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Existence of Solutions for Boundary Value Problems for First Order Impulsive Difference Equations

Lydia Bouchal and Karima Mebarki and Svetlin Georgiev Georgiev

ABSTRACT: In this work, we establish sufficient conditions for the existence of nonnegative solutions for a class of first order impulsive difference equations with a family of nonlinear boundary conditions. To prove our main result we use a new topological approach on the fixed point index theory for the sum of two operators in Banach spaces. An example is given to illustrate the main result.

Key Words: Fixed point index, cones, sum of operators, nonnegative solution, impulsive difference equations, nonlinear boundary conditions.

Contents

1. Introduction

Many phenomena studied in applied sciences are represented by differential equations and difference equations. However, many of them have a sudden change in their states such as neural networks models, population models, models in economics, etc. For more details, we refer the reader to the book [1].
An impulsive difference equation is described by three components:
a difference equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active.

This paper is devoted to investigate the following boundary value problem for impulsive difference equations with nonlinear two point functional boundary conditions

$$\begin{aligned} \Delta x(n) &= f(n, x(n)), \quad n \neq n_k, \quad n \in J, \\ \Delta x(n_k) &= I_k(x(n_k)), \quad n = n_k, \\ Mx(0) - Nx(T) &= g(x(0), x(T)), \end{aligned} \tag{1.1}$$

where Δ is the forward difference operator, i.e., $\Delta u(n) = u(n+1) - u(n)$, $J = [0, T] \cap \mathbb{N}$, $T \in \mathbb{N}$, $\mathbb{N}$ is the set of natural numbers, $M, N > 0$, $f \in \mathcal{C}(J \times \mathbb{R})$, $g \in \mathcal{C}(\mathbb{R} \times \mathbb{R})$, $I_k \in \mathcal{C}(\mathbb{R})$, $k \in \{1, \ldots, p\}$, $\{n_k\}_{k=1}^{p}$ are fixed points such that

$$0 < n_1 < n_2 < \ldots < n_p < T, \quad p \in \mathbb{N}.$$

In this paper we propose a new approach to ensure the existence of at least one nonnegative solution to the BVP (1.1). The nonlinear terms in the equation and in the boundary conditions as well as the jump function satisfy a general polynomial growth conditions. Our existence result is based on a recent fixed point index theory developed in [2,3] for the sum of two operators on cones of a Banach space. Precisely,

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our method involves the fixed point index for the sum of two operators $T+F$ on cones of a Banach space, where $I-T$ is Lipschitz invertible and F is a k-set contraction.

When the function g is a constant, criteria on the existence of minimal and maximal solutions to the BVP (1.1) are obtained in [7] by using a comparison theorem and the method of upper and lower solutions coupled with the monotone iterative technique. These two methods as well as the fixed point theory are the most common techniques used in the literature to investigate the existence of solutions for first order impulsive difference equations. Tian et al. [6] investigated periodic boundary value problems for first order impulsive difference equations with time delay. The authors in [4,9] analyzed the existence of solutions for a first order functional difference equations without impulses with nonlinear functional boundary conditions. In [8,10] they studied the existence of solutions for difference equations involving causal operators without impulses with nonlinear boundary conditions. The authors in [5] obtained the existence of positive solutions for a class of first order impulsive difference equations with periodic boundary value conditions using fixed point theorems of Krasnosel'skii and Leggett Williams.
Motivated by the previous works, the boundary conditions considered in this paper involving nonlinear functional at two point are more general. They include, among others, periodic, multipoint boundary value conditions and integral boundary value conditions as particular cases.

The paper is organized as follows. In the next section, we give some preliminary results. In Section 3, we give some auxiliary results and we prove our main result. In Section 4, we give an example that illustrates our main result.

2. Preliminaries

Definition 2.1. *Let E be a Banach space. A nonempty closed convex set $\mathcal{P} \subset E$ is called a cone provided:*

1. *$\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,*
2. *$x \in \mathcal{P}$, $-x \in \mathcal{P}$ implies $x = 0$.*

Definition 2.2. *Let E be a real Banach space. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.*

In all what follows, $\mathcal{P}$ will refer to a cone in a Banach space $(E, \|.\|)$, and U is a bounded open subset of $\mathcal{P}$, $\Omega \subset \mathcal{P}$. The fixed point index $i_*(T+F, U\cap\Omega, \mathcal{P})$ defined by

$$i_*\left(T+F, U\cap\Omega, \mathcal{P}\right) = \begin{cases} i\left((I-T)^{-1}F, U, \mathcal{P}\right), & \text{if } U\cap\Omega \neq \emptyset \\ 0, & \text{if } U\cap\Omega = \emptyset, \end{cases} \tag{2.1}$$

is well defined whenever $T : \Omega \to E$ is such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$ and $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$ and $F(\overline{U}) \subset (I-T)(\Omega)$. For more details see [2,3].

The following result (see details of its proof in [3] and [2]) will be used to prove Theorem 3.1.

Proposition 2.3. *[3] Assume that the mapping $T : \Omega \subset \mathcal{P} \to E$ be such that $(I-T)$ is Lipschitz invertible with constant $\gamma > 0$, $F : \overline{U} \to E$ is a k-set contraction with $0 \leq k < \gamma^{-1}$, and $tF(\overline{U}) \subset (I-T)(\Omega)$ for all $t \in [0,1]$. If $(I-T)^{-1}0 \in U$, and*

$$(I-T)x \neq \lambda Fx \quad \textit{for all } x \in \partial U \bigcap \Omega \quad \textit{and } 0 \leq \lambda \leq 1,$$

then the fixed point index $i_*\left(T+F, U\bigcap\Omega, \mathcal{P}\right) = 1$.

3. Main result

We suppose that

($\mathcal{H}_1$) The functions f, g, I_k, $k \in \{1, \ldots, p\}$, satisfy

$$\begin{aligned} 0 &\le f(n, x(n)) \le a_1(n) + a_2(n)|x(n)|^{p_1}, \\ 0 &\le g(x(0), x(T)) \le b_1 + b_2|x(0)|^{p_2} + b_3|x(T)|^{p_3}, \\ 0 &\le I_k(x(n_k)) \le a_3(n_k) + a_4(n_k)|x(n_k)|^{p_4}, \quad k \in \{1, \ldots, p\}, \end{aligned}$$

where $a_1, a_2, a_3, a_4 \in \mathcal{C}(J, \mathbb{R})$ are positive functions, b_1, b_2, b_3, p_1, p_2, p_3, p_4 are nonnegative constants, and

$$0 \le a_1(n), a_2(n), a_3(n), a_4(n), b_1, b_2, b_3 \le D, \quad n \in J,$$

for some positive constant D.

and

($\mathcal{H}_2$) The constants $c \in (0, 1)$, $B > 0, D > 0$, $M > 0$, $N > 0$, $T \in \mathbb{N}$, $p_j \ge 0$, $j \in \{1, \ldots, 4\}$, satisfy

$$M - N(1-c)^T > 0$$

and

$$\begin{aligned} B_1 &= \frac{D\left(1 + B^{p_2} + B^{p_3}\right)}{M - N(1-c)^T} \\ &\quad + 2T\frac{M+N}{M - N(1-c)^T}\left(D(1 + B^{p_1} + B^{p_4}) + cB\right) \\ &< B. \end{aligned}$$

Our main result is as follows.

Theorem 3.1. *Suppose that ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold. Then the BVP (1.1) has at least one nonnegative solution $x \in \mathcal{C}(J, \mathbb{R})$ so that*

$$0 \le x(n) < B, \quad n \in J.$$

Remark 3.2. *In [7], the BVP (1.1) is investigated when*

$$f(n, x) - f(n, y) \ge -L(x - y)$$

for $\alpha_0 \le \alpha(n) \le y \le x \le \beta(n) \le \beta_0$, $n \in J$, and

$$I_k(x) - I_k(y) \ge -L_k(x - y)$$

for $\alpha_0 \le \alpha(n_k) \le y \le x \le \beta(n_k) \le \beta_0$, $k \in \{1, \ldots, p\}$, and g is a constant, where α_0, β_0 are nonnegative constants, α and β are suitable nonnegative functions, $0 < L, L_k < 1$, $k \in \{1, \ldots, p\}$. It is given in [7] a criteria for existence of nonnegative minimal and maximal solutions. If

$$f(n, x) = I_k(x) = \frac{a}{(1+x)^2}, \quad x \ge 0, \quad a > \frac{(1+\beta_0)^4}{1+\alpha_0},$$

Then

$$\begin{aligned} I_k(x) - I_k(y) &= f(n,x) - f(n,y) \\ &= \frac{a}{(1+x)^2} - \frac{a}{(1+y)^2} \\ &= -\frac{a(x-y)(2+x+y)}{(1+x)^2(1+y)^2} \\ &\le -\frac{2a(x-y)(1+\alpha_0)}{(1+\beta_0)^4} \\ &< -2(x-y), \quad \alpha_0 \le y \le x \le \beta_0. \end{aligned}$$

Thus, the conditions in [7] are not fulfilled, but our conditions hold. Also, our main result is valid in the case when g is not a constant. Therefore we can consider our main result as a complementary and improvement result to those in [7].

3.1. Auxiliary Results

In [7], it is shown that the solution of the BVP

$$\begin{aligned} \Delta u(n) + cu(n) &= \sigma(n), \quad n \neq n_k, \quad n \in J, \\ \Delta u(n_k) &= -L_k u(n_k) + I_k(\eta(n_k)) + L_k \eta(n_k), \quad k \in \{1,\dots,p\}, \\ Mu(0) - Nu(T) &= C, \end{aligned}$$

where $0 < c < 1$, L_k, C, $k \in \{1,\dots,p\}$, are given constants, $\eta \in E_1, \sigma \in \mathcal{C}(J)$, where E_1 is the set of real-valued functions defined on J, is represented in the form

$$\begin{aligned} u(n) &= \frac{C(1-c)^n)}{M - N(1-c)^T} + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\sigma(j) \\ &+ \sum_{0<n_k\le T-1} G(n,n_k)\left((c-L_k)u(n_k) + I_k(\eta(u_k)) + L_k\eta(u_k)\right), \end{aligned}$$

where

$$G(n,j) = \frac{1}{M - N(1-c)^T} \begin{cases} M\frac{(1-c)^n}{(1-c)^{j+1}}, & 0 \le j \le n-1, \\ N\frac{(1-c)^{T+n}}{(1-c)^{j+1}}, & n \le j \le T-1. \end{cases}$$

We have that

$$G(n,j) \le \frac{M+N}{M-N(1-c)^T}, \quad n, j \in J. \tag{3.1}$$

In the Banach space $\mathcal{C}(J,\mathbb{R})$ of the continuous real-valued functions defined on J, define the norm

$$\|x\| = \max_{n\in J} |x(n)|.$$

Lemma 3.3. *Suppose that $(\mathcal{H}_1)$ holds. If $x \in \mathcal{C}(J,\mathbb{R})$, $\|x\| \le B$, then*

$$\begin{aligned} 0 &\le f(n,x(n)) \le D\left(1 + B^{p_1}\right), \, n \in J, \\ 0 &\le g(x(0),x(T)) \le D\left(1 + B^{p_2} + B^{p_3}\right), \\ 0 &\le I_k(x(n_k)) \le D\left(1 + B^{p_4}\right), \quad k \in \{1,\dots,p\}. \end{aligned}$$

Proof: By $(\mathcal{H}1)$, we get

$$\begin{aligned} 0 &\le f(n, x(n)) \\ &\le a_1(n) + a_2(n)|x(n)|^{p_1} \\ &\le D\left(1 + B^{p_1}\right), \; n \in J, \end{aligned}$$

and

$$\begin{aligned} 0 &\le g(x(0), x(T)) \\ &\le b_1 + b_2|x(0)|^{p_2} + b_3|x(T)|^{p_3} \\ &\le D\left(1 + B^{p_2} + B^{p_3}\right), \end{aligned}$$

and

$$\begin{aligned} 0 &\le I_k(x(n_k)) \\ &\le a_3(n_k) + a_4(n_k)|x(n_k)|^{p_4} \\ &\le D\left(1 + B^{p_4}\right), \quad k \in \{1, \ldots, p\}. \end{aligned}$$

This completes the proof. □

Lemma 3.4. *Suppose that* $(\mathcal{H}_1)$ *holds. Let* $x \in \mathcal{C}(J, \mathbb{R})$ *satisfies the equation*

$$\begin{aligned} x(n) &= \frac{g(x(0), x(T))(1-c)^n}{M - N(1-c)^T} + \sum_{j=0, j \neq n_k}^{T-1} G(n, j)\left(f(j, x(j)) + cx(j)\right) \\ &\quad + \sum_{0 < n_k \le T-1} G(n, n_k)\left(cx(n_k) + I_k(x(n_k))\right), \quad n \in J. \end{aligned}$$

Then it satisfies the BVP (1.1).

Proof: We have

$$\begin{aligned} x(n) &= \frac{g(x(0), x(T))(1-c)^n}{M - N(1-c)^T} \\ &\quad + \frac{M(1-c)^n}{M - N(1-c)^T} \sum_{0 \le j \le n-1, j \neq n_k} \frac{1}{(1-c)^{j+1}}\left(f(j, x(j)) + cx(j)\right) \\ &\quad + \frac{N(1-c)^{T+n}}{M - N(1-c)^T} \sum_{n \le j \le T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}}\left(f(j, x(j)) + cx(j)\right) \\ &\quad + \frac{M}{M - N(1-c)^T} \sum_{0 < n_k \le n-1} \frac{(1-c)^n}{(1-c)^{n_k+1}}\left(cx(n_k) + I_k(x(n_k))\right) \\ &\quad + \frac{N}{M - N(1-c)^T} \sum_{n \le n_k \le T-1} \frac{(1-c)^{T+n}}{(1-c)^{n_k+1}}\left(cx(n_k) + I_k(x(n_k))\right), \quad n \in J. \end{aligned}$$

Hence, for $n \neq n_k$, $k \in \{1, \dots, p\}$, we have

$$\begin{aligned}
x(n+1) &= \frac{g(x(0),x(T))(1-c)^{n+1}}{M-N(1-c)^T} \\
&\quad + \frac{M(1-c)^{n+1}}{M-N(1-c)^T} \sum_{0 \le j \le n, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M-N(1-c)^T} \sum_{n+1 \le j \le T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{M}{M-N(1-c)^T} \sum_{0 < n_k \le n} \frac{(1-c)^{n+1}}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad + \frac{N}{M-N(1-c)^T} \sum_{n+1 \le n_k \le T-1} \frac{(1-c)^{T+n+1}}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&= \frac{g(x(0),x(T))(1-c)^{n+1}}{M-N(1-c)^T} + \frac{M}{M-N(1-c)^T} \left(f(n,x(n)) + cx(n)\right) \\
&\quad + \frac{M(1-c)^{n+1}}{M-N(1-c)^T} \sum_{0 \le j \le n-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad - \frac{N(1-c)^T}{M-N(1-c)^T} \left(f(n,x(n)) + cx(n)\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M-N(1-c)^T} \sum_{n \le j \le T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{M(1-c)^{n+1}}{M-N(1-c)^T} \sum_{0 < n_k \le n-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad + \frac{N(1-c)^{T+n+1}}{M-N(1-c)^T} \sum_{n \le n_k \le T-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&= (1-c)\Bigg(\frac{g(x(0),x(T))(1-c)^{n}}{M-N(1-c)^T} \\
&\quad + \frac{M(1-c)^{n}}{M-N(1-c)^T} \sum_{0 \le j \le n-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{N(1-c)^{T+n}}{M-N(1-c)^T} \sum_{n \le j \le T-1, j \neq n_k} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{M(1-c)^{n}}{M-N(1-c)^T} \sum_{0 < n_k \le n-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad + \frac{N(1-c)^{T+n}}{M-N(1-c)^T} \sum_{n \le n_k \le T-1} \frac{1}{(1-c)^{n_k+1}} \left(cx(n_k) + I_k(x(n_k))\right) \Bigg) \\
&\quad + f(n,x(n)) + cx(n)
\end{aligned}$$

$$= (1-c)x(n) + f(n,x(n)) + cx(n)$$

$$= f(n,x(n)) + x(n).$$

So,

$$\Delta x(n) = f(n,x(n)), \quad n \neq n_k.$$

Next, for $n = n_k$, $k \in \{1,\dots,p\}$, we have

$$\begin{aligned}
\Delta x(n_k) &= x(n_k+1) - x(n_k) \\
&= \frac{g(x(0),x(T))(1-c)^{n_k+1}}{M-N(1-c)^T} \\
&\quad + \frac{M(1-c)^{n_k+1}}{M-N(1-c)^T} \sum_{0\le j\le n_k, j\neq n_l} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{N(1-c)^{T+n_k+1}}{M-N(1-c)^T} \sum_{n_k+1\le j\le T-1, j\neq n_l} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{M(1-c)^{n_k+1}}{M-N(1-c)^T} \sum_{0<n_l\le n_k} \frac{1}{(1-c)^{n_l+1}} \left(cx(n_l) + I_l(x(n_l))\right) \\
&\quad + \frac{N(1-c)^{T+n_k+1}}{M-N(1-c)^T} \sum_{n_k+1\le n_l\le T-1} \frac{1}{(1-c)^{n_l+1}} \left(cx(n_l) + I_k(x(n_l))\right) \\
&\quad - x(n_k) \\
&= (1-c)\Bigg(\frac{g(x(0),x(T))(1-c)^{n_k}}{M-N(1-c)^T} \\
&\quad + \frac{M(1-c)^{n_k}}{M-N(1-c)^T} \sum_{0\le j\le n_k-1, j\neq n_l} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{N(1-c)^{T+n_k}}{M-N(1-c)^T} \sum_{n_k\le j\le T-1, j\neq n_l} \frac{1}{(1-c)^{j+1}} \left(f(j,x(j)) + cx(j)\right) \\
&\quad + \frac{M(1-c)^{n_k}}{M-N(1-c)^T} \sum_{0<n_l\le n_k-1} \frac{1}{(1-c)^{n_l+1}} \left(cx(n_l) + I_k(x(n_l))\right) \\
&\quad + \frac{N(1-c)^{T+n_k}}{M-N(1-c)^T} \sum_{n_k\le n_l\le T-1} \frac{1}{(1-c)^{n_l+1}} \left(cx(n_l) + I_k(x(n_l))\right)\Bigg) \\
&\quad + \frac{M}{M-N(1-c)^T} \left(cx(n_k) + I_k(x(n_k))\right) \\
&\quad - \frac{N(1-c)^T}{M-N(1-c)^T} \left(cx(n_k) + I_k(x(n_k))\right) - x(n_k) \\
&= (1-c)x(n_k) + cx(n_k) + I_k(x(n_k)) - x(n_k) \\
&= I_k(x(n_k)).
\end{aligned}$$

Moreover,

$$\begin{aligned} Mx(0) &= \frac{g(x(0),x(T))M}{M-N(1-c)^T} \\ &+\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le j\le T-1, j\neq n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right) \\ &+\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le n_k\le T-1}\frac{1}{(1-c)^{n_k+1}}\left(cx(n_k)+I_k(x(n_k))\right), \end{aligned}$$

$$\begin{aligned} Nx(T) &= \frac{g(x(0),x(T))N(1-c)^T}{M-N(1-c)^T} \\ &+\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le j\le T-1, j\neq n_k}\frac{1}{(1-c)^{j+1}}\left(f(j,x(j))+cx(j)\right) \\ &+\frac{MN(1-c)^T}{M-N(1-c)^T}\sum_{0\le n_k\le T-1}\frac{1}{(1-c)^{n_k+1}}\left(cx(n_k)+I_k(x(n_k))\right). \end{aligned}$$

Therefore

$$Mx(0)-Nx(T)=g(x(0),x(T)).$$

This completes the proof. □

For $x\in\mathcal{C}(J,\mathbb{R})$, define the operator

$$\begin{aligned} Fx(n) &= \frac{g(x(0),x(T))}{M-N(1-c)^T}+\sum_{j=0, j\neq n_k}^{T-1}G(n,j)\left(f(j,x(j))+cx(j)\right) \\ &+\sum_{0<n_k\le T-1}G(n,n_k)\left(cx(n_k)+I_k(x(n_k))\right), \ n\in J. \end{aligned}$$

By Lemma 3.4, it follows that any fixed point $x\in\mathcal{C}(J,\mathbb{R})$ of the operator F is a solution to the BVP (1.1).

Lemma 3.5. *Suppose that $f\in\mathcal{C}(J\times\mathbb{R})$, $g\in\mathcal{C}(\mathbb{R}\times\mathbb{R})$ and $I_k\in\mathcal{C}(\mathbb{R})$, $k\in\{1,\dots,p\}$. Then $F:\mathcal{C}(J,\mathbb{R})\to\mathcal{C}(J,\mathbb{R})$ is a continuous operator.*

Proof: (a) Since $G\in\mathcal{C}(J\times J)$, $f\in\mathcal{C}(J\times\mathbb{R})$, $g\in\mathcal{C}(\mathbb{R}\times\mathbb{R})$ and $I_k\in\mathcal{C}(\mathbb{R})$, $k\in\{1,\dots,p\}$, the operator F maps $\mathcal{C}(J,\mathbb{R})$ into $\mathcal{C}(J,\mathbb{R})$.

(b) F is continuous. In fact, take $\{x_l\}_{l\in\mathbb{N}}\subset\mathcal{C}(J,\mathbb{R})$ such that $x_l\to x$, as $l\to+\infty$ in $\mathcal{C}(J,\mathbb{R})$. Fix $\varepsilon>0$ arbitrarily. Then there is a $\delta=\delta(\varepsilon)\in\mathbb{N}$ such that

$$\begin{aligned} |x_l(n)-x(n)| &< \varepsilon, \\ |f(n,x_l(n))-f(n,x(n))| &< \varepsilon, \\ |I_k(x_l(n))-I_k(x(n))| &< \varepsilon \end{aligned}$$

for any $n\in J$, $k\in\{1,\dots,p\}$, and for any $l\ge\delta$.

We have

$$
\begin{aligned}
|Fx_l(n) - Fx(n)| &= \Bigg| \frac{g(x_l(0), x_l(T)) - g(x(0), x(T))}{M - N(1-c)^T} \\
&\quad + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\left((f(j, x_l(j)) - f(j, x(j))) + c(x_l(j) - x(j))\right) \\
&\quad + \sum_{0<n_k\leq T-1} G(n, n_k)\left(c(x_l(n_k) - x(n_k)) + (I_k(x_l(n_k)) - I_k(x(n_k)))\right)\Bigg| \\
&\leq \frac{|g(x_l(0), x_l(T)) - g(x(0), x(T))|}{M - N(1-c)^T} \\
&\quad + \sum_{j=0, j\neq n_k}^{T-1} G(n,j)\left(|f(j, x_l(j)) - f(j, x(j))| + c|x_l(j) - x(j)|\right) \\
&\quad + \sum_{0<n_k\leq T-1} G(n, n_k)\left(c|x_l(n_k) - x(n_k)| + |I_k(x_l(n_k)) - I_k(x(n_k))|\right) \\
&< \frac{\varepsilon}{M - N(1-c)^T} \\
&\quad + \sum_{j=0, j\neq n_k}^{T-1} \frac{M+N}{M - N(1-c)^T}\left(\varepsilon + c\,\varepsilon\right) \\
&\quad + \sum_{0<n_k\leq T-1} \frac{M+N}{M - N(1-c)^T}\left(\varepsilon + c\,\varepsilon\right) \\
&\leq \varepsilon\Bigg(\frac{1}{M - N(1-c)^T} \\
&\quad +2T\frac{M+N}{M - N(1-c)^T}\left(1+c\right)\Bigg), \quad n \in J, \quad l \geq \delta.
\end{aligned}
$$

This completes the proof. □

Lemma 3.6. *Suppose that* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$ *hold. For* $x \in \mathcal{C}(J, \mathbb{R})$, $\|x\| \leq B$, *we have*

$$
Fx(n) \leq B_1, \quad |\Delta Fx(n)| \leq 2B_1, \quad n \in J.
$$

Proof: We have

$$
\begin{aligned}
Fx(n) &\leq \frac{D\left(1 + B^{p_2} + B^{p_3}\right)}{M - N(1-c)^T} \\
&\quad + \sum_{j=0, j\neq n_k}^{T-1} \frac{M+N}{M - N(1-c)^T}\left(D\left(1 + B^{p_1} + B^{p_4}\right) + cB\right) \\
&\quad + \sum_{0<n_k\leq T-1} \frac{M+N}{M - N(1-c)^T}\left(D\left(1 + B^{p_1} + B^{p_4}\right) + cB\right)
\end{aligned}
$$

$$\begin{aligned} &\le \frac{D\left(1+B^{p_2}+B^{p_3}\right)}{M-N(1-c)^T} \\ &\quad +2T\frac{M+N}{M-N(1-c)^T}\left(D(1+B^{p_1}+B^{p_4})+cB\right) \\ &= B_1, \quad n\in J. \end{aligned}$$

Next,

$$\begin{aligned} |\Delta Fx(n)| &= |Fx(n+1)-Fx(n)| \\ &\le Fx(n+1)+Fx(n) \\ &\le 2B_1, \quad n\in J. \end{aligned}$$

This completes the proof. □

3.2. Proof of the Main Result

Take $\epsilon>0$ arbitrarily. Let $E=\mathcal{C}(J,\mathbb{R})$ be endowed with the norm $\|x\|=\max_{n\in J}|x(n)|$, and

$$\begin{aligned} \mathcal{P} &= \{x\in E: x(n)\ge 0, \quad n\in J\}, \\ \Omega &= \mathcal{P}_{2B}=\{x\in\mathcal{P}:\|x\|<2B\}, \\ U &= \mathcal{P}_{B}=\{x\in\mathcal{P}:\|x\|<B\}. \end{aligned}$$

For $x\in E$, define the operators

$$\begin{aligned} T_1x(n) &= (1+\epsilon)x(n), \\ F_1x(n) &= -\epsilon Fx(n), \quad n\in J. \end{aligned}$$

Note that for any fixed point $x\in E$ of the operator T_1+F_1 we have that $x\in E$ and it is a solution of the BVP (1.1).

1. For $x,y\in E$, we have
$$\|(I-T_1)^{-1}x-(I-T_1)^{-1}y\|=\frac{1}{\epsilon}\|x-y\|,$$
i.e., $(I-T_1):E\to E$ is Lipschitz invertible with constant $\frac{1}{\epsilon}$.
2. According to the Arzelà-Ascoli compactness criteria, by Lemma 3.5 and Lemma 3.6, it follows that $F_1:\overline{U}\to E$ is a completely continuous operator. Therefore $F_1:\overline{U}\to E$ is a 0-set contraction.
3. Let $t\in[0,1]$ and $x\in\overline{U}$ be arbitrarily chosen. Then
$$z=tFx\in E$$
and
$$\begin{aligned} z(n) &\le tB_1 \\ &< tB \\ &\le B, \quad n\in J, \end{aligned}$$

i.e., $z \in \Omega$. Next,

$$\begin{aligned} tF_1x(n) &= -t\epsilon Fx(n) \\ &= -\epsilon z(n) \\ &= (I - T_1)z(n), \quad n \in J. \end{aligned}$$

Thus, $tF_1(\overline{U}) \subset (I - T_1)(\Omega)$.

4. Note that
$$(I - T_1)^{-1}0 = 0 \in U.$$

5. Assume that there are $x \in \partial U \bigcap \Omega$ and $\lambda \in [0, 1]$ such that
$$(I - T_1)x = \lambda F_1 x.$$
If $\lambda = 0$, then
$$0 = (I - T_1)x = -\epsilon x \quad \text{on} \quad J,$$
whereupon $x(n) = 0$, $n \in J$. This is a contradiction because $x \in \partial U$. Therefore $\lambda \in (0, 1]$. Let $n_1 \in J$ be such that $x(n_1) = B$. Then
$$\begin{aligned} (I - T_1)x(n_1) &= -\epsilon x(n_1) \\ &= -\epsilon B \\ &= -\epsilon\lambda Fx(n_1), \end{aligned}$$
whereupon
$$\begin{aligned} B &= \lambda Fx(n_1) \\ &\leq \lambda B_1 \\ &< \lambda B \\ &\leq B, \end{aligned}$$
i.e., $B < B$, which is a contradiction.

Consequently, from Proposition 2.3 and the existence property of the fixed point index, it follows that the operator $T_1 + F_1$ has a fixed point in U. Denote it by x. We have

$$0 \leq x(n) < B, \quad n \in J,$$

and $x \in E$ is a solution of the BVP (1.1).

4. Example

Let

$$D = \frac{1}{10^{10000}}, \quad B = 1, \quad p = 4, \quad p_1 = p_2 = p_3 = p_4 = 2, \quad T = 20,$$

and

$$a_1(n) = a_2(n) = a_3(n) = a_4(n) = \frac{1}{10^{10000}}, \quad n \in [0, 20], \quad b_1 = b_2 = b_3 = \frac{1}{10^{10000}},$$

$$n_1 = 1, \quad n_2 = 3, \quad n_3 = 7, \quad n_4 = 11,$$

and

$$N = 1, \quad c = \frac{1}{10^{10000}}, \quad M = 10^{100}.$$

Then

$$B_1 = \frac{\frac{3}{10^{10000}}}{10^{100} - \left(1 - \frac{1}{10^{10000}}\right)^{20}} + 40\frac{10^{100}+1}{10^{100} - \left(1 - \frac{1}{10^{10000}}\right)^{20}} \left(\frac{4}{10^{10000}}\right) < 1 = B$$

and the BVP

$$\begin{aligned} \Delta x(n) &= \frac{(x(n))^2}{10^{10000}(n^2+1)}, \quad n \in [0, 20], \\ \Delta x(n_k) &= \frac{(x(n_k))^2}{10^{10000}}, \quad k \in \{1, 2, 3, 4\}, \\ 10^{100}x(0) - x(20) &= \frac{(x(0))^2}{10^{10000}(1 + x(20) + (x(20))^2)} \end{aligned}$$

has a solution $x \in \mathcal{C}([0, 20] \cap \mathbb{N}, \mathbb{R})$ so that

$$0 \le x(n) < 1, \quad n \in \{0, 1, \ldots, 20\}.$$

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Lydia Bouchal,
Department of Mathematics,
University of Bejaia, Faculty of exacte sciences, Bejaia
Algeria.
E-mail address: `lydia.bouchal@univ-bejaia.dz`

and

Karima Mebarki,
Department of Mathematics,
University of Bejaia, Faculty of exacte sciences, Bejaia
Algeria.
E-mail address: `karima.mebarki@univ-bejaia.dz`

and

Svetlin Georgiev Georgiev,
Department of Differential Equations,
University of Sofia, Faculty of Mathematics and Informatics, Sofia
Bulgaria.
E-mail address: `svetlingeorgiev1@gmail.com`