containing $g(p)$, there exists a δ_s-open set U containing p such that $g(U) \subseteq V$..*

(5) *$g[Cl_{\delta_s}(G_1)] \subseteq Cl[g(G_1)]$ for each $G_1 \subseteq G$.*

(6) *$Cl_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}(Cl(K_1))$.*

(7) *$Bd_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}[Bd(K_1)]$, for every $K_1 \subseteq K$.*

(8) *$g[D_{\delta_s}(G_1)] \subseteq Cl[g(G_1)]$, for every $G_1 \subseteq G$.*

(9) *$g^{-1}[Int(K_1)] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$, for every $K_1 \subseteq K$.*

Proof. (1) $\implies$ (2) Let F be closed subset of K, then its complement is open in K. By using assumption, $g^{-1}(K/F) = g^{-1}(K)/g^{-1}(F) = G/g^{-1}(F)$ is δ_s-open which implies that $g^{-1}(F)$ is δ_s-closed in G.
(2) $\implies$ (1) Let F be an open set in K then K/F is closed in K, by using assumption, $g^{-1}(K/F)$ is δ_s-closed in G, which implies $g^{-1}(F)$ is δ_s- open in G. Hence g is δ_s-continuous.

(2) $\implies$ (3) Let K_1 be basic open set in K. Then K/K_1 is closed in K, therefore $g^{-1}(G/K_1)$ is δ_s-closed in G, which implies $g^{-1}(K_1)$ is δ_s-open.

(3) $\implies$ (4) For each $p \in G$ and every open set V of K containing $g(p)$. Then $U = g^{-1}(V)$ is δ_s- open in G, which implies $g(U) \subseteq V$

(4) $\implies$ (5) Let $G_1 \subseteq G$ and $p \in Cl_{\delta_s}(G_1)$. Let V be an open neighbourhood of $g(p)$ and U be δ_s-open set in G containing p, such that $g(U) \subseteq V$. Since $p \in Cl_{\delta_s}(G_1)$ implies $U \cap G_1 \neq \emptyset$. Hence $\emptyset \neq g(U \cap G_1) \subseteq g(U) \cap g(G_1) \subseteq V \cap g(G_1)$. Since choice of V is arbitrary $\implies$ every neighbourhood of $g(p)$ intersect $g(G_1) \implies g(p) \in Cl(g(G_1))$. Hence $g[Cl_{\delta_s}(G_1)] \subseteq Cl[g(G_1)]$ for each $G_1 \subseteq G$.

(5) $\implies$ (6) Let $G_1 = g^{-1}(K_1)$ then using assumption, $g[Cl_{\delta_s}(G_1)] \subseteq Cl[g(G_1)] = Cl[g(g^{-1}(K_1))] = Cl(K_1)$. Hence $Cl_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}[Cl(K_1)]$.

(7) $\implies$ (9) Let $K_1 \subseteq K$. Then by hypothesis, $Bd_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}[Bd(K_1)]$
$\implies g^{-1}(K_1) - Int_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}[K_1 - Int(K_1)] = g^{-1}(K_1) - g^{-1}[Int(K_1)]$
$\implies g^{-1}[Int(K_1)] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$.

(9) $\implies$ (7) Let $K_1 \subseteq K$. Then by hypothesis, $g^{-1}[Int(K_1)] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$
$\implies g^{-1}(K_1) - Int_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}(K_1) - g^{-1}[Int(K_1)] = g^{-1}[K_1 - Int(K_1)]$
$\implies Bd_{\delta_s}[g^{-1}(K_1)] \subseteq g^{-1}[Bd(K_1)]$.

(1) $\implies$ (8) It is obvious, since g is δ_s-continuous, by (5), $g(Cl_{\delta_s}(G_1)) \subseteq Cl(g(G_1))$ for each $G_1 \subseteq G$. So $g[D_{\delta_s}(G_1)] \subseteq Cl[g(G_1)]$.

(8) $\implies$ (1) Let $K_1 \subseteq K$ be an open set, $V = K - K_1$ and $g^{-1}(V) = W$. Then by hypothesis, $g[D_{\delta_s}(W)] \subseteq Cl[g(W)]$. Thus $g[D_{\delta_s}(g^{-1}(V))] \subseteq Cl[g(g^{-1}(V))] \subseteq Cl(V) = V$. Then $D_{\delta_s}[g^{-1}(V)] \subseteq g^{-1}(V)$ and $g^{-1}(V)$ is δ_s-closed. Therefore g is δ_s-continuous.

(1) $\implies$ (9) Let $K_1 \subseteq K$. Then $g^{-1}[Int(K_1)]$ is δ_s-open in G. Thus $g^{-1}[Int(K_1)] = Int_{\delta_s}[g^{-1}(Int(K_1))] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$. Therefore $g^{-1}[Int(K_1)] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$.

(9) $\implies$ (1) Let $K_1 \subseteq K$ be an open set. Then $g^{-1}(K_1) = g^{-1}[Int(K_1)] \subseteq Int_{\delta_s}[g^{-1}(K_1)]$. Therefore $g^{-1}(K_1)$ is δ_s- open. Hence g is δ_s-continuous. □

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