Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: `www.spm.uem.br/bspm`

(3s.) **v. 2026 (44)** : 1–14.
ISSN-0037-8712
doi:10.5269/bspm.62856

Controllability and Optimal Control for Fractional Neutral Evolution Systems With Caputo Derivative

Zoubida Ech-chaffani, Ahmed Aberqi, Touria Karite *

ABSTRACT: The purpose of this paper is to study the controllability of a class of fractional neutral evolution equations involving Caputo fractional derivative of order $q \in]0,1[$. First, we prove the result with the help of fractional calculus theory, semi-group theory and Banach fixed point theorem under some assumptions, then we concentrate on the determination of a control achieving this kind of controllability with minimum energy. An example is given to validate the obtained results.

Keywords: Controllability, fractional systems, neutral evolution systems, optimal control, minimum energy.

Contents

1. Introduction

In the past decades, the fractional evolution equations have attracted increasing interests and have been used as a valuable tool to model various real-world dynamic systems [1,2,3,4]. There has been a significant development in fractional evolution equations. For instance, the fundamental solutions of fractional evolution equations in a Banach space are presented in [5,6] and the mild solutions of non-local Cauchy problems for fractional evolution equation are obtained in [7], for more details on the investigation of fractional evolution equations, we refer the reader to [8,9,10] and the references therein.

The recent researches show that fractional differential equations accurately and perfectly can be applicable in many fields, such as, mathematical modelling of physical, engineering and biological phenomena, and also have motivated several researchers to explore theoretical as well as practical aspects of the subject.

In the last decade, many authors devoted their works to study the controllability for fractional differential equations since the controllability notion has extensive industrial and biological applications [11,12,13].

The concept of controllability has been extensively studied in the exact, approximate, complete and relative sense. Generally, it aims to show the existence of a control function that steers the solution of the system from its initial state to a final one, where the initial and final states may vary over the entire state [10,14,15,16,17,18]. Recently, attention has been paid to establish sufficient conditions for the existence and controllability of fractional differential equations (see for instance [19,20,21]).

Lions proposed the Hilbert space Uniqueness Method (HUM), as a tool for studying Hilbert spaces of controllable states for a variety of linear partial differential equations, notably the wave equation [22]. In [23], Komornik developed a general and constructive approach to improve the usual estimates of the exact controllability time. It was inspired by the estimation method set by Haraux [24]. Balachandran

---

* Corresponding author.

2020 *Mathematics Subject Classification:* 93B05, 49J20, 26A33, 34K40.
Submitted March 10, 2022. Published March 19, 2026

et al. [25] studied the controllability of fractional integro-differential systems in Banach spaces using semigroup theory and the fixed point theorem. Sakthivel et al. [26] have proved the controllability of nonlinear fractional systems by using the fixed point theorem. Mahmudov [27] derived the conditions of Approximate controllability for fractional Sobolev-type evolution equations in Banach spaces. Fudong Ge et al. [28] studied the regional controllability of the fractional order sub-diffusion systems with Caputo fractional derivative. More recently, Yong Zhou et al [7] obtained a new set of sufficient conditions for the exact controllability of a class of fractional systems by using contraction principle and the Shauder fixed point theorem. For more works about controllability for fractional system, we refer the reader to [29,30,31]. Inspired by the work of Yong Zhou, we study the controllability of a class of fractional neutral evolution equations in Caputo sense using fractional calculus theory, semi-group theory and Banach fixed point theorem under some assumptions, then we determine the optimal control achieving exact controllability with minimum energy.

In this paper, we assume that X is a Banach space with the norm $|\cdot|$. Let $J = [0, T] \subset \mathbb{R}$. Denote $C(J, X)$ to be the Banach space of continuous functions from J into X with the norm $\|x\| = \sup_{t \in J} |x(t)|$, where $x \in C(J, X)$.

Let $a > 0$ and $\mathcal{C} = C([-a, 0], X)$ be the space of continuous functions from $[-a, 0]$ into X. For any element $z \in \mathcal{C}$, define the norm $\|z\|_\infty = \sup_{v \in [-a,0]} |z(v)|$.

The contribution of this paper is to study the exact controllability and the optimal control of a class of fractional neutral evolution equations with non-local conditions:

$$\begin{cases} {}^cD_t^q \left[y(t) - h\left(t, y_t\right)\right] = \mathbf{A}y(t) + f(t, y_t) + \mathbf{B}u(t) & t \in J = (0, T], \\ y_0(v) + (g(y_{t_1}), \cdots, (y_{t_n}))(v) = \varphi(v), & v \in [-a, 0], \end{cases} \tag{1.1}$$

where ${}^CD_t^q$ denotes Caputo fractional derivative of order $q \in (0, 1)$, $0 < t_1 < \cdots < t_n \leq T$, the final time $T > 0$; the dynamic of system (1.1) $\mathbf{A} : D(\mathbf{A}) \subseteq X \to X$ is a closed linear operator with dense domain $D(\mathbf{A})$ and generates a compact and uniformly bounded C_0 semi-group $\{T(t)\}_{t \geq 0}$ on a Banach space X. The control function $u(\cdot)$ is given in $L^2(0, T; U)$; U is a reflexif Banach space. The control operator $\mathbf{B} \in \mathcal{L}(U, X)$ is a linear continuous bounded operator from U to X, i.e., there exists a positive constant $\mathcal{M}_1$ with

$$|\mathbf{B}| \leq \mathcal{M}_1. \tag{1.2}$$

$f, h : [0, T] \times \mathcal{C} \to X$ and $g : \mathcal{C}^n \to \mathcal{C}$ are given functions satisfying some assumptions, $\varphi \in \mathcal{C}$ and define y_t by $y_t(v) = y(t + v)$, for $v \in [-a, 0]$.

In the case where $y(0) = y_0$ and $f = 0$ the exact controllability and the optimal control problem of (1.1) has been studied (see for instance [32]). The remaining of this paper is organized as follows: some preliminary results are recalled in the next section, which will be used throughout this paper. In section 3, our main results on the controllability of system (1.1) are presented and proved and in section 4, the existence of an optimal control is proven for a closed convex set of admissible controls. an example is presented to illustrate our main results in the last section. And we end up by some conclusions and future perspectives.

2. Preliminaries and Background Material

In this section, we state some definitions, notations and preliminary results about fractional calculus needed to establish our main results.

Throughout this paper, let A be the infinitesimal generator of a compact and uniformly bounded C_0 semi-group $\{T(t)\}_{t \geq 0}$ of operators on X. Let $0 \in \rho(\mathbf{A})$, where $\rho(\mathbf{A})$ is the resolvent set of $\mathbf{A}$. Then for $0 \leq \eta \leq 1$, it is possible to define the fractional power $\mathbf{A}^\eta$ as a closed linear operator on it's domain $D(\mathbf{A}^\eta)$. For a compact semi-group $\{T(t)\}_{t \geq 0}$, the following properties will be used.

(i) There is a $\mathcal{M}_T \geq 1$ such that

$$\mathcal{M}_T = \sup_{t \geq 0} |T(t)|, \tag{2.1}$$

(ii) for any $\eta \in (0,1]$, there is a positive constant C_η such that

$$|\mathbf{A}^\eta T(t)| \leq \frac{C_\eta}{t^\eta}, \qquad 0 \leq t \leq T \tag{2.2}$$

Some basic definitions and properties about fractional calculus theory are needed and used further in this paper. For more details, see [32,33].

Definition 2.1 *[33] The left-sided Caputo fractional derivative of order $q > 0$ of a function $z \in L^1([0,T])$ is*

$$ {}_0^C D_t^q z(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} \frac{d^n}{ds^n} z(s) ds, \tag{2.3}$$

where $t \geq 0$, $n-1 < q < n$, $n \in \mathbb{N}$, and $\Gamma(\cdot)$ is the Euler gamma function.

Now, using the probability density function and its Laplace transform developed in [7] (see also [34,35]), we can give the following definition of mild solutions to system (1.1).

Definition 2.2 *[7] For $t \in]0,T]$ and any given $u \in U$, a function $y \in C(0,T;X)$ is said to be a mild solution of system (1.1), if*

$$\begin{aligned} y_u(t) = {} & S_q(t)\left[\varphi(0) - (g(y_{t_1}, \cdots, y_{t_n}))(0) - h(0, y_0)\right] + h(t, y_t) + \int_0^t (t-s)^{q-1} A K_q(t-s) h(s, y_s)\,\mathrm{d}s \\ & + \int_0^t (t-s)^{q-1} K_q(t-s) f(s, y_s) \mathrm{d}s + \int_0^t (t-s)^{q-1} K_q(t-s) B u(s) \mathrm{d}s \quad t \in [0,T]. \end{aligned}$$

Where $S_q(\cdot)$ and $K_q(\cdot)$ are called characteristic solution operators and given by

$$S_q(t) = \int_0^\infty \phi_q(\theta) T(t^q\theta)\ \mathrm{d}\theta \quad \text{and} \quad K_q(t) = q\int_0^\infty \theta\phi_q(\theta) T(t^q\theta)\ \mathrm{d}\theta.$$

Here

$$\phi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\psi_q(\theta^{-\frac{1}{q}}).$$

Note that ψ_q is a probability density function defined as

$$\psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in (0,\infty).$$

Furthere, we have

$$\int_0^\infty \psi_q(\theta)d\theta = 1 \ \text{and} \ \int_0^\infty \theta^\lambda \phi_q(\theta) d\theta = \frac{\Gamma(1+\lambda)}{\Gamma(1+q\lambda)}, \qquad \lambda \in [0,1].$$

The following properties of $S_q(\cdot)$ and $K_q(\cdot)$ will be used throughout this paper.

Lemma 2.1 *[7].*

1. *For any $t \geq 0$, $S_q(t)$ and $K_q(t)$ are linear bounded operators, i.e for any $y \in X$,*

   $$|S_q(t)y| \leq \mathcal{M}_T|y| \ \text{and} \ |K_q(t)y| \leq \frac{q\mathcal{M}_T}{\Gamma(1+q)}|y|,$$

   where $\mathcal{M}_T = \sup_{t\geq 0}|T(t)|$.

2. *For $t > 0$, $S_q(t)$ and $K_q(t)$ are all compact operators if, $T(t)$ is compact.*

Lemma 2.2 *[7] For any $y \in X$, $\beta \in (0,1)$ and $\eta \in (0,1]$, we have*

(i) $\mathbf{A}K_q(t)y = \mathbf{A}^{1-\beta}K_q(t)\mathbf{A}^\beta y, \quad 0 \leq t \leq T,$

(ii) $|\mathbf{A}^\eta K_q(t)| \leq \dfrac{qC_\eta}{t^{q\eta}}\dfrac{\Gamma(2-\eta)}{\Gamma(1+q(1-\eta))}, \quad 0 < t \leq T.$

3. Controllability Results

In this section, we will define the notion of exact controllability for system (1.1), and we will give the necessary hypotheses to prove our main results cited in theorem (3.1).

Definition 3.1 *[36] System (1.1) is said to be exactly controllable in X on $[0,T]$, if for any given initial state $y_0 \in X$ and any given final state $y_d \in X$, there exists a control $u(\cdot) \in L^2(0,T;U)$ such that the mild solution $y \in C(0,T;X)$ of system (1.1) satisfies $y_u(T) = y_d$.*

In the next, we adopt the following assumptions.

- $(\mathcal{A}_1)$: $T(t)$ is a compact operator for every $t > 0$.
- $(\mathcal{A}_2)$: $h : [0,T] \times C(0,T;X) \to X$ is continuous function and there exists a constant $\beta \in]0,T[$ and $c_1, c_2 > 0$ such that $h \in D(\mathbf{A}^{\beta})$. Moreover, for any $z, y \in C(0,T;X)$, $t \in [0,T]$, the function $\mathbf{A}^{\beta}h(\cdot,z)$ is strongly measurable and $\mathbf{A}^{\beta}h(t,\cdot)$ satisfies the Lipschitz condition.

$$\left|\mathbf{A}^{\beta}h(t,z) - \mathbf{A}^{\beta}h(t,y)\right| \leq c_1\|z-y\|, \tag{3.1}$$

  and the inequality

$$\left|\mathbf{A}^{\beta}h(t,z)\right| \leq c_2\left(\|z\|+1\right). \tag{3.2}$$

- $(\mathcal{A}_3)$: The function $f : J \times X \to X$ is continuous and there exists a constant $L_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq L_f\|x-y\|, \quad \textit{for each } \ t \in J \ \textit{ and for all } \ x,\ y \in X,$$

  and $M_f = \sup\limits_{t \in J}|f(t,0)|$.

- $(\mathcal{A}_4)$: There exists a constant $L > 0$ such that

$$\|g(x_{t_1}, \cdots, x_{t_n}) - g(y_{t_1}, \cdots, y_{t_n})\|_{\infty} \leq L\|x-y\| \qquad \textit{for } \ x,y \in C([-a,T],X).$$

- $(\mathcal{A}_5)$: g is continuous and there exist positive constants L_1, L_1' such that

$$\|g(x_{t_1}, \cdots, x_{t_n})\|_{\star} \leq L_1\|x\| + L_1', \quad \textit{for all } \ x \in C([-a,T],X).$$

Let $H_q : L^2(J,U) \to X$ be the linear operator defined by

$$H_q u = \int_0^T (T-s)^{q-1}K_q(T-s)\mathbf{B}u(s)\mathrm{d}s,$$

has a invertible operator H_q^{-1} (see [37], [41]) which takes values in $\left(L^2(J,U)\Big/\ker H_q\right)$; and there exists a positive constant $\mathcal{M}_2 \geq 0$ such that

$$\left|H_q^{-1}\right|_{\mathcal{L}\left(X, L^2(J,U)\big/\ker H_q\right)} \leq \mathcal{M}_2. \tag{3.3}$$

For each positive constant $k \geq 0$, let $B_k = \{x \in C(0,T;X) : \|x\| \leq k\}$ which is clearly a bounded closed and convex subset in $C(0,T;X)$.

Theorem 3.1 *If $(\mathcal{A}_1) - (\mathcal{A}_5)$ are satisfied, then the evolution system (1.1) is controllable in $[0,T]$ providing that*

$$\Bigg[\mathcal{M}_T L + (\mathcal{M}_T+1)c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f + \frac{\mathcal{M}_1\mathcal{M}_2 M}{\Gamma(1+q)}T^q\Big[\mathcal{M}_T(L + c_1|\mathbf{A}^{-\beta}|)$$
$$+ c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\Big]\Bigg] < 1.$$

Proof: For an arbitrary function y, we define the control $u_y(t)$ by

$$\begin{aligned} u_y(t) = H_q^{-1}\Big[& y_d - S_q(T)\left[\varphi(0) - (gy_{t_1}, \cdots, y_{t_n}))(0) - h\left(0, y_0\right)\right] - h\left(T, y_T\right) \\ & - \int_0^T (T-s)^{q-1}\mathbf{A}K_q(T-s)h\left(s, y_s\right)\mathrm{d}s - \int_0^T (T-s)^{q-1}K_q(T-s)f\left(s, y_s\right)\mathrm{d}s\Big](t). \end{aligned} \tag{3.4}$$

In the following, we will prove that, the operator $\mathcal{G} : C(J, X) \to C(J, X)$ defined by

$$\begin{aligned} (\mathcal{G}y)(t) = {} & S_q(t)\left[\varphi(0) - (g(y_{t_1}, \cdots, y_{t_n}))(0) - h\left(0, y_0\right)\right] + h\left(t, y_t\right) + \int_0^t (t-s)^{q-1}\mathbf{A}K_q(t-s)h\left(s, y_s\right)\mathrm{d}s \\ & + \int_0^t (t-s)^{q-1}K_q(t-s)f\left(s, y_s\right)\mathrm{d}s + \int_0^t (t-s)^{q-1}K_q(t-s)\mathbf{B}u_y(s)\mathrm{d}s, \quad t \in [0, T] \end{aligned}$$

has a fixed point y for the control u_y that steers system (1.1) from the initial state y_0 to y_d in time T. From (1.2), (3.3), we have

$$\begin{aligned} |\mathbf{B}u_y(t)| \leq \mathcal{M}_1\mathcal{M}_2\Bigg(& |y_d| + |S_q(T)|\Big[|\varphi(0)| + \big|g(y_{t_1}, \cdots, y_{t_n})(0)\big| + \big|\mathbf{A}^{-\beta}A^{\beta}h(0, y_0)\big|\Big] + |\mathbf{A}^{-\beta}\mathbf{A}^{\beta}h(T, y_T)| \\ & + \Big|\int_0^T (T-s)^{q-1}\mathbf{A}K_q(T-s)h(s, y_s)ds\Big| + \Big|\int_0^T (T-s)^{q-1}K_q(T-s)f(s, y_s)ds\Big|\Bigg). \end{aligned}$$

According to lemma (2.1) and (3.2), we have

$$\begin{aligned} |\mathbf{B}u_y(t)|\mathcal{M}_1\mathcal{M}_2\Bigg(& |y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + \|g(y_{t_1}, \cdots, y_{t_n})\|_\infty + \Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta}|\Big] + \Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta}| \\ & + \int_0^T |(T-s)^{q-1}\mathbf{A}^{1-\beta}K_q(T-s)\mathbf{A}^{\beta}h(s, y_s)|ds + \int_0^T |(T-s)^{q-1}K_q(T-s)[f(s, y_s) - f(s, 0)]ds \\ & + \int_0^T |(T-s)^{q-1}K_q(T-s)f(s, 0)]|ds\cdot \end{aligned}$$

Bylemma (2.1), (ii) of lemma (2.2), and the condition $(\mathcal{A}_4)$, it follows that

$$\begin{aligned} |Bu_y(t)| \leq \mathcal{M}_1\mathcal{M}_2\Bigg(& |y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1\|y\| + L_1' + \Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta|}\Big] + \Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta}| \\ & + \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}c_2\Big(\|y\| + 1\Big)\int_0^T (T-s)^{q\beta-1}ds + \frac{q\mathcal{M}_T}{\Gamma(1+q)}\int_0^T (T-s)^{q-1}|f(s, y_s) - f(s, 0)|ds \\ & + \frac{q\mathcal{M}_T}{\Gamma(1+q)}\int_0^T (T-s)^{q-1}|f(s, 0)|ds\cdot \end{aligned}$$

By using the condition $(\mathcal{A}_3)$, we get

$$\begin{aligned} |\mathbf{B}u_y(t)| \leq \mathcal{M}_1\mathcal{M}_2\Bigg(& |y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1\|y\| + L_1'\Big] + \Bigg(\mathcal{M}_T + 1\Bigg)\Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta}|\Big] + \\ & + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}\Big(\|y\| + 1\Big)c_2T^{q\beta} + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q\Big(L_f\|y\| + M_f\Big)\Bigg). \end{aligned}$$

Now, to prove that $\mathcal{G}$ has a unique fixed point on B_k, we will proceed in two steps.
Step I. $\mathcal{G}y \in B_k$ whenever $y \in B_k$.

For any fixed $y \in B_k$ and $0 \leq t \leq T$, we get that

$$\begin{aligned}
|(\mathcal{G}y)\,(t)| \leq{}& |S_q(t)[\varphi(0) - (g(y_{t_1}, \cdots, y_{t_n}))(0) - h(0, y_0)]| + |h(t, y_t)| + |\int_0^t (t-s)^{q-1}\mathbf{A}K_q(t-s)h(s, y_s)ds| \\
& |\int_0^t (t-s)^{q-1}K_q(t-s)f(s, y_s)ds| + |\int_0^t (t-s)^{q-1}K_q(t-s)Bu_y(s)ds| \\
& \leq |S_q(t)|\Big[|\varphi(0)| + \big|g(y_{t_1}, \cdots, y_{t_n})(0)\big| + \big|\mathbf{A}^{-\beta}\mathbf{A}^{\beta}h(0, y_0)\big|\Big] + |\mathbf{A}^{-\beta}\mathbf{A}^{\beta}h(t, y_t)| \\
& + \int_0^t |(t-s)^{q-1}\mathbf{A}^{1-\beta}K_q(t-s)\mathbf{A}^{\beta}h(s, y_s)|ds + \int_0^t (t-s)^{q-1}K_q(t-s)|f(s, y_s) - f(s, 0)|ds \\
& + \int_0^t (t-s)^{q-1}K_q(t-s)|f(s, 0)|ds + \int_0^t (t-s)^{q-1}K_q(t-s)|\mathbf{B}u_y(s)|ds\cdot
\end{aligned}$$

Lemma (2.1), (3.2), (ii) of lemma (2.2), the condition $(\mathcal{A}_3)$ and the condition $(\mathcal{A}_4)$, yield to

$$\begin{aligned}
|\mathcal{G}(t)| \leq{}& \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1 k + L_1' + \Big(k+1\Big)c_2|A^{-\beta}|\Big] + \Big(k+1\Big)c_2|A^{-\beta}| \\
& + \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}c_2\Big(k+1\Big)\int_0^T (T-s)^{q\beta-1}ds + \frac{q\mathcal{M}_T}{\Gamma(1+q)}L_f k\int_0^T (T-s)^{q-1}ds \\
& + \frac{q\mathcal{M}_T}{\Gamma(1+q)}M_f\int_0^T (T-s)^{q-1}ds + \frac{q\mathcal{M}_T}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}|\mathbf{B}u_y|ds.
\end{aligned}$$

However,

$$\begin{aligned}
|\mathbf{B}u_y(t)| \leq \mathcal{M}_1\mathcal{M}_2\Bigg(&|y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1\|y\| + L_1'\Big] + \Bigg(\mathcal{M}_T + 1\Bigg)\Big(\|y\| + 1\Big)c_2|\mathbf{A}^{-\beta}|\Big] + \\
& + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}\Big(\|y\| + 1\Big)c_2T^{q\beta} + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q\Big(L_f\|y\| + M_f\Big)\Bigg).
\end{aligned}$$

As $y \in B_k$, we obtain

$$\begin{aligned}
|\mathbf{B}u_y(t)| \leq \mathcal{M}_1\mathcal{M}_2\Bigg(&|y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1 k + L_1'\Big] + \Bigg(\mathcal{M}_T + 1\Bigg)\Big(k + 1\Big)c_2|\mathbf{A}^{-\beta}|\Big] + \\
& + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}\Big(k + 1\Big)c_2T^{q\beta} + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q\Big(L_f k + M_f\Big)\Bigg).
\end{aligned}$$

Let

$$\begin{aligned}
C' = \mathcal{M}_1\mathcal{M}_2\Bigg(&|y_d| + \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1 k + L_1'\Big] + \Bigg(\mathcal{M}_T + 1\Bigg)\Big(k + 1\Big)c_2|\mathbf{A}^{-\beta}|\Big] \\
& + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}\Big(k + 1\Big)c_2T^{q\beta} + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q\Big(L_f k + M_f\Big)\Bigg).
\end{aligned}$$

So

$$\begin{aligned}
|(\mathcal{G}y)\,(t)| \leq{}& \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1 k + L_1' + \Big(k+1\Big)c_2|A^{-\beta}|\Big] + \Big(k+1\Big)c_2|\mathbf{A}^{-\beta}| \\
& + \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}c_2\Big(k+1\Big)\int_0^T (T-s)^{q\beta-1}ds + \frac{q\mathcal{M}_T}{\Gamma(1+q)}L_f k\int_0^T (T-s)^{q-1}ds \\
& + \frac{q\mathcal{M}_T}{\Gamma(1+q)}M_f\int_0^T (T-s)^{q-1}ds + \frac{q\mathcal{M}_T}{\Gamma(1+q)}C'\int_0^t (t-s)^{q-1}ds. \\
& \leq \mathcal{M}_T\Big[\|\varphi\|_\infty + L_1 k + L_1' + \Big(k+1\Big)c_2|\mathbf{A}^{-\beta}|\Big] + \Big(k+1\Big)c_2|\mathbf{A}^{-\beta}| \\
& + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_2\Big(k+1\Big) + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f k + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q M_f + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q C'.
\end{aligned}$$

For any fixed $y \in B_k$, $0 \leq t \leq T$, and by choosing

$$\begin{aligned} k = {} & \mathcal{M}_T \Big[\|\varphi\|_\infty + L_1 k + L_1' + \Big(k+1\Big) c_2 |\mathbf{A}^{-\beta}| \Big] + \Big(k+1\Big) c_2 |\mathbf{A}^{-\beta}| \\ & + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)} T^{q\beta} c_2 \Big(k+1\Big) + \frac{\mathcal{M}_T}{\Gamma(1+q)} T^q L_f k + \frac{\mathcal{M}_T}{\Gamma(1+q)} T^q M_f + \frac{\mathcal{M}_T}{\Gamma(1+q)} T^q C'. \end{aligned}$$

We get that $(\mathcal{G}y)(t) \in B_k$ for all $y \in B_k$ and $0 \leq t \leq T$.
Step II. $\mathcal{G}$ is a contraction on B_k.
For any $y, z \in B_k$ and $0 \leq t \leq T$, according to (3.4), we have

$$\begin{aligned} |(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| \leq {} & \Big| S_q(t)[(g(y_{t_1}, \cdots, y_{t_n}))(0) - (g(z_{t_1}, \cdots, z_{t_n}))(0)] \Big| \\ & + \Big| S_q(t)[h(0, y_0) - h(0, z_0)] \Big| + \Big| h(t, y_t) - h(t, z_t) \Big| \\ & + \Big| \int_0^t (t-s)^{q-1} \mathbf{A} K_q(t-s) \Big(h\big(s, y(s)\big) - h\big(s, z(s)\big) \Big) \Big| ds \\ & + \Big| \int_0^t (t-s)^{q-1} K_q(t-s) [f(s, y_s) - f(s, z_s)] ds \Big| \\ & + \Big| \int_0^t (t-s)^{q-1} K_q(t-s) \mathbf{B} H_q^{-1} \Big[S_q(T) \Big[(g(z_{t_1}, \cdots, z_{t_n}))(0) - (g(y_{t_1}, \cdots, y_{t_n}))(0) \Big] \\ & + h(0, z_0) - h(0, y_0) \Big] + h(T, y_T) - h(T, z_T) + \int_0^T (T-\tau)^{q-1} \times \mathbf{A} K_q(T-\tau) \\ & \Big(h\big(\tau, y(\tau)\big) - h\big(\tau, z(\tau)\big) \Big) d\tau + \int_0^T (T-s)^{q-1} K_q(T-\tau) \Big[f(\tau, y_\tau) - f(\tau, z_\tau) \Big] d\tau \Big] (s) \Big| ds. \end{aligned}$$

In view of lemma (2.1), (i) of lemma (2.2) and $(\mathcal{A}_3)$, we get

$$\begin{aligned} |(\mathcal{G}y)(t) - (\mathcal{G}z)(t)| \leq {} & \mathcal{M}_T \|g(y_{t_1}, \cdots, y_{t_n}) - g(z_{t_1}, \cdots, z_{t_n})\|_\infty + \mathcal{M}_T |A^{-\beta}| \Big| A^\beta h(0, y_0) - \mathbf{A}^\beta h(0, z_0) \Big| \\ & + |A^{-\beta}| \Big| \mathbf{A}^\beta h(t, y_t) - \mathbf{A}^\beta h(t, z_t) \Big| + \int_0^t (t-s)^{q-1} \Big| \mathbf{A}^{1-\beta} K_q(t-s) [\mathbf{A}^\beta h(s, y_s) - \mathbf{A}^\beta h(s, z_s)] \Big| ds \\ & + \frac{q\mathcal{M}_T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} L_f \|y - z\| ds + \frac{q\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \Big[\mathcal{M}_T \Big[\|g(y_{t_1}, \cdots, y_{t_n}) \\ & - g(z_{t_1}, \cdots, z_{t_n})\|_\infty + |\mathbf{A}^{-\beta}| \Big| \mathbf{A}^\beta h(0, z_0) - \mathbf{A}^\beta h(0, y_0) \Big| \Big] + |\mathbf{A}^{-\beta}| |\mathbf{A}^\beta h(T, y_T) - \mathbf{A}^\beta h(T, z_T)| \\ & + \int_0^T (T-\tau)^{q-1} \Big| \mathbf{A}^{1-\beta} K_q(T-\tau) \Big(\mathbf{A}^\beta h(\tau, y(\tau)) - \mathbf{A}^\beta h(\tau, z(\tau)) \Big) \Big| \\ & + \frac{q\mathcal{M}_T}{\Gamma(1+q)} \int_0^T (T-\tau)^{q-1} L_f \|y - z\| d\tau \Big] (s) ds. \end{aligned}$$

By (3.1), (ii) of lemma (2.2) and $(\mathcal{A}_4)$, one has

$$\begin{aligned}
|(\mathcal{G}y)(t)-(\mathcal{G}z)(t)| \leq{}& \mathcal{M}_T L\|y-z\| + c_1\mathcal{M}_T|A^{-\beta}|\|y-z\| + c_1|\mathbf{A}^{-\beta}|\|y-z\| + \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}\int_0^t (t-s)^{q\beta-1}c_1\|y-z\|ds \\
&+ \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\|y-z\| + \frac{q\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\Bigg[\mathcal{M}_T\Big[L+c_1|\mathbf{A}^{-\beta}|\Big]\|y-z\| + c_1|A^{-\beta}|\|y-z\| \\
&+ \frac{qC_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+q\beta)}\int_0^T (T-\tau)^{q\beta-1}c_1\|y-z\|d\tau + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\|y-z\|\Bigg]. \\
\leq{}& \mathcal{M}_T L\|y-z\| + (\mathcal{M}_T+1)c_1|\mathbf{A}^{-\beta}|\|y-z\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1\|y-z\| + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\|y-z\| \\
&+ \frac{\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\Gamma(1+q)}T^q\Big[\mathcal{M}_T\Big[L+c_1|\mathbf{A}^{-\beta}|\Big]\|y-z\| + c_1|\mathbf{A}^{-\beta}|\|y-z\| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1\|y-z\| \\
&+ \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q \L_f\|y-z\|\Big] \\
\leq{}& \Bigg[\mathcal{M}_T L + (\mathcal{M}_T+1)c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f \\
&+ \frac{\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\Gamma(1+q)}T^q\Big[\mathcal{M}_T(L+c_1|\mathbf{A}^{-\beta}|) \\
&+ c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\Big]\Bigg]\|y-z\|
\end{aligned}$$

And using theorem (3.1), we have

$$\begin{aligned}
&\Bigg[\mathcal{M}_T L + (\mathcal{M}_T+1)c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f + \frac{\mathcal{M}_1\mathcal{M}_2\mathcal{M}}{\Gamma(1+q)}T^q\Big[\mathcal{M}_T(L+c_1|\mathbf{A}^{-\beta}|) \\
&+ c_1|\mathbf{A}^{-\beta}| + \frac{C_{1-\beta}\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}T^{q\beta}c_1 + \frac{\mathcal{M}_T}{\Gamma(1+q)}T^q L_f\Big]\Bigg] < 1.
\end{aligned}$$

Which yields to

$$|(\mathcal{G}y)(t)-(\mathcal{G}z)(t)| < \|y-z\|.$$

Hence $\mathcal{G}$ is a contraction on B_k. So the Banach fixed point theorem shows that $\mathcal{G}$ has a unique fixed point $y(\cdot)$ in $C(0,T;X)$. A direct calculation shows that u_y is the solution of the controllability problem of system (1.1), which completes the proof. □

4. Optimal Control Problem

The purpose in this section is to find the minimum energy control which steers system (1.1) from the initial state y_0 to a target state y_d at the final time T. The existence of an optimal control is proven for a closed convex set of admissible control.

Let $\mathcal{U}_{ad}$ be the set of admissible controls defined by

$$\mathcal{U}_{ad} = \left\{u \in L^2(0,T;U) : y_u(T) = y_d\right\}.$$

For a desired state y_d, the optimal control problem considered here consists in finding within $\mathcal{U}_{ad}$ a control that minimizes the following functional

$$J(u) = \frac{\beta}{2}\int_0^T \|y(t)-y_d\|_X^2\,dt + \frac{\varepsilon}{2}\int_0^T |u(t)|_U^2 dt,$$

where y is the mild solution of system (1.1), associated to u. ε and β are non-negative constants. The section aims at solving the following minimization problem:

$$\left\{\begin{array}{l}\inf J(u)\\ u\in\mathcal{U}_{ad}.\end{array}\right. \tag{4.1}$$

The following result shows the condition under which we get the control of minimum energy.

Theorem 4.1 *There exists* $u^* \in \mathcal{U}_{ad}$ *solution of problem* (3.1) *if,*

$$1 - c_1 \left|\mathbf{A}^{-\beta}\right| > 0.$$

Proof: Since system (1.1) is exactly controllable, the set $\{J(u) \mid u \in \mathcal{U}_{ad}\}$ is non-empty then it has a non-negative infimum.

Let $(u^p)_{p\in\mathbb{N}}$ be a minimizing sequence in $\mathcal{U}_{ad}$. Since $|u^p|^2 \leq \frac{2}{\varepsilon} J(u^p)$ then $(u^p)_{p\in\mathbb{N}}$ is bounded.
Then there exists a subsequence, still denoted $(u^p)_{p\in\mathbb{N}}$, such that

$$u^p \rightharpoonup u^*.$$

$\mathcal{U}_{ad}$ is closed and convex, then it is closed for the weak topology, which implies that $u^* \in \mathcal{U}_{ad}$.
Let y^p be the unique solution of system (1.1) associated to u^p, and y^* be the unique solution of system (1.1) associated to u^*, then

$$\begin{aligned}
|y^p(t) - y^*(t)| \leq{}& \left|S_q(t)[g(y^p_{t_1}, \cdots, y^p_{t_n})(0) - g(y^*_{t_1}, \cdots, y^*_{t_n})(0)]\right| + \left|S_q(t)[h(0, y^p_0) - h(0, y^*_0)]\right| \\
&+ \left|h\left(t, y^p(t)\right) - h\left(t, y^*(t)\right)\right| + \left|\int_0^t (t-s)^{q-1} \mathbf{A} K_q(t-s)[h\left(s, y^p(s)\right) - h\left(s, y^*(s)\right)]\mathrm{d}s\right| \\
&+ \left|\int_0^t (t-s)^{q-1} K_q(t-s)[f(s, y^p(s)) - f(s, y^*(s))]\right| + \left|\int_0^t (t-s)^{q-1} K_q(t-s) B[u^p(s) - u^*(s)]\mathrm{d}s\right| \quad t \in [0,T] \\
\leq{}& c_1 \left|\mathbf{A}^{-\beta}\right| |y^p(t) - y^*(t)| + \int_0^t (t-s)^{q-1} \left|\mathbf{A}^{1-\beta} K_q(t-s)[\mathbf{A}^{\beta} h\left(s, y^p(s)\right) - \mathbf{A}^{\beta} h\left(s, y^*(s)\right)]\right| \mathrm{d}s \\
&+ \left|\int_0^t (t-s)^{q-1} K_q(t-s) \mathbf{B}[u^p(s) - u^*(s)]\mathrm{d}s\right| \quad t \in [0,T].
\end{aligned} \tag{4.2}$$

According to lemmas (2.1) and (2.2) and $(\mathcal{A}_4)$, we obtain that

$$\begin{aligned}
|y^p(t) - y^*(t)| \leq{}& \mathcal{M}_T L |y^p(t) - y^*(t)| + \mathcal{M}_T c_1 |\mathbf{A}^{-\beta}| |y^p(t) - y^*(t)| + c_1 |\mathbf{A}^{-\beta}| |y^p(t) - y^*(t)| \\
&+ \frac{q C_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+q\beta)} \int_0^t (t-s)^{q\beta-1} c_1 |y^p(t) - y^*(t)| ds + \frac{q \mathcal{M}_T}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} L_f |y^p(t) - y^*(t)| ds \\
&+ \left|\int_0^t (t-s)^{q-1} K_q(t-s) \mathbf{B}[u^p(s) - u^*(s)]\mathrm{d}s\right| \\
\leq{}& \left(\mathcal{M}_T L + \mathcal{M}_T c_1 |\mathbf{A}^{-\beta}| + c_1 |\mathbf{A}^{-\beta}|\right) |y^p(t) - y^*(t)| + \int_0^t \left(\frac{q C_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+q\beta)} c_1 (t-s)^{q\beta-1}\right. \\
&\left.+ \frac{q \mathcal{M}_T}{\Gamma(1+q)} L_f (t-s)^{q-1}\right) |y^p(s) - y^*(s)| ds + \left|\int_0^t (t-s)^{q-1} K_q(t-s) B[u^p(s) - u^*(s)]\mathrm{d}s\right| \cdot
\end{aligned}$$

Let set

$$K^* = \mathcal{M}_T L + (\mathcal{M}_T + 1) c_1 |\mathbf{A}^{-\beta}| \cdot$$

Which yields to

$$\begin{aligned}
(1 - K^*) |y^p(t) - y^*(t)| \leq{}& +q \int_0^t \left(\frac{C_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+q\beta)} c_1 (t-s)^{q\beta-1} + \frac{\mathcal{M}_T}{\Gamma(1+q)} L_f (t-s)^{q-1}\right) |y^p(s) - y^*(s)| ds \\
&+ \left|\int_0^t (t-s)^{q-1} K_q(t-s) \mathbf{B}[u^p(s) - u^*(s)]\mathrm{d}s\right| \quad t \in [0,T].
\end{aligned} \tag{4.3}$$

Let set $K' = \dfrac{1}{1 - K^*}$. Then

$$\begin{aligned}
|y^p(t) - y^*(t)| \leq{}& \int_0^t q K' \left(\frac{C_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+q\beta)} c_1 (t-s)^{q\beta-1} + \frac{\mathcal{M}_T}{\Gamma(1+q)} L_f (t-s)^{q-1}\right) |y^p(s) - y^*(s)| ds \\
&+ K' \left|\int_0^t (t-s)^{q-1} K_q(t-s) \mathbf{B}[u^p(s) - u^*(s)]\mathrm{d}s\right| \quad t \in [0,T].
\end{aligned} \tag{4.4}$$

Using the Gronwall lemma, we obtain

$$\begin{aligned}|y^p(t)-y^*(t)| &\leq K'\left|\int_0^t (t-s)^{q-1}K_q(t-s)B[u^p(s)-u^*(s)]\mathrm{d}s\right| \exp\left(K'\frac{q\Gamma(1+\beta)}{\Gamma(1+q\beta)}C_{1-\beta}c_1\int_0^t (t-s)^{q\beta-1}\mathrm{d}s\right.\\ &\left.+K'\frac{q\mathcal{M}_T L_f}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}ds\right)\\ &\leq K'\left|\int_0^t (t-s)^{q-1}K_q(t-s)B[u^p(s)-u^*(s)]\mathrm{d}s\right| \exp\left(K'\frac{\Gamma(1+\beta)}{\beta\Gamma(1+q\beta)}C_{1-\beta}c_1T^{q\beta}\right.\\ &\left.+K'\frac{\mathcal{M}_T L_f}{\Gamma(1+q)}T^q\right).\end{aligned} \tag{4.5}$$

Now, by the weak convergence, $u^p \rightharpoonup u^*$ in $L^2(0,T,U)$ and using lemma (2.1), we obtain

$$\left|\int_0^t (t-s)^{q-1}K_q(t-s)\mathbf{B}[u^p(s)-u^*(s)]\mathrm{d}s\right| \leq \frac{q\mathcal{M}_1\mathcal{M}_T}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\left|u^p(s)-u^*(s)\right|_{L^2(0,T,U)}\mathrm{d}s. \tag{4.6}$$

Which gives

$$y^p \to y^* \quad \textit{strongly} \quad \textit{in} \quad L^2(0,T;X).$$

Hence,

$$\lim_{n\to\infty}\int_0^T \|y^p(t)-y_d\|_X^2\,dt = \int_0^T \|y(t)-y_d\|_X^2\,dt.$$

Using the lower semi-continuity of norms, the weak convergence of $(u^p)_n$ gives

$$|u^*| \leq \lim_{n\to\infty} \mathit{inf}\,|u^p|\,.$$

Therefore $J(u^*) \leq \lim\limits_{n\to\infty} \mathit{inf}\, J(u^p)$, leading to $J(u^*) = \inf\limits_{u\in\mathcal{U}_{ad}} J(u^p)$. And this establishes the optimality of u^*. □

5. Example

Let $X = (L^2[0,\pi];\mathbb{R})$. As an application of theorem (3.1), we consider the following fractional differential system of the form

$$\begin{cases} {}^cD_t^{1/2}\Big(y(t,z)-h(t,y_t)\Big) = \Delta y(t,z)+\lambda sin(y(t,z))+\mathbf{B}u(t,z) & t\in[0,1],\\ y(t,0)=y(t,1)=0, & t\in[0,1],\\ y(v,z)+(g(y_{t_1},\cdots,y_{t_n}))(v) = (\varphi(v))(z), & v\in[-a,0], \end{cases} \tag{5.1}$$

where ${}^cD_t^{1/2}$ is a Caputo fractional partial derivative, λ is a positive number, n is a positive integer, $0<t_0<t_1<\cdots<t_n<1$, $\varphi\in C([-a,0],X)$, that is $\varphi(v)\in X = L^2([0,1],\mathbb{R})$ and $y_t(v,z)=y(t+v,z)$, $t\in[0,1], v\in[-a,0]$, the function $h:[0,1]\times\mathcal{C}\to X$ is given by

$$h(t,y_t)(x) = \int_0^\pi U(x,z)u_t(v,z)dz.$$

Let

$$(U_h v')(x) = \int_0^\pi U(x,z)v'(z)dz, \quad \mathit{for} \quad v'\in X, \quad z\in[0,\pi].$$

The function $g:\mathcal{C}^n\to\mathcal{C}$ is given by

$$(g(y_{t_1},\cdots,y_{t_n}))(v) = \sum_{i=0}^n K_g y_{t_i}(v),$$

where $(K_g v')(x) = \displaystyle\int_0^\pi k(x,z)v'(z)dz$ and $k(z,y) \in L^2([0,1]\times[0,1],\mathbb{R})$. The function g verify hypotheses $\mathcal{A}_4$ and $\mathcal{A}_5$.
Let $\mathbf{A} : D(\mathbf{A}) \subseteq X \to X$ be defined by

$$\mathbf{A}x = -x'',$$

with the domain

$$D(\mathbf{A}) = \{x(\cdot) \in X; x, x' are \ \ absolutely \ \ continuous, x'' \in X, x(0) = x(1) = 0\}.$$

Clearly the operator $\mathbf{A}$ is self-adjoint, with compact resolvent and is the infinitesimal generator of an analytic semi-group $T(t)$. Furthermore, $\mathbf{A}$ has a discrete spectrum, the eigenvalues are $\lambda_n = n^2\pi^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $e_n(z) = \sqrt{\frac{2}{\pi}} sin(n\pi z)$, $\{e_i\}_{i=1}^{i=\infty}$ form an orthonormal basis of X. Then

$$\mathbf{A}x = -\sum_{n=1}^{n=\infty} \lambda_n (x, e_n)e_n \qquad x \in D(\mathbf{A}),$$

and

$$T(t)x(s) = \sum_{i=1}^{i=\infty} \exp(\lambda_i t)(x, e_i)e_i(s) \qquad x \in X.$$

$T(\cdot)$ is a uniformly stable semi-group and $\|T(t)\|_{L^2[0,\pi]} \leq \exp(-t)$.
We use the following properties

(i) $\mathbf{A}^{-\frac{1}{2}}x = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(x, e_n)e_n$, $\|\mathbf{A}^{\frac{-1}{2}}\|_{L^2[0,\pi]}$.

(ii) the operator $\mathbf{A}^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}x = \sum_{n=1}^{\infty} n(x, e_n)e_n$$

The space $D(\mathbf{A}^{\frac{1}{2}}) = \{x(\cdot) \in X, \displaystyle\sum_{n=1}^{\infty} n(x, e_n)e_n \in X\}$.
Clearly (2.1), (2.2), and $\mathcal{A}_1$ are satisfied.
Moreover we assume that the following conditions hold

(a) The function $U(x,z)$, $x, z \in [0,\pi]$ is measurable and

$$\int_0^\pi \int_0^\pi U^2(x,z)dz < \infty,$$

(b) The function $\partial x U(x,z)$ is measurable, $U(0,z) = U(\pi,z) = 0$, and

$$\left(\int_0^\pi \int_0^\pi \left(\partial x U(x,z)\right)^2 dzdx\right)^{1/2} < \infty.$$

Then (3.1) and (3.2) are satisfied.

The control operator $B : U \to X$ is defined by

$$\mathbf{B}u = \sum_{n=1}^{n=\infty} \lambda_n(\bar{u}, e_n)e_n,$$

where

$$\bar{u} = \begin{cases} u_n, & n = 1, 2, \cdots N \\ 0, & n = N+1, N+2, \cdots, \end{cases}$$

We see that B is a bounded continuous operator with $\mathcal{M}_1 = N\lambda_N$.
for $N \in \mathbb{N}$ and $H_{1/2} : L^2([0,1], U) \to X$ as follows:

$$H_{1/2}u = \int_0^1 (1-s)^{1/2} P_{1/2}(1-s)Bu(s)ds.$$

Then

$$\begin{aligned}
H_{1/2} &= \int_0^1 (1-s)^{1/2} \frac{1}{2} \int_0^\infty \theta\phi_{1/2}(\theta) T((1-s)^{1/2}\theta)\mathbf{B}u(s)d\theta ds \\
&= \int_0^1 (1-s)^{1/2} \frac{1}{2} \int_0^\infty \theta\phi_{1/2}(\theta) \sum_{i=1}^{i=\infty} \exp(\lambda_i (1-s)^{1/2}\theta)(\mathbf{B}u, e_i)e_i(s)d\theta ds \\
&= \int_0^1 (1-s)^{1/2} \sum_{i=1}^{\infty} \int_0^\infty \frac{1}{2}\theta\phi_{1/2}(\theta) \sum_{j=0}^{\infty} \frac{\lambda_i (1-s)^{1/2}\theta)^j}{j!}(u, e_i)e_i(s)d\theta ds \\
&= \int_0^1 (1-s)^{1/2} \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(\lambda_i (1-s)^{1/2})^j}{\Gamma\left(\frac{1}{2} + \frac{1}{2}j\right)}(u, e_i)e_i(s)ds \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \int_0^1 \frac{\lambda_i^j}{\Gamma\left(\frac{1}{2} + \frac{1}{2}j\right)}(1-s)^{\frac{1+j}{2}}(u, e_i)e_i(s) \\
&= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{2\lambda_i^j}{\Gamma\left(\frac{1}{2} + \frac{1}{2}j\right)(3+j)}(u, e_i)e_i(s)\cdot
\end{aligned}$$

Then the assumptions $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ hold, then by Theorem (3.1) the system (5.1)is exactly controllable. Furthermore if $c_1 < \dfrac{1}{\left|\mathcal{A}^{-\frac{1}{2}}\right|}$, where c_1 is a Lipshitz constant of the operator h, we deduce from theorem (4.1) the existence of an optimal control for (4.1).

Conclusion

In this paper, we have established the exact controllability and we have treated the control optimal problem for a class of fractional neutral evolution equations with non-local conditions involving Caputo fractional derivative of order $q \in]0,1[$. Our work can be extended to the case of enlarged controllability using different fractional derivatives such as in the works of Karite et al [38,39,40], where the authors resolved the minimum energy problem using RHUM and a penalization approaches. Many questions remain open, this is the case of regional controllability and regional optimal control for this kind of problem and impulsive linear/semilinear systems with/without delay.

References

1. M. Ichise, Y. Nagayanagi and T. Kojima, An analog simulation of non integer order transfer functions for analysis of electrode processes, J. Elect. Chemistry **13** (1971), 253–265.
2. J. A. Tenreiro Machado, I. S. Jesus, A. Galhano and J. Boaventura Cunha, Fractional order electromagnetics, Signal Process. **86** (2006), 2637–2644.
3. J. L. Battaglia, O. Cois, L. Puigsegur and A. Oustaloup, Solving an inverse heat conduction problem using a non-integer identified model, Int. J. Heat Mass Transfer **44** (2001), no. 14, 2671–2680.
4. G. Catania and S. Sorrentino, Analytical modelling and experimental identification of viscoelastic mechanical systems, in *Advances in fractional calculus*, 403-416, Springer, Dordrecht, 2007.
5. F. Mainardi, P. Paradisi, R. Gorenflo, Probability distributions generated by fractional diffusion equations, in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer, Dordrecht, 2000.
6. M.M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos Solitons Fractals 14 (2002) 433-440.

7. Y. Zhou, F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl. 59 (2010), pp. 1063-1077.

8. L. Kexue, P. Jigen, Fractional resolvents and fractional evolution equations, Applied Mathematics Letters 25, Issue 5 (2012), pp. 808-812.

9. J. R. Wang, Y. Zhou, A class of fractional evolution equations and optimal controls, Nonlinear Analysis: Real World Applications 12, Issue 1 (2011), pp. 262-272.

10. J.R. Wang, Y. Zhou, Complete controllability of fractional evolution systems, Commun. Nonlinear Sci. Numer. Simul. 17 (2012), pp. 4346-4355.

11. D. Baleanu, J. A. Tenreiro Machado, A. C. J. Luo, Fractional Dynamics and Control. Springer, New York (2012), https://doi.org/10.1007/978-1-4614-0457-6

12. A. Khan, H. Ullah, M. Zahri, U. Wannasingha Humphries, T. Karite, A. Yusuf, H. Ullah, M. Fiza, Stationary distribution and extinction of stochastic coronavirus (COVID-19) epidemic model, Fractals 30, No. 1 (2022) 2240050 (15 pages).

13. R. Zarin, A. Khan, M. Inc, U. Wannasingha Humphries, T. Karite Dynamics of five grade leishmania epidemic model using fractional operator with Mittag?Leffler kernel, Chaos, Solitons & Fractals 147, June 2021, 110985, ISSN 0960-0779 (2021).

14. K. Balachandran, J.P. Dauer, Controllability of nonlinear systems in Banach space: a survey, J. Optim. Theory Appl. 115 (2002) 7-28.

15. J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations. Springer (1971).

16. K. Surendra, N. Sukavanam, Approximate controllability of fractional order semilinear delayed control systems, Nonlinear Stud. 20 (2013) 73-83.

17. R. Sakthivel, Yong Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, Comput. Math. Appl. 62 (2011), pp. 1451-1459.

18. S.C. Ji, Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method, Appl. Math. Comput. 236 (2014) 43-53.

19. N. Abada, M. Benchohra, H. Hammouche, Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions, J. Differ. Equ. 246 (10) (2009), pp. 3834-3863.

20. M. Benchohra, E.P. Gatsori, S.K. Ntouyas, Controllability results for semilinear evolution inclusions with nonlocal conditions, J. Optim. Theory Appl. 118 (3) (2003), pp. 493-513.

21. R. Sakthivel, N.I. Mahmudov, J.J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, Appl. Math. Comput. 218 (20) (2012) pp. 10334-10340.

22. J. Lions, Exact controllability, stabilization and perturbations for distributed systems. SIAM Review 30, No 1 (1988), 1-68.

23. V. Komornik, Exact Controllability and Stabilization: The Multiplier Method. Université Louis Pasteur, Strasbourg (1993).

24. A. Haraux, Séries lacunaires et contrôle semi-interne des vibrations d?une plaque rectangulaire, J. Math. Pures Appl. 68 (1989), 457-465.

25. K. Balachandran, J. Y. Park, Controllability of fractional integrodifferential systems in Banach spaces, Nonlinear Analysis: Hybrid Systems 3 (2009) 363-367.

26. R. Sakthivel, N. I. Mahmudov, J. J. Nieto, Controllability for a class of fractional-order neutral evolution control systems, Applied Mathematics and Computation 218, No. 20 (2012), pp. 10334?10340.

27. N.I. Mahmudov, Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces. Abstr. Appl. Anal. (2013), pp. 1-9.

28. Fudong Ge, YangQuan Chen, Chunhai Kou, Igor Podlubny, On the regional controllability of the Sub-diffusion process with Caputo fractional derivative, fractional calculus and applied analysis, volume 19, number 5 (2016).

29. S. Rathinasamy, R. Yong, Approximate controllability of fractional differential equations with state-dependent delay, Results Math. 63 (3-4) (2013), pp. 949-963.

30. R. Sakthivel, R. Ganesh, S.M. Anthoni, Approximate controllability of fractional nonlinear differential inclusions, Appl. Math. Comput. 225 (2013), pp. 708-717.

31. R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of semilinear fractional differential systems, Comput. Math. Appl. 62 (3) (2011), pp. 1451-1459.

32. I. Podlubny, Fractional Differential Equations. Academic Press (1999).

33. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier (2006).

34. M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos, Solitons Fractals, vol. 14 (2002), no. 3, pp. 433-440.

35. S. Kumar, N. Sukavanam, Approximate controllability of fractional order semilinear systems with bounded delay, J. Differ. Equ. 252 (2012), pp. 6163-6174.

36. F. Ge, H. Zhou, C. Kou, Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique. Appl. Math. and Computation 275 (2016), pp. 107-120.

37. Y. Zhou, J. Wei He, New results on controllability of fractional evolution systems with order $q \in]0,2[$. Vol. 10, no. 3, September 2021.

38. T. Karite, A. Boutoulout and D. F. M. Torres, Enlarged Controllability and Optimal Control of Sub-Diffusion Processes with Caputo Fractional Derivatives, Progr. Fract. Differ. Appl. 6 (2020), no. 2, 81–93.

39. T. Karite, A. Boutoulout and D. F. M. Torres, Enlarged Controllability of Riemann-Liouville Fractional Differential Equations, J. Comput. Nonlinear Dynam. 13 (2018), no. 9, 090907, 6 pp.

40. T. Karite, A. Boutoulout, A. Khan (2020) Minimum Energy Control of Fractional Linear Systems Using Caputo-Fabrizio Derivative. In: Zerrik E., Melliani S., Castillo O. (eds) Recent Advances in Modeling, Analysis and Systems Control: Theoretical Aspects and Applications. Studies in Systems, Decision and Control, vol 243. Springer, Cham.

41. Ech-chaffani, Z.;Aberqi, A.; Karite, T.; Torres,D.F.M. Minimum Energy Problem in the Sence of Caputo of fractional Neutral Evolution Systems in Banach Spaces. Axiome 2022, 11, 379. https://doi.org/10.3390/axioms11080379.

Zoubida Ech-chaffani,
Applied Sciences and Emerging Technologies Laboratory,
National School of Applied Sciences of Fez,
Sidi Mohamed Ben Abdellah University,
Ave. My Abdallah, Km 5 Route d'Imouzzer, BP 72, Fez, Morocco
E-mail address: `zoubida.echchaffani@usmba.ac.ma`

and

Ahmed Aberqi,
Applied Sciences and Emerging Technologies Laboratory,
National School of Applied Sciences of Fez,
Sidi Mohamed Ben Abdellah University,
Ave. My Abdallah, Km 5 Route d'Imouzzer, BP 72, Fez, Morocco
E-mail address: `ahmed.aberqi@usmba.ac.ma`

and

Touria Karite,
Applied Sciences and Emerging Technologies Laboratory,
National School of Applied Sciences of Fez,
Sidi Mohamed Ben Abdellah University,
Ave. My Abdallah, Km 5 Route d'Imouzzer, BP 72, Fez, Morocco
E-mail address: `touria.karite@usmba.ac.ma`