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Some Common Fixed Point Results on (ψ,ϕ)-contraction

M. C. Arya, N. Chandra and Mahesh C. Joshi

ABSTRACT: The aim of the paper is to obtain common fixed point theorems for (ψ,ϕ)-contraction under the generalized rational type condition in a complete metric space. Moreover, these theorems generalize recent well known results in the literature.

Key Words: Weak compatibility, (ψ,ϕ)-contraction, common fixed point.

Contents

1. Introduction

In 1976, Jungck [9] initiated the notion of commutativity of mappings and established a common fixed point theorem on a complete metric space. In 1982, Sessa [18] also introduced the concept of weak commutativity by weakening the commutativity and obtained some interesting results on the existence of common fixed points. Further, Jungck [10] generalized the weak commutativity by a new notion of compatible mappings. However, in 1996, Jungck [11] again introduced a more generalized concept known as weakly compatiblity, and defined as follows.

Definition 1.1 ([11])**.** *Let f and g be self mappings of a set X. Then the pair $\{f,g\}$ is said to be weakly compatible if they commute on the set of coincidence points, i.e., $fgx = gfx$ whenever $fx = gx$ for some $x \in X$.*

On the other hand, generalizing Banach contraction condition, Boyd and Wong [5] defined a new class of contractive condition which is generally known as ϕ-contraction. Further Alber et al. [2] generalized this concept by introducing weak ϕ-contraction and established a fixed point theorem for the mapping satisfying such type of contractive condition. By the way, a self mapping T on a metric space (X,d) is said to be *weak ϕ-contractive* if there exists a function $\phi:[0,+\infty)\to[0,+\infty)$ with $\phi(0)=0$ and $\phi(t)>0$ for all $t>0$, such that $d(Tx,Ty)\le d(x,y)-\phi(d(x,y))$ for each $x,y\in X$. Thereafter, Rhoades [15] again generalized the result of Alber et al. [2] and obtained the following interesting theorem.

Theorem 1.2 ([15])**.** *Let (X,d) be a complete metric space and $T:X\to X$ such that, for every $x,y\in X$,*

$$d(Tx,Ty)\le d(x,y)-\phi(d(x,y)), \tag{1.1}$$

where $\phi:[0,+\infty)\to[0,+\infty)$ is a continuous and non-decreasing function with $\phi(0)=0$ and $\phi(t)>0$ for all $t>0$. Then T has a unique fixed point.

Now, for further discussions, we consider following classes of functions:

(C_1) $\Phi=\{\phi\,|\,\phi:[0,+\infty)\to[0,+\infty)$ is lower semi continuous with $\phi(t)>0$ for all $t>0$ and $\phi(0)=0\}$.

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(C_2) $\Psi = \{\psi \,|\, \psi : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing with $\psi(t) = 0$ if and only if $t = 0\}$.

Moreover, in 2008, Dutta et al. [17] generalized the ϕ-contraction by a new extended class contractive mappings known as (ψ, ϕ)-contraction and established the following result.

Theorem 1.3 ([17]). *Let X be a complete metric space and $T : X \to X$ such that, for every $x, y \in X$,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)), \tag{1.2}$$

where $\phi \in \Phi$, $\psi \in \Psi$. Then T has a unique fixed point in X.

Furthermore, in 2015, Murty et al. [14] also obtained the following common fixed point theorem for (ψ, ϕ)-contraction which generalizes various results in the literature.

Theorem 1.4 ([14]). *Suppose that A, B, S and T are self mappings of a complete metric space (X, d), $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. If, for every $x, y \in X$ with $x \neq y$,*

$$\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y)), \tag{1.3}$$

where $\psi \in \Psi$, $\phi \in \Phi$ such that ϕ is discontinuous at $t = 0$, and

$$M(x, y) = \max\left\{d(Sx, Ty), \frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Sx, By) + d(Ax, Ty)}{2}\right\}$$

and

$$N(x, y) = \min\left\{d(Sx, Ty), \frac{d(Ax, Sx) + d(By, Ty)}{2}, \frac{d(Sx, By) + d(Ax, Ty)}{2}\right\}.$$

Then A, B, S, and T have a unique common fixed point in X.

Moreover, during last three decades, a number of researchers have extended and weakened (ψ, ϕ)-contractive condition in different settings and obtained several common fixed point theorems for pairs of mappings (see, [1,3,4,6,7,8,12,13,14,16,17,19] and references therein).

Now, we are in a position to state and prove our results which have been obtained for mappings satisfying a generalized rational type condition under the weak compatibility and (ψ, ϕ)-contraction in complete metric spaces as follow.

2. Main Results

Theorem 2.1. *Suppose that A, B, S and T are self mappings of a complete metric space (X, d), $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. If, for every $x, y \in X$,*

$$\psi(d(Ax, By)) \leq \psi(M_1(x, y)) - \phi(N_1(x, y)), \tag{2.1}$$

where $\psi \in \Psi$, $\phi \in \Phi$ such that ϕ is discontinuous at $t = 0$, and

$$\begin{aligned} M_1(x, y) \;=\; & \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \\ & \left(\frac{d(By, Sx) + d(Ax, Ty)}{2}\right), \left(\frac{d(Sx, Ax) + d(Ty, By)}{2}\right), \\ & d(By, Ty)\left(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\right), d(Ax, Sx)\left(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\right)\Big\}. \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} N_1(x, y) \;=\; & \min\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \\ & \left(\frac{d(By, Sx) + d(Ax, Ty)}{2}\right), \left(\frac{d(Sx, Ax) + d(Ty, By)}{2}\right), \\ & d(By, Ty)\left(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\right), d(Ax, Sx)\left(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\right)\Big\}. \end{aligned} \tag{2.3}$$

Then A, B, S, and T have a unique common fixed point in X, whenever one of the range $A(X)$, $B(X)$, $S(X)$, $T(X)$ is closed in X.

Proof. Let x_0 be an arbitrary point in X. Since $A(X) \subseteq T(X)$, we can choose an $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. Similarly, since $B(X) \subseteq S(X)$, there exists an $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$. Continuing in this way, we construct a sequence $\{y_n\}_{n\in\mathbb{N}_0}$ in X, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is a set of natural numbers, such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$.

We shall now show that $\{y_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. If $y_{2n} = y_{2n+1}$ for some $n \in \mathbb{N}_0$, it is obvious to say that $\{y_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. So, we assume the case when $y_{2n} \neq y_{2n+1}$ for every $n \in \mathbb{N}_0$. Then, by taking $x = x_{2n}, y = x_{2n+1}$ in (2.2) and (2.3), we have

$$\begin{aligned}
M_1(x_{2n}, x_{2n+1}) &= \max\Big\{d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad \left(\frac{d(Bx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})}{2}\right), \left(\frac{d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})}{2}\right), \\
&\quad d(Bx_{2n+1}, Tx_{2n+1})\left(\frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Sx_{2n}, Tx_{2n+1})}\right), d(Ax_{2n}, Sx_{2n})\left(\frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})}\right)\Big\} \\
&= \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\
&\quad \left(\frac{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})}{2}\right), \left(\frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\right), \\
&\quad d(y_{2n+2}, y_{2n+1})\left(\frac{1 + d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})}\right), d(y_{2n+1}, y_{2n})\left(\frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right)\Big\} \\
&= \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), \left(\frac{d(y_{2n+2}, y_{2n})}{2}\right), \\
&\quad \left(\frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\right), d(y_{2n+1}, y_{2n})\left(\frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right)\Big\}
\end{aligned}$$

and

$$\begin{aligned}
N_1(x_{2n}, x_{2n+1}) &= \min\Big\{d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad \left(\frac{d(Bx_{2n+1}, Sx_{2n}) + d(Ax_{2n}, Tx_{2n+1})}{2}\right), \left(\frac{d(Sx_{2n}, Ax_{2n}) + d(Tx_{2n+1}, Bx_{2n+1})}{2}\right), \\
&\quad d(Bx_{2n+1}, Tx_{2n+1})\left(\frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Sx_{2n}, Tx_{2n+1})}\right), d(Ax_{2n}, Sx_{2n})\left(\frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})}\right)\Big\} \\
&= \min\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\
&\quad \left(\frac{d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})}{2}\right), \left(\frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\right), \\
&\quad d(y_{2n+2}, y_{2n+1})\left(\frac{1 + d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})}\right), d(y_{2n+1}, y_{2n})\left(\frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right)\Big\} \\
&= \min\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), d(y_{2n+2}, y_{2n+1}), \left(\frac{d(y_{2n+2}, y_{2n})}{2}\right), \\
&\quad \left(\frac{d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})}{2}\right), d(y_{2n+1}, y_{2n})\left(\frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right)\Big\}.
\end{aligned}$$

Now, if $M_1(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2})$ then

$$\begin{aligned}
\psi(d(y_{2n+1}, y_{2n+2})) &= \psi(d(Ax_{2n}, Bx_{2n+1})) \\
&\leq \psi(M_1(x_{2n}, x_{2n+1})) - \phi(N_1(x_{2n}, x_{2n+1})) \\
&= \psi(d(y_{2n+1}, y_{2n+2})) - \phi(N_1(x_{2n}, x_{2n+1})) \\
&< \psi(d(y_{2n+1}, y_{2n+2}),
\end{aligned}$$

which is a contradiction. Therefore $d(y_{2n+2}, y_{2n+1}) \leq d(y_{2n}, y_{2n+1})$, and

$$d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1})\left(\frac{1 + d(y_{2n+2}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})}\right),$$

which implies

$$d(y_{2n},y_{2n+1})\left(\frac{1+d(y_{2n+2},y_{2n+1})}{1+d(y_{2n},y_{2n+1})}\right)\leq d(y_{2n},y_{2n+1}).$$

Hence, $M_1(x_{2n},x_{2n+1})=d(y_{2n},y_{2n+1})$ and $N_1(x_{2n},x_{2n+1})=\frac{d(y_{2n},y_{2n+2})}{2}$. Using (2.1), we have

$$\begin{aligned}\psi(d(y_{2n+1},y_{2n+2})) &= \psi(d(Ax_{2n},Bx_{2n+1}))\\ &\leq \psi(M_1(x_{2n},x_{2n+1}))-\phi(N_1(x_{2n},x_{2n+1}))\\ &\leq \psi(d(y_{2n},y_{2n+1}))-\phi(N_1(x_{2n},x_{2n+1})).\end{aligned}\tag{2.4}$$

This implies $\psi(d(y_{2n},y_{2n+1}))$ for all $n\in\mathbb{N}_0$, and the sequence is monotonically decreasing of non-negative real numbers. Hence, there exists $r>0$ such that $\lim_{n\to+\infty}d(y_{2n},y_{2n+1})=r$. Moreover, $\lim_{n\to+\infty}\psi(M_1(x_{2n},x_{2n+1}))=\psi(r)$. Now, taking upper limits on each side of (2.4) to obtain the following inequality

$$\limsup_{n\to+\infty}\psi(d(y_{2n+1},y_{2n+2}))\leq\limsup_{n\to+\infty}\psi(d(y_{2n+1},y_{2n}))-\limsup_{n\to+\infty}\phi(N_1(x_{2n},x_{2n+1})).$$

Thus, the lower semi continuity of ϕ gives

$$\psi(r)\leq\psi(r)-\limsup_{n\to+\infty}\phi(N_1(x_{2n},x_{2n+1})).$$

Therefore, by the property of ϕ, we get a contradiction. Hence, we have

$$\lim_{n\to+\infty}d(y_{2n},y_{2n+1})=0.$$

Similarly, taking $x=x_{2n+1}$ and $y=x_{2n+2}$ in (2.1) and arguing as above, we have

$$\lim_{n\to+\infty}d(y_{2n+1},y_{2n+2})=0.$$

Therefore, for all $n\in\mathbb{N}_0$, we have

$$\lim_{n\to+\infty}d(y_{2n},y_{2n+1})=0.\tag{2.5}$$

Next, we prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For this, it is sufficient to show $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. To the contrary, suppose $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, then there exists an $\epsilon>0$ and the sequence of natural numbers $\{2m_k\}$, $\{2n_k\}$ with $2m_k>2n_k>k$ such that

$$d(y_{2m_k},y_{2n_k})\geq\epsilon \text{ and } d(y_{2m_k-2},y_{2n_k})<\epsilon.$$

Using (2.5) and the inequality

$$\epsilon\leq d(y_{2m_k},y_{2n_k})\leq d(y_{2n_k},y_{2m_k-2})+d(y_{2m_k-1},y_{2m_k-2})+d(y_{2m_k-1},y_{2m_k})$$

we get

$$\lim_{k\to+\infty}d(y_{2m_k},y_{2n_k})=\epsilon.\tag{2.6}$$

Also (2.5), (2.6) and the inequality, $d(y_{2m_k},y_{2n_k})\leq d(y_{2m_k},y_{2m_k+1})+d(y_{2m_k+1},y_{2n_k})$, yield

$$\epsilon\leq\lim_{k\to+\infty}d(y_{2m_k+1},y_{2n_k}),$$

and (2.5), (2.6) and the inequality, $d(y_{2m_k+1}, y_{2n_k}) \leq d(y_{2m_k+1}, y_{m_k}) + d(y_{2m_k}, y_{2n_k})$, yield that

$$\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k}) \leq \epsilon.$$

Hence, we obtain

$$\lim_{k\to+\infty} d(y_{2m_k+1}, y_{2n_k}) = \epsilon. \tag{2.7}$$

In similar manner, it can be shown that

$$\lim_{k\to+\infty} d(y_{2m_k}, y_{2n_k-1}) = \lim_{k\to+\infty} d(y_{2n_k-1}, y_{2n_k+1}) = \epsilon. \tag{2.8}$$

Now, we find

$$\begin{aligned} M_1(x_{2m_k-1}, x_{2n_k-1}) &= \max\Big\{d(Sx_{2m_k-1}, Tx_{2n_k-1}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \\ &\quad \left(\frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2}\right), \\ &\quad \left(\frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2}\right), \\ &\quad d(Bx_{2n_k-1}, Tx_{2n_k-1})\left(\frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right), \\ &\quad d(Ax_{2m_k-1}, Sx_{2m_k-1})\left(\frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right)\Big\} \\ &= \max\Big\{d(y_{2m_k+1}, y_{2n_k}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \\ &\quad \left(\frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2}\right), \\ &\quad \left(\frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2}\right), \\ &\quad d(Bx_{2n_k-1}, Tx_{2n_k-1})\left(\frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right), \\ &\quad d(Ax_{2m_k-1}, Sx_{2m_k-1})\left(\frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right)\Big\} \end{aligned}$$

and

$$\begin{aligned} N_1(x_{2m_k-1}, x_{2n_k-1}) = \min\Big\{&d(Sx_{2m_k-1}, Tx_{2n_k-1}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}), \\ &\left(\frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2}\right), \\ &\left(\frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2}\right), \\ &d(Bx_{2n_k-1}, Tx_{2n_k-1})\left(\frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right), \\ &d(Ax_{2m_k-1}, Sx_{2m_k-1})\left(\frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right)\Big\} \end{aligned}$$

$$= \min\Big\{d(y_{2m_k+1}, y_{2n_k}), d(Ax_{2m_k-1}, Sx_{2m_k-1}), d(Bx_{2n_k-1}, Tx_{2n_k-1}),$$
$$\left(\frac{d(Bx_{2n_k-1}, Sx_{2m_k-1}) + d(Ax_{2m_k-1}, Tx_{2n_k-1})}{2}\right),$$
$$\left(\frac{d(Sx_{2m_k-1}, Ax_{2m_k-1}) + d(Tx_{2n_k-1}, Bx_{2n_k-1})}{2}\right),$$
$$d(Bx_{2n_k-1}, Tx_{2n_k-1})\left(\frac{1 + d(Ax_{2m_k-1}, Sx_{2m_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right),$$
$$d(Ax_{2m_k-1}, Sx_{2m_k-1})\left(\frac{1 + d(Bx_{2n_k-1}, Tx_{2n_k-1})}{1 + d(Sx_{2m_k-1}, Tx_{2n_k-1})}\right)\Big\}.$$

Thus, using (2.2), (2.5), (2.6), (2.7) and (2.8), we have $\lim_{k\to+\infty} M_1(x_{2m_{k-1}}, x_{2n_{k-1}}) = \epsilon$ and $\lim_{k\to+\infty} N_1(x_{2m_{k-1}}, x_{2n_{k-1}}) = 0$ Moreover, by taking $x = x_{2m_k-1}$ and $y = x_{2n_k-1}$ in (2.1), we get

$$\begin{aligned} \psi(d(y_{2m_k}, y_{2n_k+1})) &= \psi(d(Ax_{2m_{k-1}}, Bx_{2n_{k-1}})) \\ &\le \psi(M_1(x_{2m_{k-1}}, x_{2n_{k-1}}) - \phi(N_1(x_{2m_{k-1}}, x_{2n_{k-1}}). \end{aligned}$$

Therefore, taking the limit as $k \to +\infty$, we get $\psi(\epsilon) \le \psi(\epsilon) - \phi(N_1(x_{2m_{k-1}}, x_{2n_{k-1}}))$, which is a contradiction for $\epsilon > 0$ (due to discontinuity of ϕ at $t = 0$). Hence $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.

Thus, in both cases, it has been shown that $\{y_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in X. Since X is complete, it has a limit in X, say z. We shall now show that z is a common fixed point for mappings A and S. It is clear that

$$\lim_{n\to+\infty} y_{2n+1} = \lim_{n\to+\infty} Ax_{2n} = \lim_{n\to+\infty} Tx_{2n+1} = z,$$

and

$$\lim_{n\to+\infty} y_{2n+2} = \lim_{n\to+\infty} Bx_{2n+1} = \lim_{n\to+\infty} Sx_{2n+2} = z.$$

Assuming that $S(X)$ is closed, there exists a $u \in X$ such that $z = Su$. We claim that $Au = z$. If not, then

$$\begin{aligned} M_1(u, x_{2n+1}) = \max\Big\{&d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\Big(\frac{d(Bx_{2n+1}, Su) + d(Au, Tx_{2n+1})}{2}\Big), \Big(\frac{d(Su, Au) + d(Tx_{2n+1}, Bx_{2n+1})}{2}\Big), \\ &d(Bx_{2n+1}, Tx_{2n+1})\Big(\frac{1 + d(Au, Su)}{1 + d(Su, Tx_{2n+1})}\Big), d(Au, Su)\Big(\frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})}\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} N_1(u, x_{2n+1}) = \min\Big\{&d(Su, Tx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\Big(\frac{d(Bx_{2n+1}, Su) + d(Au, Tx_{2n+1})}{2}\Big), \Big(\frac{d(Su, Au) + d(Tx_{2n+1}, Bx_{2n+1})}{2}\Big), \\ &d(Bx_{2n+1}, Tx_{2n+1})\Big(\frac{1 + d(Au, Su)}{1 + d(Su, Tx_{2n+1})}\Big), d(Au, Su)\Big(\frac{1 + d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})}\Big)\Big\}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} M_1(u, x_{2n+1}) = d(Au, z) \text{ and } \lim_{n\to+\infty} N_1(u, x_{2n+1}) = 0.$$

Therefore, by (2.1), we have

$$\psi(d(Au, Bx_{2n+1})) \le \psi(M_1(u, x_{2n+1})) - \phi(N_1(u, x_{2n+1}),$$

which, taking the limit as $n \to +\infty$, implies that

$$\psi(d(Au,z)) \le \psi(d(Au,z)) - \phi(N_1(u,x_{2n+1}),$$

a contradiction for $d(Au,z) > 0$. Hence $Au = z$, and $Au = Su = z$. Since the mappings A and S are weakly compatible, $Az = ASu = SAu = Sz$.

Next we claim that $Az = z$. If not, we find

$$\begin{aligned} M_1(z,x_{2n+1}) &= \max\Big\{d(Sz,Tx_{2n+1}), d(Az,Sz), d(Bx_{2n+1},Tx_{2n+1}), \\ &\Big(\frac{d(Bx_{2n+1},Sz)+d(Az,Tx_{2n+1})}{2}\Big), \Big(\frac{d(Sz,Az)+d(Tx_{2n+1},Bx_{2n+1})}{2}\Big), \\ &d(Bx_{2n+1},Tx_{2n+1})\Big(\frac{1+d(Az,Sz)}{1+d(Sz,Tx_{2n+1})}\Big), d(Az,Sz)\Big(\frac{1+d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sz,Tx_{2n+1})}\Big)\Big\} \end{aligned}$$

and

$$\begin{aligned} N_1(z,x_{2n+1}) &= \min\Big\{d(Sz,Tx_{2n+1}), d(Az,Sz), d(Bx_{2n+1},Tx_{2n+1}), \\ &\Big(\frac{d(Bx_{2n+1},Sz)+d(Az,Tx_{2n+1})}{2}\Big), \Big(\frac{d(Sz,Az)+d(Tx_{2n+1},Bx_{2n+1})}{2}\Big), \\ &d(Bx_{2n+1},Tx_{2n+1})\Big(\frac{1+d(Az,Sz)}{1+d(Sz,Tx_{2n+1})}\Big), d(Az,Sz)\Big(\frac{1+d(Bx_{2n+1},Tx_{2n+1})}{1+d(Sz,Tx_{2n+1})}\Big)\Big\}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} M_1(z,x_{2n+1}) = d(Sz,z) = d(Az,z).$$

Using (2.1), we have

$$\psi(d(Az,Bx_{2n+1})) \le \psi(M_1(z,x_{2n+1})) - \phi(N_1(z,x_{2n+1}))$$

which, on taking limit as $n \to +\infty$, gives

$$\psi(d(Az,z)) \le \psi(d(Az,z)) - \phi(d(Az,z)),$$

a contradiction for $d(Az,z) > 0$. Therefore $Az = z$.

Moreover, we show that z is a fixed point for mappings B and T. Since $A(X) \subseteq T(X)$, there is some $v \in X$ such that $Az = Tv$. Then $Az = Tv = Sz = z$. We claim that $Bv = z$. If not then by (2.1), we have

$$\begin{aligned} \psi(d(z,Bv)) &= \psi(d(Az,Bv)) \\ &\le \psi(M_1(z,v)) - \phi(N_1(z,v)) \\ &= \psi(d(Bv,z)), \end{aligned}$$

a contradiction for $d(Bv,z) > 0$, hence $Bv = z$. Thus $Bv = Tv = z$, and by the weak compatibility of mappings B and T, we get $Bz = BTv = TBv = Tz$. If $Bz \ne z$ then by (2.1), we have

$$\begin{aligned} \psi(d(z,Bz)) &= \psi(d(Az,Bz)) \\ &\le \psi(M_1(z,z)) - \phi(N_1(z,z)) \\ &= \psi(d(z,Tz)) - \phi(N_1(z,z)) = \psi(d(z,Bz)) - \phi(N_1(z,z)), \end{aligned}$$

a contradiction for $d(z,Bz) > 0$. Hence $Az = Bz = Sz = Tz = z$. A similar analysis is also valid for the case in which $T(X)$ is closed as well as for the cases in which $A(X)$ or $B(X)$ is closed. Also, the uniqueness of the common fixed point z follows from (2.1). □

Remark 2.2. *Our Theorem 2.1 also generalizes the results in [3], [4], [7], [13], ([14], Theorem 1.4), ([17], Theorem 1.3), ([15], Theorem 1.2) and many others.*

Now, we obtain some special cases of our Theorem 2.1 in the form of corollaries as follow.

Corollary 2.3. *Suppose that A, B, S and T are self mappings of a complete metric space (X, d), $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. If, for every $x, y \in X$,*

$$\begin{aligned}\psi(d(Ax, By)) \quad \leq \quad & \psi\left(\max\left\{d(By, Ty)\Big(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\Big), d(Ax, Sx)\Big(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\Big), d(Sx, Ty)\right\}\right) \\ & -\phi\left(\min\left\{d(By, Ty)\Big(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\Big), d(Ax, Sx)\Big(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\Big), d(Sx, Ty)\right\}\right),\end{aligned}$$

where $\phi \in \Phi$, $\psi \in \Psi$ Then A, B, S, and T have a unique common fixed point in X, whenever one of the range $A(X), B(X), S(X), T(X)$ is closed in X.

Corollary 2.4. *Suppose A, B, S and T are self mappings of a complete metric space (X, d), $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. If, for every $x, y \in X$,*

$$\psi(d(Ax, By)) \quad \leq \quad \psi\left(\max\left\{d(By, Ty)\Big(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\Big)\right\}\right) - \phi\left(\min\left\{d(By, Ty)\Big(\frac{1 + d(Ax, Sx)}{1 + d(Sx, Ty)}\Big)\right\}\right)$$

or

$$\psi(d(Ax, By)) \leq \psi\left(\max\left\{d(Ax, Sx)\Big(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\Big)\right\}\right) - \phi\left(\min\left\{d(Ax, Sx)\Big(\frac{1 + d(By, Ty)}{1 + d(Sx, Ty)}\Big)\right\}\right),$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Then A, B, S, and T have a unique common fixed point in X, whenever one of the range $A(X), B(X), S(X), T(X)$ is closed in X.

Also, by taking $S = T = I$ (identity mapping) in Theorem 2.1, we obtain the following.

Corollary 2.5. *Let (X, d) be a complete metric space and let $A, B : X \to X$ be two mappings such that, for every $x, y \in X$,*

$$\begin{aligned}\psi(d(Ax, By)) \quad \leq \quad & \psi\left(\max\left\{d(y, By)\Big(\frac{1 + d(x, Ax)}{1 + d(x, y)}\Big), d(x, y)\right\}\right) \\ & -\phi\left(\min\left\{d(y, By)\Big(\frac{1 + d(x, Ax)}{1 + d(x, y)}\Big), d(x, y)\right\}\right)\end{aligned}$$

or

$$\begin{aligned}\psi(d(Ax, By)) \quad \leq \quad & \psi\left(\max\left\{d(x, Ax)\Big(\frac{1 + d(y, By)}{1 + d(x, y)}\Big), d(x, y)\right\}\right) \\ & -\phi\left(\min\left\{d(x, Ax)\Big(\frac{1 + d(y, By)}{1 + d(x, y)}\Big), d(x, y)\right\}\right),\end{aligned}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Then A and B have a unique common fixed point in X.

However, if we assume $A = B$ and $S = T = I$ (identity mapping) in Theorem 2.1, we have the following.

Corollary 2.6. *Let (X, d) be a complete metric space and let $A : X \to X$ be a mapping such that, for every $x, y \in X$,*

$$\begin{aligned}\psi(d(Ax, Ay)) \quad \leq \quad & \psi\left(\max\left\{d(y, Ay)\Big(\frac{1 + d(x, Ax)}{1 + d(x, y)}\Big), d(x, Ax)\Big(\frac{1 + d(y, Ay)}{1 + d(x, y)}\Big), d(x, y)\right\}\right) \\ & -\phi\left(\min\left\{d(y, Ay)\Big(\frac{1 + d(x, Ax)}{1 + d(x, y)}\Big), d(x, Ax)\Big(\frac{1 + d(y, Ay)}{1 + d(x, y)}\Big), d(x, y)\right\}\right),\end{aligned}$$

where $\phi \in \Phi$ and $\psi \in \Psi$. Then A has a unique fixed point in X.

Hereto, we have obtained another result for mappings satisfying a generalized rational type condition under the weak compatibility and (ψ,ϕ)-weak contraction in complete metric spaces. This result also generalizes many other results in the literature.

Theorem 2.7. *Suppose that A, B, S and T are self mappings of a complete metric space (X, d), $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. If, for every $x, y \in X$,*

$$\psi(d(Ax, By)) \leq \psi(M_1(x, y)) - \phi(M_1(x, y)), \tag{2.9}$$

where $\psi \in \Psi$, $\phi \in \Phi$ such that ϕ is discontinuous at $t = 0$, and $M_1(x, y)$ is defined by (2.2). Then A, B, S, and T have a unique common fixed point in X, whenever one of the range $A(X), B(X), S(X), T(X)$ is closed in X.

Proof. By property of function ψ, we have $\phi(N_1(x, y)) \leq \phi(M_1(x, y))$, and therefore

$$\begin{aligned} \psi(d(Ax, By)) &\leq \psi(M_1(x, y)) - \phi(M_1(x, y)) \\ &\leq \psi(M_1(x, y)) - \phi(N_1(x, y)). \end{aligned}$$

Hence, Theorem 2.1 completes the proof. □

Here, we give the following example for the vindication of our result (Theorem 2.1) on (ψ,ϕ)-contraction.

Example 2.8. *Let $X = \{(1, 1), (1, 4), (4, 1), (4, 5), (5, 4)\}$ be endowed with metric d defined by*

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Suppose $A, B, S, T : X \to X$ are such that

$$\begin{aligned} A(x_1, x_2) &= B(x_1, x_2) = \begin{cases} (x_1, 1) & \text{if } x_1 \leq x_2 \\ (1, x_2) & \text{if } x_1 > x_2. \end{cases} \\ S(x_1, x_2) &= T(x_1, x_2) = (x_1, x_2). \end{aligned}$$

Choose $\psi(t) = t$ and $\phi(t) = \frac{t}{6}$. Clearly, mappings A, B, S, T do not satisfy the condition (1.3) of Theorem 1.4. To see this, at $x = (4, 5)$ and $y = (5, 4)$, we have $d(Ax, By) = 6, M(x, y) = 4$, $N(x, y) = 2$, $M_1(x, y) = \frac{20}{3}$ and $N_1(x, y)) = 2$. Then, $\psi(d(Ax, By)) \leq \psi(M(x, y)) - \phi(N(x, y))$ implies $6 \leq 4 - \frac{1}{3}$, which is not possible. Hence the condition (1.3) is not satisfied. However, the condition (2.1) of our Theorem 2.1 is satisfied for all $x, y \in X$ and $(1, 1)$ is the only common fixed point. Moreover, it is clear that $A(X) \subseteq T(X), B(X) \subseteq S(X)$, and the pairs $\{A, S\}, \{B, T\}$ are weakly compatible.

3. Compliance with Ethical Standards

3.1. Conflict of interest

The authors declare that they have no conflict of interest.

3.2. Ethical approval

This article does not contain any studies with human participants or animals performed by any of the authors.

3.3. Informed consent

Informed consent was obtained from all individual participants included in the study.

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M. C. Arya,
Department of Mathematics,
B. S. R. Govt. Degree College, Rikhanikhal, Pauri Garwal, Uttarakhand,
India.
E-mail address: mcarya1986@gmail.com

and

N. Chandra,
Department of Mathematics,
H. N. B. Garhwal University, B. G. R. Campus, Pauri, Uttarakhand,
India.
E-mail address: cnaveen329@gmail.com

and

Mahesh C. Joshi,
Department of Mathematics,
D. S. B. Campus, Kumaun University, Nainital, Uttarakhand,
India.
E-mail address: mcjoshi69@gmail.com