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Neutrosophic G - Co - compactness and Neutrosophic G - Co - paracompactness

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ABSTRACT: This paper reveals the study of a new type of compactness via grills in neutrosophic topological spaces. We define neutrosophic co - compactness and neutrosophic co - paracompactness via grills in neutrosophic topological spaces. We simply call them neautrosophic G - co - compactness and neautrosophic G - co - paracompactness. We investigate some of their basic properties and results in neutrosophic topological spaces. We establish some of their relationships with other known compact spaces.

Key Words: Neutrosophic set, compactness, neutrosophic G - co - compact, neutrosophic G-co-paracompact, G - cocover, neutrosophic topological space.

Contents

1. Introduction

The history of grill was initiated by Choquet [2] in 1947. Thereafter, it had been turned out a very useful tool for exploring several topological problems. In 1968, Singal and Rani [13] introduced one of the most interesting generalized forms of compactness, namely, almost compactness. Mashhour and Atis [6] introduced a new version of compactness, namely, co - compactness and co - paracompactness. Roy and Mukherjee [15] extended the concept of paracompactness in terms of grills.

We are facing real life problems due to uncertainty in our ever day life. The notion of fuzzy sets, introduced by Zadeh [16] associating with only membership values, was not sufficient to deal with all types of real life problems due to uncertainty. Therafter, Atanaosv [1] initiated the notion of intuitionistic fuzzy sets associating with membership and non - membership values.

In order to overcome all sorts of difficulty to handle all types of problems under uncertainty, Smarandache [12] considered the elements with membership, non - membership and indeterministic values and introduced the notion of neutrosophic sets. The notion of neutrosophic topological space was first introduced by Salama and Alblowi [9], followed by Salama and Alblowi [10]. Pal et al. [7] introduced the notion of grill in neutrosophic topological spaces. Karthika and Arockiarani [5] introduced the notion of co - compactness and co - paracompactness via grills in topological spaces. Further several researchers [3, 4, 8, 11, 14] contributed themselves to investigate neautrosophic topological spaces in different directions. Following their works, we introduce and study co - compactness and co - paracompactness via grills in neutrosophic topological spaces. Throughout this paper, we call them neautrosophic G - co - compactness and neautrosophic G - co - paracompactness respectively. The section wise subdivision of the paper is as follows. The next section briefly presents some known definitions and results which are relevant to the study. In section 3, we introduce G - co - compactness and investigate some of its basic properties in neautrosophic topological spaces. We introduce G - co - paracompactness and study some of its basic properties in neutrosophic topological spaces in section 4. Conclusion appears in last section.

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2. Preliminaries

Here we procure some basic concepts to reveal the study.

Definition 2.1. [2] Let X be a non - empty set and G be a collection of non - empty subsets of it. Consider $A, B \subseteq X$. Then G is called a grill on X if
(1) $A \in G$ and $A \subseteq B$ implies that $B \in G$ and
(2) $A \cup B \in G$ implies that $A \in G$ or $B \in G$

Definition 2.2. [15] Let G be a grill on a topological space (X, τ). A cover $\{U_\alpha : \alpha \in \Lambda\}$ of X is said to be a G - cover if there exists a finite subset Λ_0 of Λ such that $X \setminus \cup_{\alpha \in \Lambda_0} U_\alpha \notin G$.

Definition 2.3. [9] Let X be an universal set. A neutrosophic set (in short NS) A in X is a set contains triplet having truthness, falseness and indeterminacy membership values in [0,1]. These can be characterized independently and denoted by T_A, F_A, I_A respectively . The neutrosophic set is denoted as follows

$$A = \{(x, T_A(x), F_A(x), I_A(x)) : x \in X \text{ and } T_A(x), F_A(x), I_A(x) \in [0, 1]\}.$$

There is no restriction on the sum of $T_A(x), F_A(x)$ and $I_A(x)$, so

$$0 \leq T_A(x) + F_A(x) + I_A(x) \leq 3$$

Definition 2.4. [9] Let X be a non - empty set and T be the collection of neutrosophic subsets of X. Then T is said to be a neutrosophic topology (in short NT) on X if the following properties hold:
(1)$0_N, 1_N \in T$.
(2)$U_1, U_2 \in T \implies U_1 \cap U_2 \in T$.
(3) $\cup_{i \in \Delta} U_i \in T$, for every $\{U_i : i \in \Delta\} \subseteq T$.
Then (X, T) is called a neutrosophic topological space (in short NTS) over X. The members of T are called neutrosophic open sets (in short NOS). A neutrosophic set D is called neutrosophic closed set (in short NCS) if and only if D^c is a neutrosophic open set.

Definition 2.5. [7] Let X be a non - empty set and $P(N)$ be denote of all neutrosophic subsets of X. A subcollection G (not containing 0_N) of $P(N)$ is called a grill on X if G satisfies the following conditions:
(1)$A \in G$ and $A \subseteq B$ implies $B \in G$.
(2)$A, B \subseteq X$ and $A \cup B \in G$ implies that $A \in G$ or $B \in G$.

3. Co - compactness in neutrosophic topological space via grills

In this section, we introduce the notion of co - compactness via grills in neutrosophic topological spaces. We prove some of its basic properties and results in neutrosophic topological spaces. We also establish some of its relationships with other known compact spaces.

Definition 3.1. Let G be a grill on a neutrosophic topological space (X, T). A cover $\{U_\alpha : \alpha \in \Lambda\}$ of X is said to be a neautrosophic G - cover if there exists a finite subset Λ_0 of Λ such that $X \setminus \cup_{\alpha \in \Lambda_0} U_\alpha \notin G$.

Definition 3.2. A subset A of a neutrsophic topological space (X, T) is neautrosophic co - subset if cl$A$ is neautrosophic open.

Definition 3.3. A subset A of a neutrsophic topological space (X, T) is neautrosophic Ic - subset if int$A$ is neautrosophic closed.

Definition 3.4. A cover η of a neutrsophic topological space (X, T) is called neautrosophic co - open cover if each member of η is neautrosophic co - open set.

Definition 3.5. Let (X, T, G) be a neutrosophic topological space with a grill G and A be a subset of it. Then A is said to be neautrosophic G - compact if for every cover U of A formed by elements of T, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $X \setminus \bigcup_{i=1}^{n} A_i \notin G$. The grill neutrosophic topological space (X, T, G) is said to be neautrosophic G - compact if X is neautrosophic G - compact.

Definition 3.6. A grill neutrosophic topological space (X, T, G) is almost neatrosophic G - compact if each open cover of X has a finite subfamily $\{U_1, U_2, U_3, \ldots, U_\alpha\}$ such that $X \setminus \cup_{\alpha \in \Lambda_0} U_\alpha \notin G$.

Definition 3.7. Let G be a grill on a neutrsophic topological space (X, T). Then the space X is said to be neutrosophic paracompact with respect to the grill G or simply neautrosophic G - paracompact if every open cover U of X has a precise locally finite open refinement $U^\star$ (not necessarily a cover of X) such that $X \setminus \cup U^\star \notin G$, where the statement "a cover $U = \{U_\alpha : \alpha \in \Lambda\}$ has a precise refinement" means as usual, that there exists a collection $V = \{V_\alpha : \alpha \in \Lambda\}$ of subsets of X such that $V_\alpha \subseteq U_\alpha$, for all $\alpha \in \Lambda$. (Here refinement need not be a cover).

Definition 3.8. Let G be a grill on a neutrosophic topological space (X, T). A cover $\{U_\alpha : \alpha \in \Lambda\}$ of X, whose elements are neutrosophic co - open sets is said to be a neutrosophic G - co - compact if there exists a finite subset Λ_0 of Λ such that $X \setminus \cup_{\alpha \in \Lambda_0} U_\alpha \notin G$.

Definition 3.9. A grill neutrosophic topological space (X, T, G) is almost neutrosophic G - co - compact if each neutrsophic co - open cover of X has a finite subfamily $\{U_1, U_2, U_3, \ldots, U_\alpha\}$ such that $X \setminus \cup_{\alpha \in \Lambda_0}$ $\mathrm{cl}(U_\alpha) \notin G$.

Remark 3.10. Let (X, T, G) be a neutrosophic topological space with the grill G and $A \subseteq X$. Then
(1) X is neutrosophic G - compact $\implies$ X is neutrosophic G - co - compact $\implies$ X is almost neutrosophic G - co - compact.
(2) X is almost neutrosophic G - compact $\implies$ X is almost neutrosophic G - co - compact.

Theorem 3.11. A neutrosophic topological space (X, T, G) with the grill G is neutrosophic G - co - compact if and only if each family of neutrosophic Ic - closed sets which has a neutrosophic G - finite intersection property has the arbitrary intersection belongs to the grill, that is $\cap_{\alpha \in \Lambda} A_\alpha \in G$.

Proof: If $\eta = \{A_\alpha : \alpha \in \Lambda\}$ is a family of neutrosophic Ic - closed subsets of a grill neutrosophic topological space (X, T, G) with neutrosophic G finite intersection property and the arbitrary intersection also belongs to grill, then $\cap_{\alpha \in \Lambda} A^c \in G \implies X \setminus \cup_\alpha A^c \notin G \implies$ if (X, T, G) is not neutrosophic G - co - compact, η has a finite sub - collection such that $X \setminus \bigcup_{i=1}^{n} A_i \in G \implies \bigcap_{i=1}^{n} A_i \notin G$ where A (complement set of co - open set) is neutrosophic Ic - closed set, which is a contradiction. So, for every cover of neutrosophic co - open sets, there exists a sub - collection such that $X \setminus \bigcup_{i=1}^{n} A_i \notin G$. Hence X is neutrosophic G - co - compact.

Conversely, let the space (X, T, G) be neutrosophic G - co - compact. If each family η of neutrosophic co - open sets such that no finite subfamily of η is a cover of X,then it fails to be a neutrosophic G - co - cover of X. This is true if and only if each family of neutrosophic Ic - closed sets which satisfies $\bigcap_{i=1}^{n} A_i \in G \implies \cap_{\alpha \in \Lambda} A_\alpha \in G$. Hence the proof. □

Theorem 3.12. For a grill neutrosophical topological space (X, T, G), the following statements are equivalent:
(1) (X, T, G) is a almost neutrosophic G - co - compact space.
(2) Each cover of X whose members are both neutrosophic open and closed has a finite sub - collection $\{A_1, A_2, ..., A_n\}$ such that $X \setminus \bigcup_{i=1}^{n} A_i \notin G$.
(3) For each family J of neutrosophic co - open subsets of X having the G finite intersection property, $\cap\{F^c : F \in J\} \in G$.

Proof: **(1)** $\implies$ **(2)**: Since a neutrosophic open and closed set is neutrosophic co - open, we can consider the cover as a co - cover. By (1) $X\backslash \cup_{\alpha\in\Lambda_0} clU_\alpha \notin G$. But the sets are both neutrosophic open and closed $\implies$ $\mathrm{cl}U_\alpha = U_\alpha$. Therefore $X\backslash \cap_{\alpha\in\Lambda_0} U_\alpha \notin G$. Hence the proof.

(2) $\implies$ **(3)**: Let J be a family of neutrosophic co - open subsets of X having the neutrosophic G finite intersection property with the intersection belongs to the grill......(i). Suppose that $\cap_{\alpha\in\Lambda} F \notin G \implies X\backslash \cup_{\alpha\in\Lambda} F \notin G \implies \{F_\alpha : \alpha \in \Lambda\}$ is a neutrosophic G - co - cover of X in which every member is both neutrosophic open and closed. So, there exists a finite subfamily $\{F_1, F_2, ..., F_n\}$ such that $X\backslash\{\bigcup_{i=1}^n F_i\} \notin G \implies \cap F_i \notin G$, which is a contradiction to the fact (1). Hence the result.
(3) $\implies$ **(1)**: If X is not a almost neutrosophic G - co - compact space, then there is a neutrosophic G - co - cover η of X which has no finite subfamily $\{U_1, U_2, ..., U_n\}$ such that $X\backslash \bigcup_{i=1}^n U_i \notin G \implies$ we have $\bigcap_{i=1}^n U_i \notin G$, which gives a contradiction to the fact that the collection $\{U_\alpha : \alpha \notin \Lambda\}$ has neutrosophic G finite intersection property. Hence X is a almost neutrosophic G - co - compact space. □

Theorem 3.13. A neutrosophic Ic - closed subset of a neutrosophic G - co - compact space (X, T) is G - co - compact neutrosophic topological space.

Proof: Let A be a neutrosophic Ic - closed subset of the G - co - compact neutrosophic topological space (X, T) and η be any T_A co - cover of A. Every member $U \in \eta$ is of the form $U = A \cap V$ where V is T - co - open. Let $W = \{U_\alpha : \alpha \in \Lambda\} \cup (X\backslash A)$ where $X\backslash A$ is also neutrosophic co - open and hence it is a neutrosophic G - co - cover of X. Since X is neutrosophic G - co - compact, W has a finite subfamily $\{U_1, U_2, ...U_n\}$ such that $X\backslash \cup_{i=1}^n U_i \cup (X\backslash A) \notin G$. W is a neutrosophic G - co - cover. It also should cover $A \subseteq X$. But A is not a subset of $X\backslash A \implies A \subseteq \cup_{i=1}^n U_i \implies A\backslash \cup_{i=1}^n U_i \notin G \implies A$ is neutrosophic G - co - compact.

Corollary 3.14. A neutrosophic Ic - closed subset of a almost neutrosophic G - co - compact space is also almost neutrosophic G - co - compact space. □

Theorem 3.15. Let $f : (X, T_1, G_1) \to (Y, T_2, G_2)$ be a homeomorphism from X to Y, where X is alomost neutrosophic G_1 - co - compact space. Then the image $f(X)$ with the properties $G_2 \subseteq f(G_1)$, is almost neutrosophic G_2 - co - compact.

Proof: Let $f : (X, T_1, G_1) \to (Y, T_2, G_2)$ be a co - continuous map from the almost neutrosophic G_1 co - compact space X on to a grill neutrosophic topological space Y and let $\eta = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of Y whose members are both neutrosophic open and closed. $\{f^{-1}(U_\alpha) : U \in \eta\}$is a cover of X, whose members are both neutrosophic open and closed. Then there is a finite subfamily $\{f^{-1}(U_i) : i \in I\}$ such that $X\backslash \cup_{i=1}^n f^{-1}(U_i) \notin G_1 \implies f(X)\backslash \cup_{i=1}^n U_i \notin G_1$. It is given that $G_2 \subseteq f(G_1)$. That is $f(X)\backslash \cup_{i=1}^n U_i \notin G_2$. Hence $f(X)$ is almost neutrosophic G_2 - co - compact space. □

Theorem 3.16. An almost neutrosophic G - co - compact space which is neutrosophic semiregular is neutrosophic G - co - compact space.

Proof: Let (X, T, G) be a semi - regular grill neutrosophic topological space. Then every neutrosophic open set is neutrosophic regular open. Suppose there is a collection $\{U_\alpha : \alpha \in \Lambda\}$ of neutrosophic co - open sets which covers X. A neutrosophic co - open set which is neutrosophic regular open is both neutrosophic open and closed. The space is almost neutrosophic G - co - compact. So, there exists a sub - collection $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $X\backslash \cup_{i=1}^n \mathrm{cl}(U_i) \notin G$. Since the neutrosophic co - open sets are both neutrosophic open and closed, $\mathrm{cl}(U_\alpha) = U_\alpha$ for every $\alpha \implies X\backslash \cup_{i=1}^n U_i \notin G$. Hence the space is neutrosophic G - co - compact. □

4. Co - paracompactness in neutrosophic topological space via grills

In this section, we introduce the notion of co - paracompactness via grills in neutrosophic topological spaces. We also investigate some of its basic properties and results in neutrosophic topological spaces.

Definition 4.1. A neutrosophic topological space (X, T, G) with grill G is neutrosophic G - co - paracompact if each co - cover $U = \{U_\alpha : \alpha \in \Lambda\}$ of X has an open locally finite refinement $U^\star$ such that $X \backslash \cup_{\alpha\in\Lambda} U^\star \notin G$ where $U^\star = \{V_\alpha : \alpha \in \Lambda\}$ and for every $U_\alpha \in U$, we can find a $V_\alpha \in V$ such that $V_\alpha \subseteq U_\alpha$. Here refinement need not be a cover.

Definition 4.2. A neutrosophic topological space (X, T, G) with grill G is almost neutrosophic G - co - paracompact if for each co - cover U of X, there exists a locally finite family $U^\star$ of neutrosophic co - open sets refines U such that $X \backslash \cup_{\alpha\in U^\star} \mathrm{cl}(U_\alpha) \notin G$.

Remarks 4.3. Let (X, T, G) be a neutrosophic topological space with grill G. Then
(1) X is almost neutrosophic G - co - compact $\implies$ X is almost neutrosophic G - co - paracompact.
(2) X is neutrosophic G - co - compact $\implies$ X is neutrosophic G - co - paracompact $\implies$ X is almost neutrosophic G - co - paracompact.
(3) X is neutrosophic G - paracompact $\implies$ X is neutrosopic G - co -paracompact.
(4) X is almost neutrosophic G - paracompact $\implies$ X is almost neutrosophic G - co - paracompact.

Theorem 4.4. A neutrosophic Ic - closed subset of a neutrosophic G - co - paracompact is neutrosophic G - co - paracompact.

Proof: Let A be a neutrosophic Ic - closed subset of the neutrosophic G - co - paracompact space (X, T, G). Let μ be the family defined by $\mu = \{U = A \cap V :$ where V is T - co - open$\}$ and denote by W, the family of all members of μ and $(X\backslash A)$. Since A is neutrosophic Ic - closed, $(X\backslash A)$ is neutrosophic co - open set. Then W is a T - co - open cover of X. The family W has a T - co - open locally finite refinement K.
Denoting by γ, the members of K which refines μ. In this case $C = \{A \cap H : H \in \gamma\}$ refines U and C is T_A - locally finite. But this collection C should be a co - cover of A. That is $A \backslash \cup \{A \cap H : H \in \gamma\} \notin G \implies A \backslash \cup \{D : D \in C\} \notin G$. Thus A is neutrosophic G - co - paracompact. □

Theorem 4.5. A neutrosophic G - co - paracompact space, which is lightly compact is neutrosophic G - co - compact space.

Proof: Let X be a neutrosophic G - co - paracompact space and let γ be any neutrosophic co - open cover of $X \implies \gamma$ has a neutrosophic co - open locally finite refinement $\gamma^\star = \{A_\alpha : \alpha \in \Lambda\}$ such that $X \backslash \cup_{\alpha\in\Lambda} A_\alpha \notin G$. Since X is lightly compact, so every locally finite collection is finite. That is $X \backslash \cup_{i=1}^{n} A_i \notin G$. Hence γ is a neutrosophic G - co - cover. Finally X is neutrosophic G - co - compact space. □

Theorem 4.6. An almost neutrosophic G - co - paracompact space which is lightly compact is almost neutrosophic G - co - compact space.

Proof: Let X be a almost neutrosophic G - co - paracompact space and let η be any co - cover of X. Then there exists a locally finite family J of neutrosophic co - open sets which refines η such that $X \backslash \cup_{\alpha\in J} \mathrm{cl}(A_\alpha) \notin G$. Since X is lightly compact, J is finite. So, there exists a family $\{U_1, U_2, ..., U_n\}$ such that $X \backslash \cup_{i=1}^{n} \mathrm{cl}(U_i) \notin G$. Hence η is a neutrosophic G - co - cover and so X is almost neutrosophic G - co -paracompact space. □

Theorem 4.7. An almost neutrosophic G - co - paracompact space which is neutrosophic semi regular is neutrosophic G - co - paracompact space.

Proof: Let X be a neutrosophic semi regular space. Then every neutrosophic open set is neutrosophic regular open. Since a neutrosophic co - open set which is neutrosophic regular open is both neutrosophic open and closed, every co - cover consists of sets which are both neutrosophic open and closed. If X is an almost neutrosophic G - co - paracompact and V is a neutrosophic co - cover of X, then there exists a locally finite refinement $V^\star$ such that $X \backslash \cup_{A \in V^\star} \mathrm{cl}(A) \notin G$. But $\mathrm{cl}A = A \implies X$ is neutrosophic G - co - paracompact space. □

5. Conclusion

We have introduced neutrosophic G - co - compactness and neutrosophic G - co - paracompactness as a generalization of compactness in neutrosophic topological spaces. We have investigated their basic properties and characterization theorems. We have established their relationships with other compact spaces. This paper will help the researchers for further investigation of the compactness in neutrosophic topological spaces. .

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