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Multiplicative b-Generalized Derivations on Prime Ideals

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ABSTRACT: The aim of this manuscript is to investigate the relationship between the behaviour of multiplicative b-generalized derivations and the commutativity of quotient ring. In particular, we study certain algebraic identities like $F(z) \pm z \in P \ \forall \ z \in R$, without considering the primeness of R.

Key Words: Prime ring, prime ideal, derivation, multiplicative b-generalized derivation, martindale rings of quotients

Contents

1. Introduction

Throughout, R will be denoted as an associative ring and $Z(R)$ its center, unless otherwise stated. Q_{mr} and Q_s denotes the martindale right and symmetric ring of quotients of R, respectively. $C = Z(Q_{mr}) = Z(Q_s)$ is a center of Q_{mr} and Q_s and its also known as extended centroid of R. Since, $R \subseteq Q_{mr}$, and the overrings Q_{mr} of R is prime with the same center C. Also, if R is a prime ring then C is a field, converse is also true. For more information, we recommend the book [4]. For any $a, b \in R$, an ideal $P \neq R$ is called prime if $aRb \subseteq P$ we get either $a \in P$ or $b \in P$. If $P = (0)$, the ring R becomes prime and converse also holds. The commutator (Lie products) is represented by $[a, b] = ab-ba$, while the anti-commutator (Jordan products) is represented by $a \circ b = ab + ba \ \forall \ a, b \in R$. A map $g : R \to R$ given by $g(ab) = g(a)b + ag(b) \ \forall \ a, b \in R$ is termed as derivation if it is additive. I_a called an inner derivation of R induced by a, for all $x \in R$ we have $I_a(x) = [a, x]$. For all $a, b \in R$, an additive map $f : R \to R$ is termed as generalized derivation if a derivation g of R exists such that $f(ab) = f(a)b + ag(b)$. The example of generalized derivation is a map of the form $f(x) = ax + xb$ associated with inner derivation d induced by b. If the map f is not additive then we called this map as multiplicative generalized derivation.

Several authors have recently looked into the connection between the lattice of ring and various kind of maps on R. (see [5,10,17] for further references therein). For a subset $(0) \neq S$ and for any map f on S is called centralizing if $[f(a), a] \in Z(R) \ \forall \ a \in S$, and if $[f(a), a] = 0 \ \forall \ a \in S$ then it is called commuting on S. Posner [17] was the first who initiated the study of commuting and centralizing mappings, he shows that "A prime ring R must be commutative if R possesses a nonzero derivation d such that $[d(x), x] \in Z(R) \ \forall \ x \in R$". Following this, several authors have expanded Posner's findings in a variety of ways. In [12], Lanski generalizes the result of Posner to a Lie ideal. In addition, for commutativity in prime ring, in [3], Ashraf et al. shows that R must be commutative, If $G(xy) - xy \in Z(R)$ where G is a generalized derivation in a prime ring R. Various authors looked at a parallel condition in which they used generalized derivation or multiplicative generalized derivation instead of derivation d.

Recently, Almahdi et al. [1] were interested in the identities over prime rings R/P. In their theorem they states that "If $[[x, d(x)], y] \in P \ \forall \ x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative". The Posner's finding is the outcome of that result. In [9], Herstein proved that "R is commutative if a prime ring R

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with a characteristic other than two admits a nonzero derivation d such that $[d(x), d(y)] = 0$". In [15], Mamouni et al. goes on to generalize Herstein's earlier result in two ways. First, they used two derivation on more general form $[d_1(x), d_2(y)]$. Second, rather than 0 they assumes that $[d_1(x), d_2(y)]$ belong to a prime ideal P. In [16], Mamouni et al. takes a novel method, using generalized derivations on prime ideals P to study the algebraic identities without considering the primeness of R. They proved that "Let R be a ring, P be a prime ideal of R. If F is a generalized derivation of R associated with a derivation d such that $F(xy) \in Z(R/P)$ for all $x, y \in R$, then $F(R) \subset P$ or R/P is a commutative integral domain".

Let R be any ring and P be a prime ideal of R, we investigate the relationship between the commutativity and the lattice of the ring R/P without assuming primeness condition of ring R. In this manuscript, we generalize the result of Dhara and Ali *[8, Theorem 2.1]*. Precisely, for every $r \in R$ we consider the algebraic identities of the form $F(r) \pm r \in P$. Here, F is a multiplicative b-generalized derivation from $R \to Q_{mr}$ associated with any map $g : R \to R$. In 2014, Koşan and Lee [11] suggested the following new terminology. They define as "In a semiprime ring R, an additive mapping $F : R \to Q$ is called a left b-generalized derivation associated with derivation $d : R \to Q$ if $F(xy) = F(x)y + bxd(y)\ \forall\ x, y \in R$ and $b \in Q$", similarly they define right b-generalized derivation. Clearly, for $b = 1$, any generalized derivation is an b-generalized derivation. In the present manuscript, we have consider $F : R \to Q_{mr}$ (not necessarily additive) is called a multiplicative b-generalized derivation associated with any map (need not be additive) $g : R \to R$ satisfying $F(xy) = F(x)y + bxg(y)\ \forall\ x, y \in R$ and for fixed $b \in R \subset Q_s$. Koşan and Lee [11] were the first to establish the concept of b-generalized derivation. In [6,11,13,14] and references therein. For simplicity of notation, a multiplicative b-generalized derivation always means a multiplicative left b-generalized derivation with associated pair (b, d).

2. Example

Example 2.1 *For any $a, c \in R$, we defined a map F from R to R given by $F(s_1) = as_1 + bs_1c\ \forall\ s_1 \in R$ and d is a derivation from R to R (need not be additive) given by $d(s_2) = [s_2, c]\ \forall\ s_2 \in R$. Then, clearly we observe that the map $F(s_1s_2) = F(s_1)s_2 + bs_1d(s_2)\ \forall\ s_1, s_2 \in R$, is a b-generalized derivation.*

Example 2.2 *Let $F : R \to R$ be a map defined by $F(s_1) = as_1 + bd(s_1)$, where $d : R \to R$ is a multiplicative derivation and $0 \neq b \in R$ and for all $a, s_1 \in R$. Then, clearly observe that the map $F(s_1s_2) = F(s_1)s_2 + bs_1d(s_2)\ \forall\ s_1, s_2 \in R$, is a multiplicative b-generalized derivation.*

Example 2.3 *Let $R = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \Big| a, b, c \in \mathbb{Z} \right\}$ be a ring under usual matrix operations, where $\mathbb{Z}$ is the set of integers. F and d is a map from $R \to R$ such that $F\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $d\left(\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then it is easy to verify that F is a multiplicative b-generalized derivation associated with a multiplicative derivation d and for any $b \in R$.*

3. Main results

Throughout the paper, we use one of the properties of martindale right symmetric ring of quotient which states as follows: for any $q \in Q_s$, there exist a dense right ideal I such that $qI \cup Iq \subseteq R$. In our case, for $b \in R \subset Q_s$, we assume that, there exists a dense right ideal R such that $bR \cup Rb \subseteq R$ i.e., bs_1 or $s_1b \in R\ \forall\ s_1 \in R$. The general element of R/P will be written as $\bar{s_1}$ where $\bar{s_1} = s_1 + P\ \forall\ s_1 \in R$. In this section, we give some well known basic identities which will be used extensively in the forthcoming sections.

(*i*) $[s_1, s_2s_3] = s_2[s_1, s_3] + [s_1, s_2]s_3$.

(*ii*) $[s_1s_2, s_3] = s_1[s_2, s_3] + [s_1, s_3]s_2$.

(ii) $(s_1 \circ s_2 s_3) = (s_1 \circ s_2)s_3 - s_2[s_1, s_3] = s_2(s_1 \circ s_3) + [s_1, s_2]s_3$.

(iv) $(s_1 s_2 \circ s_3) = s_1(s_2 \circ s_3) - [s_1, s_3]s_2 = (s_1 \circ s_3)s_2 + s_1[s_2, s_3]$.

In 2010, Dhara [7] studied the identities of the form $F(r) \pm r = 0$ on semiprime rings having nonzero two sided ideal I, where F is a generalized derivation of R. In the above context, Ali et al. [2], studied the above mentioned case of Dhara in semiprime rings replacing two-sided ideal I by left sided ideal λ and generalized derivation with multiplicative (generalized)-derivation. Motivated by the above result, we present the following theorems. In addition, the following lemmas will be required.

Lemma 3.1 [1, Lemma 2.1] *"Let R be a ring and P be a prime ideal of R. If d is a derivation of R satisfies the condition $[d(s_1), s_1] \in P \ \forall \ s_1 \in R$, then $d(R) \subseteq P$ or R/P is commutative".*

Lemma 3.2 *If $[s_1, s_2] \in P \ \forall \ s_1, s_2 \in R$, then $\bar{s_1} \in Z(R/P)$ i.e., R/P is commutative.*

Proof: Since P is prime ideal, then it is abelian group also with binary operation addition. Then by using certain properties of group, we have

$$[s_1, s_2] \in P \text{ implies } [s_1, s_2] + P = P \ \forall \ s_1, s_2 \in R. \tag{3.1}$$

That is

$$s_1 s_2 - s_2 s_1 + P = P \text{ implies } s_1 s_2 + P = s_2 s_1 + P \ \forall \ s_1, s_2 \in R. \tag{3.2}$$

This implies that

$$(s_1 + P)(s_2 + P) = (s_2 + P)(s_1 + P) \text{ implies that } s_1 + P \in Z(R/P). \tag{3.3}$$

That is, $\bar{s_1} \in Z(R/P)$. Since every element of R/P belongs to its center. So, R/P is commutative. □

Theorem 3.1 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with map g satisfying $F([s_1, s_2]) \pm [s_1, s_2] \in P \ \forall \ s_1, s_2 \in R$ and $b \notin P$, then either $g(R) \subset P$ or $\overline{b} \in Z(R/P)$.*

Proof: First we consider $F([s_1, s_2]) + [s_1, s_2] \in P \ \forall \ s_1, s_2 \in R$. Since the map $G(r) = F(r) \pm r \ \forall \ r \in R$ is a multiplicative b-generalized derivation, then our hypothesis becomes

$$G([s_1, s_2]) \in P \ \forall \ s_1, s_2 \in R. \tag{3.4}$$

Replacing s_2 by $s_2 s_1$ in (3.4) and using the definition of multiplicative b-generalized derivation, we have

$$(G[s_1, s_2])s_1 + b[s_1, s_2]g(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.5}$$

Using (3.4) in (3.5), we obtain

$$b[s_1, s_2]g(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.6}$$

By the mention property of martindale right symmetric ring of quotient, we substitute s_2 by bs_2 in (3.6), we obtain

$$bb[s_1, s_2]g(s_1) + b[s_1, b]s_2 g(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.7}$$

By using (3.6) in (3.7), we get

$$b[s_1, b]s_2 g(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.8}$$

That is $b[s_1, b]Rg(s_1) \in P \ \forall \ s_1 \in R$, it gives that either $b[s_1, b] \in P$ or $g(s_1) \in P \ \forall \ s_1 \in R$. As a result, R is the union of two additive subgroups A and B, where

$$A = \{s_1 \in R \mid b[s_1, b] \in P\} \text{ and } B = \{s_1 \in R \mid g(s_1) \in P\}.$$

Since a group can not be union of two its proper subgroups, it gives that either $R = A$ or $R = B$. If $R = A$, then $b[s_1, b] \in P\ \forall\ s_1 \in R$. We replace s_1 by $s_1 s_2$ in $b[s_1, b] \in P$ and using it, we get $bs_1[s_2, b] \in P\ \forall\ s_1, s_2 \in R$. That is $bR[s_2, b] \in P\ \forall\ s_2 \in R$. Since $b \notin P$, it gives that $[s_2, b] \in P$. From Lemma 3.2, we get $\overline{b} \in Z(R/P)$. If $R = B$, then $g(s_1) \in P\ \forall\ s_1 \in R$, which is conclusion.

Similarly, we can prove the result for the case $F([s_1, s_2]) - [s_1, s_2] \in P\ \forall\ s_1, s_2 \in R$. □

Corollary 3.1 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with derivation d satisfying $F([s_1, s_2]) \pm [s_1, s_2] \in P\ \forall\ s_1, s_2 \in R$ and $b \notin P$, then either $d(R) \subset P$ or $\overline{b} \in Z(R/P)$.*

Theorem 3.2 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with map g satisfying $F(s_1 \circ s_2) \pm s_1 \circ s_2 \in P\ \forall\ s_1, s_2 \in R$ and $b \notin P$, then either $g(R) \subset P$ or $\overline{b} \in Z(R/P)$.*

Proof: First we consider the situation $F(s_1 \circ s_2) + s_1 \circ s_2 \in P\ \forall\ s_1, s_2 \in R$. Since the map $G(r) = F(r) \pm r\ \forall\ r \in R$ is a multiplicative b-generalized derivation, then our hypothesis gives

$$G(s_1 \circ s_2) \in P\ \forall\ s_1, s_2 \in R. \tag{3.9}$$

Replacing s_2 by $s_2 s_1$ in (3.9) and using the definition of multiplicative b-generalized derivation, we obtain

$$G(s_1 \circ s_2)s_1 + b(s_1 \circ s_2)g(s_1) \in P\ \forall\ s_1, s_2 \in R. \tag{3.10}$$

Using (3.9) in (3.10), we have

$$b(s_1 \circ s_2)g(s_1) \in P\ \forall\ s_1, s_2 \in R. \tag{3.11}$$

Using the property of martindale right symmetric ring of quotients, we substitute s_2 by bs_2 in (3.11), we obtain

$$bb(s_1 \circ s_2)g(s_1) + b[s_1, b]s_2 g(s_1) \in P\ \forall\ s_1, s_2 \in R. \tag{3.12}$$

By using (3.11) in (3.12), we find that

$$b[s_1, b]s_2 g(s_1) \in P\ \forall\ s_1, s_2 \in R. \tag{3.13}$$

The above equation (3.13) is same as equation (3.8) of Theorem 3.1. Then by using the same argument of Theorem 1.1, we get the required result.

□

Corollary 3.2 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with derivation d satisfying $F(s_1 \circ s_2) \pm s_1 \circ s_2 \in P\ \forall\ s_1, s_2 \in R$ and $b \notin P$, then either $d(R) \subset P$ or $\overline{b} \in Z(R/P)$.*

Theorem 3.3 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with map g satisfying $F(s_1 s_2) \pm s_1 s_2 \in P\ \forall\ s_1, s_2 \in R$ and $b \notin P$, then we have $g(R) \subset P$.*

Proof: Since the map $G(r) = F(r) \pm r\ \forall\ r \in R$ is a multiplicative b-generalized derivation, then our hypothesis gives

$$G(s_1 s_2) \in P\ \forall\ s_1, s_2 \in R. \tag{3.14}$$

Substituting s_2 by $s_2 s_3$ in (3.14) for all $s_3 \in R$, and using the definition of multiplicative b-generalized derivation, we get

$$G(s_1 s_2)s_3 + bs_1 s_2 g(s_3) \in P\ \forall\ s_1, s_2, s_3 \in R. \tag{3.15}$$

Using (3.14) in (3.15), we obtain

$$bs_1 s_2 g(s_3) \in P\ \forall\ s_1, s_2, s_3 \in R. \tag{3.16}$$

From equation (3.16), we get either $b \in P$ or $s_2 g(s_3) \in P$. Since $b \notin P$, it gives that $s_2 g(s_3) \in P\ \forall\ s_2, s_3 \in R$. As P is a prime ideal of R, the relation $s_2 g(s_3) \in P$ gives that $g(s_3) \in P\ \forall\ s_3 \in R$, we are done. □

Corollary 3.3 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with derivation d satisfying $F(s_1s_2) \pm s_1s_2 \in P \ \forall\ s_1, s_2 \in R$ and $b \notin P$, then we have $d(R) \subset P$.*

The following corollary is an immediate consequence of Theorem 3.3. For that, letting $P = (0)$, we get R as a prime ring which is also a semiprime ring and we consider $b = 1$. Then our map $F : R \to Q_{mr}$ becomes multiplicative generalized derivations of R associated with any map $g : R \to R$.

Corollary 3.4 *Let R be a semiprime ring and $F : R \to Q_{mr}$ be a multiplicative generalized derivation associated with map $g : R \to R$ such that $F(s_1s_2) \pm s_1s_2 = 0 \ \forall\ s_1, s_2 \in R$, then $Rg(R) = 0$, $F(s_1s_2) = F(s_1)s_2 \ \forall\ s_1, s_2 \in R$ and F is a commuting map on R.*

Proof: By equation (3.16) of Theorem 3.3, we get $s_1s_2g(s_3) = 0 \ \forall\ s_1, s_2, s_3 \in R$. Substituting s_1 by $s_2g(s_3)s_1$ in the previous equation, we get $s_2g(s_3)s_1s_2g(s_3) = 0 \ \forall\ s_1, s_2, s_3 \in R$, since R is a semiprime ring, then we get $s_2g(s_3) = 0$, that is, $Rg(R) = \{0\}$. Moreover, if $Rg(R) = \{0\}$, then by the definition of multiplicative generalized derivation we have $F(s_1s_2) = F(s_1)s_2 \ \forall\ s_1, s_2 \in R$, and by using the hypothesis we get $(F(s_1) - s_1)s_2 = 0 \ \forall\ s_1, s_2 \in R$. This implies that $s_2(F(s_1) - s_1)Rs_2(F(s_1) - s_1) = \{0\}$. Since R is semiprime, we get $s_2(F(s_1) - s_1) = 0 \ \forall\ s_1, s_2 \in R$. Thus, we have $(F(s_1) - s_1)s_2 = 0$ and $s_2(F(s_1) - s_1) = 0 \ \forall\ s_1, s_2 \in R$. In particular for $s_2 = s_1$, we conclude that $[F(s_1), s_1] = 0 \ \forall\ s_1 \in R$. So, F is a commuting map on R. □

Theorem 3.4 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with map g satisfying $[F(s_1), s_2] \pm [F(s_2), s_1] \in P \ \forall\ s_1, s_2 \in R$ and $b \notin P$, then either $g(R) \subset P$ or $\bar{b} \in Z(R/P)$.*

Proof: First we consider the situation

$$[F(s_1), s_2] + [F(s_2), s_1] \in P \ \forall\ s_1, s_2 \in R. \tag{3.17}$$

Substituting s_2 by s_2s_1 in (3.17), we observe that

$$[F(s_1), s_2s_1] + [F(s_2s_1), s_1] \in P \ \forall\ s_1, s_2 \in R. \tag{3.18}$$

Using the definition of multiplicative b-generalized derivation and identity of commutator in (3.18), we have

$$s_2[F(s_1), s_1] + [F(s_1), s_2]s_1 + [F(s_2), s_1]s_1 + [bs_2g(s_1), s_1] \in P \ \forall\ s_1, s_2 \in R. \tag{3.19}$$

By using our hypothesis in (3.19), the above relation gives

$$s_2[F(s_1), s_1] + [bs_2g(s_1), s_1] \in P \ \forall\ s_1, s_2 \in R. \tag{3.20}$$

Using the property of martindale right symmetric ring of quotient, we substitute s_2 by bs_2 in (3.20), we obtain

$$bs_2[F(s_1), s_1] + b[bs_2g(s_1), s_1] + [b, s_1]bs_2g(s_1) \in P \ \forall\ s_1, s_2 \in R. \tag{3.21}$$

Using (3.20) in (3.21), we find that

$$[b, s_1]bs_2g(s_1) \in P \ \forall\ s_1, s_2 \in R. \tag{3.22}$$

By using P is a prime ideal, equation (3.22) gives that either $[b, s_1]b \in P$ or $g(s_1) \in P \ \forall\ s_1 \in R$. By using same technique as we have done in Theorem 3.1, we get our conclusions

Similarly, we can prove the result for the case $[F(s_1), s_2] - [F(s_2), s_1] \in P \ \forall\ s_1, s_2 \in R$. □

Corollary 3.5 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with derivation d satisfying $[F(s_1), s_2] \pm [F(s_2), s_1] \in P \ \forall\ s_1, s_2 \in R$ and $b \notin P$, then either $d(R) \subset P$ or $\bar{b} \in Z(R/P)$.*

Theorem 3.5 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with map g satisfying $F([s_1, s_2]) \pm [F(s_1), s_2] \in P \ \forall \ s_1, s_2 \in R$ and $b \notin P$, then either $g(R) \subset P$ or $\overline{b} \in Z(R/P)$.*

Proof: First we consider that

$$F([s_1, s_2]) + [F(s_1), s_2] \in P \ \forall \ s_1, s_2 \in R. \tag{3.23}$$

Now we substitute s_2 by $s_2 s_1$ in (3.23), we get

$$F([s_1, s_2]s_1) + [F(s_1), s_2 s_1] \in P \ \forall \ s_1, s_2 \in R. \tag{3.24}$$

That is

$$F([s_1, s_2])s_1 + b[s_1, s_2]g(s_1) + s_2[F(s_1), s_1] + [F(s_1), s_2]s_1 \in P \ \forall \ s_1, s_2 \in R. \tag{3.25}$$

By using our hypothesis in (3.25), we obtain

$$b[s_1, s_2]g(s_1) + s_2[F(s_1), s_1] \in P \ \forall \ s_1, s_2 \in R. \tag{3.26}$$

Using the property of martindale right symmetric ring of quotient in (3.26), replacing s_2 by bs_2, we find that

$$bb[s_1, s_2]g(s_1) + b[s_1, b]s_2 g(s_1) + bs_2[F(s_1), s_1] \in P \ \forall \ s_1, s_2 \in R. \tag{3.27}$$

Using (3.26) in (3.27), we have

$$b[s_1, b]s_2 g(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.28}$$

The above equation is same as Theorem 3.1's equation (3.8), then by the same argument we get the required result.

Similarly, we can prove the result for the case $F([s_1, s_2]) - [F(s_1), s_2] \in P \ \forall \ s_1, s_2 \in R$.

□

Theorem 3.6 *Let R be a ring, P be a prime ideal of R. If R admits a multiplicative b-generalized derivation F associated with derivation d satisfying $F(s_1)F(s_2) \pm [s_1, s_2] \in P \ \forall \ s_1, s_2 \in R$ and $b \notin P$, then either R/P is commutative or $d(R) \subseteq P$.*

Proof: First we consider that

$$F(s_1)F(s_2) + [s_1, s_2] \in P \ \forall \ s_1, s_2 \in R. \tag{3.29}$$

Now we substitute s_2 by $s_2 s_3 \ \forall \ s_3 \in R$ in (3.29), we obtain

$$F(s_1)F(s_2)s_3 + F(s_1)bs_2 d(s_3) + s_2[s_1, s_3] + [s_1, s_2]s_3 \in P \ \forall \ s_1, s_2, s_3 \in R. \tag{3.30}$$

By using our hypothesis in (3.30), we have

$$F(s_1)bs_2 d(s_3) + s_2[s_1, s_3] \in P \ \forall \ s_1, s_2, s_3 \in R. \tag{3.31}$$

Now we put $s_3 = s_1$ in (3.31), we see that

$$F(s_1)bs_2 d(s_1) \in P \ \forall \ s_1, s_2 \in R. \tag{3.32}$$

Substituting s_3 by $s_3 r \ \forall \ r \in R$ in (3.31) and using it, we find that

$$F(s_1)bs_2 s_3 d(r) + s_2 s_3[s_1, r] \in P \ \forall \ r, s_1, s_2 \in R. \tag{3.33}$$

Replacing s_3 by $d(s_1)$ in above relation and using (3.32) , we find that

$$s_2 d(s_1)[s_1, r] \in P \ \forall \ r, s_1, s_2 \in R. \tag{3.34}$$

Substituting r by $rs_3 \ \forall \ s_3 \in R$ in (3.34) and using this relation, we have

$$s_2 d(s_1) r[s_1, s_3] \in P \ \forall \ r, s_1, s_2, s_3 \in R. \tag{3.35}$$

That is

$$s_2 d(s_1) R[s_1, s_3] \subseteq P \ \forall \ s_1, s_2, s_3 \in R. \tag{3.36}$$

Since P is prime, thus for each $s_1 \in R$ either $s_2 d(s_1) \in P$ or $[s_1, s_3] \in P$. Now let $R_1 = \{s_1 \in R \mid s_2 d(s_1) \in P\}$ and $R_2 = \{s_1 \in R \mid [s_1, s_3] \in P\}$. Then R_1 and R_2 are additive subgroups of R whose union is R. But a group cannot be the union of two of its proper subgroups. Hence, either $R_1 = R$ or $R_2 = R$. If $R_1 = R$, then we have $s_2 d(s_1) \in P$. We replace s_2 by $s_2 r \ \forall \ r \in R$ and in the light of primeness of P we get either $s_2 \in P$ or $d(s_1) \in P$. Since we know that $s_2 \notin P$ otherwise we get contradiction. So, finally we have $d(R) \subseteq P$. Next, if $R_2 = R$ then we have $[s_1, s_3] \in P \ \forall \ s_3 \in R$, then by using Lemma 3.2 we get R/P is commutative.

Similarly, we can prove the result for the case $F(s_1)F(s_2) - [s_1, s_2] \in P \ \forall \ s_1, s_2 \in R$. □

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