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Three Weak Solutions for a Class of Quasilinear Choquard Equations Involving the Fractional $p(x,.)$-Laplacian Operator with Weight

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ABSTRACT: In this paper, we establish the existence of at least three weak solutions to a problem involving the fractional $p(x,.)$- Laplacian operator with weight. Our method used for obtaining the existence of three solutions for a class of Choquard equations is based on the variational method concerned a type of version of Ricceri.

Key Words: Quasilinear Choquard equations, fractional $p(x,.)$-Laplacian with weight, variational methods, nonlinear Elliptic equations.

Contents

1. Introduction

Partial differential equations ($PDEs$) and variational problems with nonstandard $p(x,.)$-growth conditions does not only have an impact in the applied sciences, but also plays a very important role in the current development of mathematics itself, both in geometry and in analysis. The study of such problems has a strong development due to the fact that they can model various phenomena, which arise in the problem of elastic mechanics [18], electrorheological fluids [17] or image restoration [8] etc.

Besides, the $p(x,.)$-Laplacian operator, where $p(x,.)$ is a continuous function, possesses more complicated properties than the p-Laplacian operator mainly due to the fact that it is not homogeneous. There has been many results devoted to the existence of solutions for variable exponent problems, for more details we refer readers to [4,3] and the references therein. Moreover, some researchers extended the integer case to the fractional one. In particular, many authors generalized the last operator to the fractional case (fractional $p(x,.)$-Laplacian operator).

The main objective of this work is to present some results about the existence of three weak solutions for the following class of nonlinear Choquard problem:

$$(P^s)\begin{cases} -(\Delta_{p(x,.)})^s u + w(x)|u|^{\overline{p}(x)-2}u = (\int_\Omega \frac{F(y,u(y))}{|x-y|^{a(x,y)}}dy)f(x,u(x)) & \\ \qquad\qquad\qquad +\lambda(\int_\Omega \frac{G(y,u(y))}{|x-y|^{b(x,y)}}dy)g(x,u(x)) & in\ \ \Omega, \\ \qquad\qquad\qquad u=0 & on\ \ \mathbb{R}^{\mathbb{N}}\backslash\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^{\mathbb{N}}$ is a Lipschitz bounded open domain, $p:\bar\Omega\times\bar\Omega\longrightarrow(1,+\infty)$ is a bounded continuous functions. We assume that

$$1<p^-=\min_{(x,y)\in\bar\Omega\times\bar\Omega}p(x,y)\le p(x,y)\le p^+=\max_{(x,y)\in\bar\Omega\times\bar\Omega}p(x,y)<+\infty. \tag{1.1}$$

and we set $\bar p(x)=p(x,x)$ for all $x\in\bar\Omega$, $s\in(0,1)$ and $a,b\in C(\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}},\mathbb{R})$, λ is a positive real number, $w\in L^\infty(\Omega)$ such that $\inf_{x\in\bar\Omega}w(x)=w^->0$ and f,g such that $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ and $g:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ are two *Carathéodory* functions with subcritical growth conditions, such that $F(x,t)=\int_0^t f(x,s)ds$ and $G(x,t)=\int_0^t g(x,s)ds$ are the primitive of f and g respectively . Here, the operator $(-\Delta_{p(x,.)})^s$ is the fractional $p(x,.)$-Laplacian operator defined as follows

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$$(-\Delta_{p(x,.)})^s u(x) = p.v. \int_{\mathbb{R}^{\mathbb{N}}} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy$$

for all $x \in \mathbb{R}^{\mathbb{N}}$,where $p.v.$ is a commonly used abbreviation in the principal value sense.

Note that $(-\Delta_{p(x,.)})^s$ is a nonlocal pseudo-differential operator of elliptic type which can be seen as a generalization of the fractional p-Laplacian operator $(-\Delta_p)^s$ in the constant exponent case (i.e., when p(x,.)=p=constant) and is the fractional version of the well-known $p(x)$-Laplacian operator $\Delta_{p(x)}u(x) = div(|\nabla u(x)|^{p(x)-2}u(x))$ (where $\overline{p}(x) = p(x,x)$) which is associated with the variable exponent Sobolev space.

One typical feature of the problem (P^s) is the nonlocality in the sense that the value of $(-\Delta_{p(x)})^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^{\mathbb{N}}$. Therfore, the Dirichlet datum given in $\mathbb{R}^{\mathbb{N}}\backslash\Omega$ (which is different from the classical case of the $p(x)$- Laplacian) and is not simply on $\partial\Omega$, which implies that the first equation in (P^s) is no longer a pointwise equation. It is no longer a pointwise identity. Therefore it is often called nonlocal problem. This causes some mathematical difficulties, which make the study of such a problem particularly interesting and challenging.

The nonlinearity on the right-hand side of (P^s) is motivated by the Choquard equation which was proposed by Choquard in 1976, and can be described an electron trapped in its own hole. Very recently Alves, *Rădulescu* and Tavares [1] studied generalized Choquard equations driven by nonhomogeneous operators. In [2] , Alves et al. proved a Hardy-Littlewood-Sobolev-type inequality for variable exponents and used it to study the quasilinear Choquard equations involving variable exponents. Further results for related problems we refer to [7,2,1] and references therein.

In recent years, the kind of problems of the form (P^s) in which a fractional variable exponent operator Quasilinear Choquard equations have been extensively studied by many authors, using various methods, we refer the reader to [15,6,9] and the references therein.

For instance, E. Azroul, A. Benkirane, M. Shimi and M. Srati in [4] used the fundamental tool for proving the existence result is a recent three critical-points theorem established by Ricceri for a problem involving the fractional $p(x,.)$-Laplacian operator with weight defined as follow:

$$(P^1) \begin{cases} -(\Delta_{p(x,.)})^s u(x) + w(x)|u(x)|^{\bar{p}(x)-2}u(x) = \lambda f(x,u) + \mu g(x,u), & in \ \ \Omega, \\ u = 0, & on \ \ \mathbb{R}^{\mathbb{N}}\backslash\Omega, \end{cases}$$

where Ω is a Lipschitz open and bounded set in $\mathbb{R}^{\mathbb{N}}$, $(N \geq 3)$, $\lambda > 0$ is a real number, $p : \bar{\Omega}\times\bar{\Omega} \longrightarrow]1,+\infty[$ is a bounded continuous functions and $\overline{p}(x) = p(x,x)$ for all $x \in \bar{\Omega}$, $s \in (0,1)$, λ, μ are two positive real numbers.

In other work of Reshmi Biswas and Sweta Tiwari [7], they established the existence and multiplicity results for the variable order nonlocal Choquard problem with variable exponents and in the same work they study the analougs Hardy-Sobolev Littlewood-type result for variable exponents suitable for the fractional Sobolev space with variable order and variable exponents. they considered the following problem

$$(P^2) \begin{cases} -(\Delta_{p(.)})^{s(.)} u(x) = \lambda|u(x)|^{\alpha(x)-2}u(x) \\ \qquad\qquad +(\int_\Omega \frac{F(y,u(y))}{|x-y|^{\mu(x,y)}}dy)f(x,u(x)) & in \ \ \Omega, \\ \qquad\qquad u = 0, & on \ \ \mathbb{R}^{\mathbb{N}}\backslash\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a smooth and bounded domain, $N \geq 2$ $p \in C(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, (1,\infty))$, $s \in C(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, (0,1))$, $\mu \in C(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}, \mathbb{R})$ and $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ are continuous functions.
The authors, considered some conditions about the funcional f and by using the variational methods, which is the mountain-pass geometry they proved the existence and multiplicity of solutions for the above problem (P^2).

In the present paper, we are concerned with the existence of solutions for quasilinear Choquard Equations involving by Fractional $p(x,.)$-Laplacian operator. Our main objective in this work is to generalize results concerning quasilinear Choquard equations involving variable exponents and the following problems (P^1) and (P^2) to the fractional case by using one type version of Ricceri note that this version was used in [12,13,14].

Throughout this paper, we suppose the following assumpions, then we discuss several settings where these assumptions are satisfied.

(H_1) p is symmetric, that is,

$$p(x,y) = p(y,x),\ \forall\ (x,y) \in \bar{\Omega} \times \bar{\Omega}. \tag{1.2}$$

(H_2) $a : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ is a symmetric function, i.e. $a(x,y) = a(y,x)$ $\forall\ (x,y) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ with $0 < a^- \leq a^+ < N$.

(H_3) $b : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ is a symmetric function, i.e. $b(x,y) = b(y,x)$ $\forall\ (x,y) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ with $0 < b^- \leq b^+ < N$.

(F_1) $f, g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ such that $\mid f(x,t) \mid, \mid g(x,t) \mid \leq A_1 + A_2 \mid t \mid^{r(x)-1}$ $\forall (x,t) \in \bar{\Omega} \times \mathbb{R}$, where $r \in C(\bar{\Omega})$, $A_1, A_2 > 0$ and $1 \leq r(x) \leq p_s^*(x)$ $\forall x \in \bar{\Omega}$, with

$$p_s^*(x) = \tfrac{N\overline{p}(x)}{N - s\overline{p}(x)}.$$

(F_2) $\lim\limits_{|t| \longrightarrow \infty} [F(x,t) - \frac{\lambda_1}{p(x)} \mid t \mid^{p^-}] = -\infty$ uniformly for almost every $x \in \bar{\Omega}$.

(F_3) There exist $x_0 \in \Omega$, $\rho_0 \in]0,1[$ and $t_0 > 1$ with $B(x_0, 2\rho_0) \subset \Omega$ such that
$F(x,t) \geq 0$ for $x \in B(x_0, 2\rho_0) \subset \Omega$ and $t \in]0, t_0]$,
$F(x, t_0) \geq A_0$ for $x \in B(x_0, \rho_0)$.
Denote

$$A_0 = [(\tfrac{2}{\rho_0})^{p^+(B(x_0,2\rho_0),B(x_0,2\rho_0))}(2^N - 1) + \mid w \mid_\infty 2^N] \tfrac{|t_0|^{p^+(B(x_0,2\rho_0),B(x_0,2\rho_0))}}{p^-(B(x_0,2\rho_0),B(x_0,2\rho_0))},$$

such that

$$\begin{aligned} p^-(B(x_0,2\rho_0),B(x_0,2\rho_0)) &= \inf_{x,y \in B(x_0,2\rho_0) \times B(x_0,2\rho_0)} p(x,y), \\ p^+(B(x_0,2\rho_0),B(x_0,2\rho_0)) &= \sup_{x,y \in B(x_0,2\rho_0) \times B(x_0,2\rho_0)} p(x,y) \end{aligned}$$

(F_4) There exist $a_0 > 0, \varepsilon > 0$ such that
$F(x,t) \leq a_0 \mid t \mid^{r_0(x)}, \forall x \in \Omega, \mid t \mid < \varepsilon$, where $r_0 \in C(\bar{\Omega})$ with
$p^+ < r_0(x) < p_s^*(x)$, for $x \in \bar{\Omega}$.

(G_1) There exist an open ball $B(x_1, \rho_1) \subset \Omega$, $\mu \in C(B(x_1, \rho_1), \mathbb{R})$ with
$1 \leq \mu(x) \leq \mu^+(B(x_1,\rho_1)) \leq p^-(B(x_1,\rho_1), B(x_1,\rho_1))$, $z > 0$ and $\theta > 0$ such that

$$G(x,t) \geq z \mid t \mid^{\mu^+(B(x_1,\rho_1))} \text{ for all } x \in B(x_1,\rho_1) \text{ and } \mid t \mid < \theta.$$

with

$$\mu^+(B(x_1,\rho_1)) = \sup_{x \in B(x_1,\rho_1)} \mu(x),\ \mu^-(B(x_1,\rho_1)) = \inf_{x \in B(x_1,\rho_1)} \mu(x).$$

Now we present an example which verifies assumptions (H_1), (H_2), (H_3), (F_1), (F_2), (F_3), (F_4) and (G_1).

Example 1.1. *We just consider Ω a convex domain (for example $\Omega = B(0,R)$), then we suppose that :*

1. *$a(x,y) = b(x,y) = p(x,y) = xy$, $\forall x, y \in \Omega$, to check assumptions (H_1), (H_2) and (H_3),*
2. *Concerning assumption (F_1), we just take f and g two polynomial functions*

3. *If* $\Omega = B(0,5)$*, we take* $x_0 = (0,1) \in \Omega$*,* $\rho_0 = \frac{1}{2} \in]0,1[$ *and* $t_0 > 1$*, then* $B(x_0, 2\rho_0) \subseteq \Omega$ *and for* $f(x,s) = 2x^2 s$ *we have* $F(x,t) \geq 0$*, this example verifies assumption* (F_3)

4. *Finally, we take* $\mu(x) = \|x\|$*,* $p(x,y) = \|x+y\|$*, for all* $x, y \in B(0,2) \times B(0,2)$ *and for* t *small enough we just take* $\theta = \varepsilon$*. We consider* $G(x,t) = 2xt$*, this example verifies assumption* (G_1)

The main result of this paper is the following theorem

Theorem 1.2. *Under assumptions* (H_1)*,* (H_2)*,* (H_3)*,* (F_1)*,* (F_2)*,* (F_3)*,* (F_4) *and* (G_1)*. Then, there exist* $\lambda_* > 0$ *such that for any* $\lambda \in]0, \lambda_*[$*, the problem* (P^s) *admits at least three weak solutions.*

This work based on three sections. In section 2, we give some preliminary important results on theory of Lebesgue-Sobolev spaces with variables exponent. Moreover, we present some fundamental propositions follows from the Hardy-Littlewood-Sobolev type of inequality of the solution of the variable order nonlocal Choquard equation with variable exponents. In section 3, we present one type version of Ricceri's variational principle that we use to prove that the problem (P^s) has three solutions.

2. Variational setting and Preliminaries Results

In this subsection, we present the main properties and results on variable exponent Lebesgue and Sobolev spaces. For more details, we refer the reader to [13,16]. Denote by $M(\Omega)$ the set of all measurable real functions on Ω and consider the set

$$C_+(\bar{\Omega}) = \{q \in C(\bar{\Omega}) : q(x) > 1 \ for \ all \ x \in \bar{\Omega}\}.$$

For all $q \in C_+(\bar{\Omega})$, we define

$$q^+ = \sup_{x \in \bar{\Omega}} q(x) \ and \ q^- = \inf_{x \in \bar{\Omega}} q(x),$$

such that

$$1 < q^- \leq q(x) \leq q^+ < +\infty. \tag{2.1}$$

For any $q \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \{u \in M(\Omega) : \textstyle\int_\Omega |u(x)|^{q(x)} dx < +\infty\}.$$

This vector space endowed with the Luxemburg norm, which is defined by

$$\|u\|_{L^{q(x)}(\Omega)} = \inf\{\lambda > 0 : \textstyle\int_\Omega |\frac{u(x)}{\lambda}|^{q(x)} dx \leq 1\}$$

is a separable reflexive Banach space.
Let $\widehat{q} \in C_+(\bar{\Omega})$ be the conjugate exponent of q, that is,

$$\tfrac{1}{q(x)} + \tfrac{1}{\widehat{q}(x)} = 1.$$

So we give the following $H\ddot{o}lder$-type inequality.

Lemma 2.1 ($H\ddot{o}lder's$ **inequality**)**.** *If* $u \in L^{q(x)}(\Omega)$ *and* $v \in L^{\widehat{q}(x)}(\Omega)$*, then*

$$|\textstyle\int_\Omega uv dx| \leq (\frac{1}{q^-} + \frac{1}{\widehat{q}^-})\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\widehat{q}(x)}(\Omega)} \leq 2\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\widehat{q}(x)}(\Omega)}$$

An immediate consequence of the $H\ddot{o}lder's$ inequality is the following.

Corollary 2.2. *[13] If* $r(.), q(.) \in C_+(\bar{\Omega})$*, define* $p(.) \in C_+(\bar{\Omega})$ *by*

$$\tfrac{1}{p(x)} = \tfrac{1}{q(x)} + \tfrac{1}{r(x)}.$$

Then there exists a positive constant C *such that, for all* $u \in L^{q(x)}(\Omega)$ *and* $v \in L^{r(x)}(\Omega)$*,* $uv \in L^{p(x)}(\Omega)$ *and*

$$\|uv\|_{L^{p(x)}(\Omega)} \leq C\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{r(x)}(\Omega)}.$$

We define the modular of the space $L^{q(x)}(\Omega)$ by

$$\rho_{q(.)}(u) : L^{q(x)}(\Omega) \longrightarrow \mathbb{R}.$$

such that

$$\rho_{q(.)}(u) = \int_\Omega |u(x)|^{q(x)} dx$$

Proposition 2.3. *[4] Let $u \in L^{q(x)}(\Omega)$ and $\{u_k\} \subset L^{q(x)}(\Omega)$ then we have*

(i) $\|u\|_{L^{q(x)}(\Omega)} < 1(resp. = 1, > 1) \Longleftrightarrow \rho_{q(.)}(u) < 1(resp. = 1, > 1)$,

(ii) $\|u\|_{L^{q(x)}(\Omega)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(x)}(\Omega)} \leq \rho_{q(.)}(u) \leq \|u\|^{q^-}_{L^{q(x)}(\Omega)}$,

(iii) $\|u\|_{L^{q(x)}(\Omega)} > 1 \Longrightarrow \|u\|^{q^-}_{L^{q(x)}(\Omega)} \leq \rho_{q(.)}(u) \leq \|u\|^{q^+}_{L^{q(x)}(\Omega)}$,

(iv) $\lim_{k\to+\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0 \Longleftrightarrow \lim_{k\to+\infty} \rho_{q(.)}(u_k - u) = 0$.

Now, let $w \in M(\Omega)$ with $w(x) > 0$ for a.e.$x \in \Omega$. We define the weighted variable exponent Lebesgue space $L^{q(x)}_w(\Omega)$ by

$$L^{q(x)}_w(\Omega) = \{u \in M(\Omega) : \int_\Omega w(x)|u(x)|^{q(x)} dx < +\infty\},$$

with the norm

$$\|u\|_{q,w} = inf\{\gamma > 0 : \int_\Omega w(x)|\frac{u(x)}{\gamma}|^{q(x)} dx \leq 1\}.$$

Then $L^{q(x)}_w(\Omega)$ is a Banach space obviously [10,5] . Besides, the weighted modular on $L^{q(x)}_w(\Omega)$ is defined as follows

$$\rho_{q,w} : L^{q(x)}_w(\Omega) \longrightarrow \mathbb{R}$$

such that

$$\rho_{q,w}(u) = \int_\Omega w(x)|u(x)|^{q(x)} dx$$

the following proposition is similar to Proposition 2.3, and it follows easily from the definition of $\|u\|_{q,w}$ and $\rho_{q,w}$.

Proposition 2.4. *[4] Let $u \in L^{q(x)}_w(\Omega)$ and $\{u_k\} \subset L^{q(x)}_w(\Omega)$ then we have*

(i) $\|u\|_{q,w} < 1(resp. = 1, > 1) \Longleftrightarrow \rho_{q,w}(u) < 1(resp. = 1, > 1)$,

(ii) $\|u\|_{q,w} < 1 \Longrightarrow \|u\|^{q^+}_{q,w} \leq \rho_{q,w}(u) \leq \|u\|^{q^-}_{q,w}$,

(iii) $\|u\|_{q,w} > 1 \Longrightarrow \|u\|^{q^-}_{q,w} \leq \rho_{q,w}(u) \leq \|u\|^{q^+}_{q,w}$,

(iv) $\lim_{n\to+\infty} \|u_n\|_{q,w} = 0 \Longleftrightarrow \lim_{n\to+\infty} \rho_{q,w}(u_n) = 0$.

(v) $\lim_{n\to+\infty} \|u_n\|_{q,w} = \infty \Longleftrightarrow \lim_{n\to+\infty} \rho_{q,w}(u_n) = \infty$.

Let Ω be a smooth bounded open set in $\mathbb{R}^{\mathbb{N}}$. We take $s \in (0, 1)$ and
$p : \bar{\Omega} \times \bar{\Omega} \longrightarrow (1, +\infty)$ be a continuous bounded function.

We define the fractional Sobolev space $W = W^{s,p(x,y)}(\Omega)$ with variable exponent via the Gagliardo approach as follows :

$$W = \{u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}} dxdy < +\infty, \text{for some } \gamma > 0\}$$

Where $L^{\bar{p}(x)}(\Omega)$ is the Lebesgue space with variable exponent. $W^{s,p(x,y)}(\Omega)$ is a Banach space, if it is equipped with the norm:

$$\|u\|_W = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[u]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = [u]_{s,p(x,y)}(\Omega) = inf\{\gamma > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}} dxdy \leq 1\}$$

$(W, \|.\|_W)$ is a separable reflexive space, see[[6], Lemma3.1].

In [15], Kaufmann, Rossi and Vidal introduced the variable exponent Sobolev fractional space $E = W^{s,q(x),p(x,y)}(\Omega)$ as follows:

$$E = \{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}} dxdy < +\infty, \text{ for some } \gamma > 0\},$$

where $q : \bar{\Omega} \longrightarrow (1, +\infty)$ is a continuous function, such that:

$$1 < q^- = \min_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} q(x) \leq q(x) \leq q^+ = \max_{(x,y)\in\bar{\Omega}\times\bar{\Omega}} q(x) < +\infty$$

Remark 2.5. *Let W_0 denote the closure of $\overline{C_0^\infty(\Omega)}^{\|.\|_W}$ Next, we introduce the fractional weighted variable exponent Sobolev space as follows:*

$$W_w = W_w^{s,p(x,y)}(\Omega) = \{u \in L_w^{\overline{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{sp(x,y)+N}} dxdy < +\infty, \textit{ for some } \gamma > 0\},$$

which endowed with the norm:

$$\begin{aligned} \|u\|_w &= \|u\|_{W_w} \\ &= inf\{\gamma > 0 : \int_{\Omega\times\Omega} \frac{\mid u(x) - u(y) \mid^{p(x,y)}}{\gamma^{p(x,y)} \mid x-y \mid^{sp(x,y)+N}} dxdy + \int_\Omega w(x) |\frac{u(x)}{\gamma}|^{q(x)} dx \leq 1\} \end{aligned}$$

The norms $\|.\|_w$ and $\|.\|_W$ are equivalent in W_w. Moreover, the space $(W_w, \|.\|_w)$ is a separable reflexive Banach space.

We set

$$\rho^w_{p(.,.)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{sp(x,y)+N}} dxdy + \int_\Omega w(x) \mid u(x) \mid^{\overline{p}(x)} dx,$$

which is a modular on W_w, and it satisfies the following inequalities.

Proposition 2.6. *[4] For all $u \in W_w$ we have*

(i) $\|u\|_w < 1(resp. = 1, > 1) \Longleftrightarrow \rho^w_{p(.,.)}(u) < 1(resp. = 1, > 1)$,

(ii) $\|u\|_w < 1 \Longrightarrow \|u\|_w^{p^+} \leq \rho^w_{p(.,.)}(u) \leq \|u\|_w^{p^-}$,

(iii) $\|u\|_w > 1 \Longrightarrow \|u\|_w^{p^-} \leq \rho^w_{p(.,.)}(u) \leq \|u\|_w^{p^+}$,

Theorem 2.7. *[15] Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in]0,1[$. Let $p : \bar{\Omega}\times\bar{\Omega} \to]1,\infty[$ be a continuous variable exponent with $sp(x,y) < N$ for all $(x,y) \in \bar{\Omega}\times\bar{\Omega}$. Let 1.1 and 2.1 are satisfied and $r : \bar{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent such that*

$$p_s^*(x) = \frac{N\overline{p}(x)}{N-s\overline{p}(x)} > r(x) \geq r_- > 1, \textit{ for all } x \in \bar{\Omega}$$

Then, there exists a constant $C = C(N,s,p,r,\Omega) > 0$, such that, for any $u \in W$, $\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_W$.

Thus, this embedding is continuous and compact for any $r \in]1, p_s^*[$

$$W \to L^{r(x)}(\Omega).$$

In order to formulate the variational approach of problem (P^s) we introduce the following defintion:

Definition 2.8. *Let $u \in W_0$, we side that u is a weak solution of problem (P^s), if for all $v \in W_0$, we have*

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp(x,y)+N}}dxdy + \int_\Omega w(x) \mid u \mid^{\bar{p}(x)-2} uvdx$$
$$= \int_{\Omega\times\Omega} \frac{F(x,u(x))f(y,u(y))v(y)}{|x-y|^{a(x,y)}}dxdy + \lambda \int_{\Omega\times\Omega} \frac{G(x,u(x))g(y,u(y))v(y)}{|x-y|^{b(x,y)}}dxdy.$$

Definition 2.9. *The energy functional $\varphi : W_w \longrightarrow \mathbb{R}$ associated to the problem (P^s) is defined as*

$$\varphi(u) = \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{\mid u(x)-u(y) \mid^{p(x,y)}}{\mid x-y \mid^{sp(x,y)+N}}dxdy + \int_\Omega \frac{w(x)}{\bar{p}(x)} \mid u \mid^{\bar{p}(x)} dx$$
$$- \frac{1}{2}\int_{\Omega\times\Omega} \frac{F(x,u(x))F(y,u(y))}{\mid x-y \mid^{a(x,y)}}dxdy - \frac{\lambda}{2}\int_{\Omega\times\Omega} \frac{G(x,u(x))G(y,u(y))}{\mid x-y \mid^{b(x,y)}}dxdy.$$

To prove theorem 1.2, we need to show some auxiliary lemmas.

So we define the functionals $\phi, \psi, J : W_w \longrightarrow \mathbb{R}$ corresponding to the problem (P^s), as follow:

$$\phi(u) = \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{\mid u(x)-u(y) \mid^{p(x,y)}}{\mid x-y \mid^{sp(x,y)+N}}dxdy + \int_\Omega \frac{w(x)}{\bar{p}(x)} \mid u \mid^{\bar{p}(x)} dx,$$

$$\psi(u) = -\int_{\Omega\times\Omega} \frac{1}{2}\frac{F(x,u(x))F(y,u(y))}{\mid x-y \mid^{a(x,y)}}dxdy,$$

and

$$J(u) = -\int_{\Omega\times\Omega} \frac{1}{2}\frac{G(x,u(x))G(y,u(y))}{\mid x-y \mid^{b(x,y)}}dxdy.$$

Lemma 2.10. *[2,7] Let f, g two Carathèodory functions, then $\psi, J \in C^1(W_w,\mathbb{R})$ with the derivatives given by*

$$\langle \psi^{'}(u), v\rangle = -\int_{\Omega\times\Omega} \frac{1}{2}\frac{F(x,u(x))f(y,u(y))v(y)}{\mid x-y \mid^{a(x,y)}}dxdy$$

and

$$\langle J^{'}(u), v\rangle = -\int_{\Omega\times\Omega} \frac{1}{2}\frac{G(x,u(x))g(y,u(y))v(y)}{\mid x-y \mid^{b(x,y)}}dxdy.$$

Let define $I(u) = \phi(u) + \psi(u)$. The critical point of the integral functional $\varphi(u) = I(u) + \lambda J(u)$ is solution of the problem (P^s).

Lemma 2.11. *[4,6] Assume that the assumptions of Theorem 1.2 are satisfied , then*

(i) the functional ϕ is sequentially weakly lower semi-continuous, and its Gâteaux derivative $\phi^{'} : W_w \longrightarrow W_w^*$ is given by
$$\langle \phi^{'}(u), v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{sp(x,y)+N}}dxdy + \int_\Omega w(x) \mid u \mid^{\bar{p}(x)-2} uvdx.$$

(ii) the functional $\phi^{'} : W_w \longrightarrow W_w^*$ is a strictly monotone, bounded, homeomorphism and is of type (S_+), i.e. if $u_k \rightharpoonup u$ in W_w and $\lim\limits_{k\to+\infty} sup\langle \phi^{'}(u_k) - \phi^{'}(u), u_k - u\rangle \leq 0$ implies $u_k \to u$, where $\langle .,.\rangle$ denotes the usual duality between W_w and its dual space W_w^*.

We get some definitions and properties of Ricceri's variational principle in the following paragraph, when we used to show the main results concerning the existence of three nontrivial solutions.

Definition 2.12. *[12] Let G a bounded subset of W_w and $r \in \mathbb{R}$. G is called a block of I with type r if $I(u) < r, \forall x \in G$ and $I(x) = r, \forall x \in \overline{\partial G}$. Where $\overline{\partial G} = \overline{G}^{W_w} \backslash G$ and $\overline{G}^{W_w}$ is the closure of G in W_w in the weak topology.*

Definition 2.13. *[12] Let G a bounded open subset of W_w and $c < c^{'}$ is called Ricceri box of I with the type $(c, c^{'})$ if*

$$c = \inf_{G} I < \inf_{\overline{\partial G}} I = c^{'}.$$

Definition 2.14. *[12] Let Y be a Banach space, G_0 and G be two bounded open subset of Y with $\overline{G_0} \subset G$ and $\phi : Y \longrightarrow \mathbb{R}$ a functional. (G_0, G) is a valley box of ϕ if*

$$\sup_{G_0} \phi < \inf_{\partial G} \phi.$$

Theorem 2.15. *[12] Assume that $I, J : W_W \longrightarrow \mathbb{R}$ are sequentially weakly lower semi continuous and G is a Ricceri block of I with type r. Let*

$$\lambda_* = \sup_{x \in G} \frac{r - I(x)}{J(x) - \inf_{\overline{G}^{W_w}} J}$$

then for each $\lambda \in]0, \lambda_*[$, the restriction of $I + \lambda J$ to $\overline{G}^{W_w}$ achieves its infimum at some $x_* \in G$, so x_* is a local minimizer of $I + \lambda J$.

Remark 2.16. *(i) let $u_* \in W_w$ a strictly local minimizer of I, then for $\varepsilon > 0$ small enough, we have $\inf_{\partial B(u,\varepsilon)} I > I(u_*)$ i.e. $B(u_*, \varepsilon)$ is a Ricceri box of I.*

(ii) So, by proposition 2.6 in [12], I, $J : W_w \longrightarrow \mathbb{R}$ are sequentially weakly lower semi continuous.

Proposition 2.17. *[12] Suppose that G is a Ricceri box of I with type $(c, c^{'})$ and $I : W_w \longrightarrow \mathbb{R}$ continuous. Then for every $r \in]c, c^{'}]$ we have $I^{-1}(] - \infty, r[) \cap G$ is a Ricceri block of I with type r.*

Proposition 2.18. *[11,12] Suppose that I, $J : W_w \longrightarrow \mathbb{R}$ are continuous. For some $\rho > 0$, $u_1 \in B(u_0, \rho)$, $I(u_0) = \inf_{B(u_0,\rho)} I = c_0$; $\inf_{\partial B(u_0,\rho)} I = c^{'} > c_0$ and u_1 is a strictly local minimizer of I and $I(u_1) = c_1 > c_0$. Then for $\varepsilon > 0$ small enough and $r_1 > c_1$, $r_0 \in]c_0, \min\{c^{'}, c_1\}[$ and $\forall \lambda \in]0, \lambda_*[$, $I + \lambda J$ has at least two local minima u_0^*, u_1^* in $B(u_0, \rho)$. Where $u_0^* \in I^{-1}(] - \infty, r_0[) \cap B(u_0, \rho)$, $u_0^* \in \overline{B(u_1, \varepsilon)}$ and $u_1^* \in I^{-1}(] - \infty, r_1[) \cap B(u_1, \rho)$*

Theorem 2.19. *[11] Let Y be a reflexive Banach space. Assume that*

(1) $\phi \in C^1(Y, \mathbb{R})$, the mapping $\phi^{'} : Y \longrightarrow Y^*$ is of type S_+.

(2) (G_0, G) is a valley box of ϕ with G_0, G being connected and $0 \in G_0$.

(3) There exist $e \in G_0$ and $\rho > 0$ such that

$$\|e\| > \rho, \ \inf_{\partial B(0,\rho)} \phi > \max\{\phi(0), \phi(e)\}.$$

Then the functional ϕ has at least a critical point $u_0 \in \overline{G}$ with $\phi(u_0) = c$, where $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \phi(\gamma(t))$ and $\Gamma = \{\gamma \in C([0, 1], G) : \gamma(0) = 0; \gamma(1) = e\}$.

Corollary 2.20. *[13] Under the same assumption as in previous theorem, furthermore, if $J : Y \longrightarrow \mathbb{R} \in C^1$ and $J^{'} : Y \longrightarrow Y^{'}$ are weakly strongly continuous. Then, for each $\lambda \in]0, \lambda_*[$, $I + \lambda J$ has still a mountain pass type critical point $u_2 \in \overline{G}$.*

3. Proof of the main results

Proof: [**Proof of Theorem 1.2**]

Define the positive parameter λ_1 as follow

$$\lambda_1 = \inf_{u\in W_w\setminus\{0\}} \frac{\phi(u)}{\int_\Omega \frac{1}{\overline{p}(x)}|u|^{\overline{p}(x)}dx}.$$

step (1) : We show that $v_0 = 0$ is strictly local minimizer of I. By (F_1) and assumption (F_4), we may find $r_1 \in C(\overline{\Omega})$ with $\frac{p^+}{2} < r_1^- \le r_1(x) < p_s^*(x)$ such that

$$F(x,t) \le a_3|t|^{r_1(x)},\ \forall x\in\Omega,\ \forall t\in\mathbb{R}. \tag{3.1}$$

We suppose that $\|u\|_w < 1$ is small enough, so by [7] and proposition 2.6 there exists positive constants c_3, c_4 and c_5 such that

$$\begin{aligned} I(u) &\ge \int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y}}dxdy + \int_\Omega \frac{w(x)}{\overline{p}(x)}|u|^{\overline{p}(x)}dx \\ &- \frac{1}{2}\int_{\Omega\times\Omega}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{a(x,y)}}dxdy \\ &\ge \frac{1}{p^+}\|u\|_w^{p^+} - c_3(max\{\|u\|_w^{2r_1^+},\|u\|_w^{2r_1^-}\}) \\ &\ge \frac{1}{p^+}\|u\|_w^{p^+} - c_3\|u\|_w^{2r_1^-}, \end{aligned} \tag{3.2}$$

because, we have

$$\begin{aligned} \left|\int_{\Omega\times\Omega}\frac{F(x,u(x))F(y,u(y))}{|x-y|^{a(x,y)}}dxdy\right| &\le c_4(\|F(.,u(.))\|^2_{L^{r_1^+}(\Omega)} + \|F(.,u(.))\|^2_{L^{r_1^-}(\Omega)}) \\ &\le c_5\{max\{\|u\|_w^{2r_1^+},\|u\|_w^{2r_1^-}\}\} \end{aligned} \tag{3.3}$$

Since $2r_1^- > p^+$ there exist $\varepsilon > 0$, such that $\forall x,y \in \overline{B(0,\varepsilon)}\setminus\{0\}$ with $x\ne y$, we have $I(u) > 0 = I(v_0)$
step (2) : we prove that the functional I has a global minimizer $v_1 \ne 0$. Set $\Re(x,t) = F(x,t) - \frac{\lambda_1}{p(x)}|t|^{p^-}$, so we use assumption (F_2) we remark that for any $M>0$, there exist R_M, such that

$$\Re(x,t) \le -M,\ \forall |t| \ge R_M,\ almost\ every\ x\in\Omega. \tag{3.4}$$

We have I is coercive, or else there exist $K\in\mathbb{R}$ and $(u_n)_n \subset W$ such that

$$\|u_n\|_w \longrightarrow \infty \quad \text{and} \quad I(u_n) \le K.$$

Define the sequence v_n as follow $v_n = \frac{u_n}{\|u_n\|_w}$, so that $\|v_n\|_w = 1$.

Then for subsequence, we may prove that for $v\in W_w$. We obtain $v_n \rightharpoonup v$ in W_w, $v_n \longrightarrow v$ in $L^{\overline{p}(x)}(\Omega)$, $v_n(x) \longrightarrow v(x)$ for almost every $x\in\Omega$.

Now, applying 3.4, we have

$$\begin{aligned}
K &\geq I(u_n) = \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y}} dxdy + \int_{\Omega} \frac{w(x)}{\overline{p}(x)} |u_n|^{\overline{p}(x)} dx \\
&- \frac{1}{2}\int_{\Omega\times\Omega} \frac{F(x,u_n(x))F(y,u_n(y))}{|x-y|^{a(x,y)}} dxdy \\
&\geq \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y}} dxdy + \int_{\Omega} \frac{w(x)}{\overline{p}(x)} |u_n|^{\overline{p}(x)} dx \\
&- -\frac{1}{2}\int_{\Omega\times\Omega} \frac{(\Re(x,u_n(x)) + \frac{\lambda_1}{p(x)} \mid u_n(x) \mid^{p^-})}{\mid x-y \mid^{a(x,y)}} dxdy \\
&\geq \frac{1}{p^+}\|u_n\|_w^{p^-} - \frac{1}{2}\int_{\Omega\times\Omega} \frac{\Re(x,u_n(x))}{\mid x-y\mid^{a(x,y)}} dxdy \\
&- \frac{\lambda_1}{2}\int_{\Omega\times\Omega} \frac{\mid u_n(x)\mid^{p^-}}{p(x)\mid x-y\mid^{a(x,y)}} dxdy, \qquad (3.5)
\end{aligned}$$

dividing 3.5 by $\|u_n\|_w^{p^-}$ and passing to the limit, we obtain

$$\tfrac{1}{p^+} + \tfrac{\lambda_1}{2p^-} \int_{\Omega\times\Omega} \tfrac{|v|^{p^-}}{|x-y|^{a(x,y)}} dxdy \leq 0,$$

then $\lambda_1 < 0$, which is a contradiction. So I is coercive and has a global minimizer v_1. When the assumption (F_3) holds, suppose that there exist $w_1 \in C_0^\infty(B(x_0, 2\rho_0))$ such that $0 \leq w_1 \leq t_0$ for all $x, y \in B(x_0, 2\rho_0)$, $w_1(x) \equiv t_0$, $w_1(y) \equiv t_0$ for $x, y \in B(x_0, \rho_0)$ and $\mid \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{sp(x,y)+N}} dxdy \mid \leq \frac{2t_0}{\rho_0}$ then $w_1 \in W_w$.

On the other hand,

$$\begin{aligned}
I(w_1) &\leq \int_{B(x_0,2\rho_0)\setminus B(x_0,\rho_0)} \mid \frac{2t_0}{\rho_0} \mid dx + \mid w \mid_\infty \int_{B(x_0,2\rho_0)} \frac{1}{\overline{p}(x)} \mid t_0 \mid^{\overline{p}(x)} dx \\
&- \frac{1}{2}\int_{\Omega\times\Omega} \frac{F(x,w_1)F(y,w_1)}{|x-y|^{a(x,y)}} dxdy \\
&\leq A_0 \mid B(x_0,\rho_0) \mid - \frac{1}{2}\int_{\Omega\times\Omega} \frac{F(x,w_1)F(y,w_1)}{|x-y|^{a(x,y)}} dxdy.
\end{aligned} \qquad (3.6)$$

Since

$$\begin{aligned}
\int_{\Omega\times\Omega} \frac{F(x,w_1)F(y,w_1)}{|x-y|^{a(x,y)}} dxdy &> \int_{B(x_0,\rho_0)\times\Omega} \frac{F(x,w_1)F(y,w_1)}{|x-y|^{a(x,y)}} dxdy \\
&\geq A_0 \mid B(x_0,\rho_0) \mid \times |\Omega|,
\end{aligned}$$

then for $|\Omega| > 1$ we obtain

$$I(w_1) \leq A_0|B(x_0,\rho_0)| - A_0|B(x_0,\rho_0)||\Omega| < 0$$

we deduce $I(w_1) < 0$ then $I(v_1) < 0 = I(v_0)$, so $v_1 \neq 0$.

step (3) : We prove that φ has two local minima.

Since I is coercive there is $\rho_0 > 0$ large enough such that $v_0, v_1 \in B(0,\rho_0)$ and $\inf_{\partial B(0,\rho_0)} I > I(v_0) > I(v_1)$.

By proposition 2.3, given any $\varepsilon > 0$, $r_1 \in]I(v_1), 0[$ and $r_2 > 0$ then $\forall \lambda \in]0, \underline{\lambda_*[, \varphi}$ has at least two local minima $u_0 \in B(0,\varepsilon) \cap I^{-1}(]-\infty, r_2[)$, $u_1 \in B(0, I^{-1}(]-\infty, r_1[)$ and $u_1 \notin \overline{B(0,\varepsilon)}$. The minimizer $u_0 \neq 0$. In consequence, when (G_1) verifies, supposing that $\omega \in C_0^\infty(B(x_1,\rho_1))$ such that $0 \leq \omega \leq 1$ and $\omega(x) \equiv 1$ for $x \in B(x_1, \frac{\rho_1}{2})$, so, it is easy to see that for $\lambda \in]0, \lambda_*[$, when $t > 0$ is small enough, we get $t\omega \in B(0,\varepsilon) \cap I^-(]-\infty, r_2[)$ and $I(u_0) + \lambda J(u_0) \leq I(t\omega) + \lambda J(t\omega) < 0$. In particular, $u_0 \neq 0$.

step (4) : φ has a mountain pass type critical point $\forall\lambda \in]0, \lambda_*[$. We take $\rho_1 > 0$ such that $B(0, \rho_1) \subset W_w$ and $I^{-1}(]-\infty, r_1[)\cup B(0,\varepsilon) \subset B(0, \rho_1)$. Since I is coercive, there exists $\rho_2 > \rho_1$ such that $\inf\limits_{\partial B(0,\rho_2)} I > \sup\limits_{B(0,\rho_1)} I$, then $(B(0, \rho_1), B(0, \rho_2))$ is a valley box of I. Since $I(v_1) < 0 = I(v_0)$ and by **step (1)**, we have that for some $\varepsilon_0 > 0$ with $\varepsilon_0 > \|v_1\|_w$ and $\inf\limits_{\partial B(0,\varepsilon)} I > 0$, then φ admits a mountain pass point u_2. Consequently, u_0, u_1 and u_2 are at least three nontrivial solutions of the problem (P^s). □

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