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Characterization of the center of a prime Banach algebra by its homoderivations

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ABSTRACT: Let $\mathcal{X}$ be a Banach algebra. The value of the present article focuses on the fact that, first, it presents detailed informations about zero-power valued maps on $\mathcal{X}$, second, it provides characterization to the center of $\mathcal{X}$ via its homoderivations. Finally, we include some examples to show that various restrictions in the hypothesis of our theorems are not superfluous.

Key Words: Nilpotent maps, Zero-power valued mappings, Homoderivations, Prime Banach algebras.

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1. Introduction and Preliminaries

All over the paper, $\mathcal{R}$ always denotes an associative ring with center $\mathrm{Z}(\mathcal{R})$. For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ will denote the commutator $xy - yx$. Recall that $\mathcal{R}$ is prime if for all $x, y \in \mathcal{R}$, $x\mathcal{R}y = 0$ yields $x = 0$ or $y = 0$, and $\mathcal{R}$ is semiprime if $x\mathcal{R}x = 0$ for $x \in \mathcal{R}$ forces $x = 0$. Let S be a part of $\mathcal{R}$, the mapping $f : \mathcal{R} \longrightarrow \mathcal{R}$ is zero-power valued on S if $f(S) \subset S$ and for evry $x \in S$ there is $n \in \mathbb{N}^*$ such that $f^n(x) = 0$. Following El Sofy [3], an additive mapping $h : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be a homoderivation on $\mathcal{R}$ if $h(xy) = h(x)h(y) + h(x)y + xh(y)$ holds for all $x, y \in \mathcal{R}$. He also uses this mapping to prove the commutativity of prime rings. In particular, he proved that every prime ring which admits a homoderivation h such that: $h([x, y]) = \pm[x, y]$ for all $x, y \in I$, where I is a nonzero bilateral ideal of the ring, is commutative.

Several authors have obtained commutativity criteria for certain prime and semiprime Banach algebras by considering specific conditions which are apparently too weak to imply the commutativity. In this direction, Yood [2] proved that if a semiprime Banach algebra $\mathcal{X}$ having two nonvoid open subsets $\mathcal{H}_1$ and $\mathcal{H}_2$ such that for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ there is two strictly positive integers n and m such that $[x^n, y^m]$ or $x^n.y^m$ is in the center of $\mathcal{X}$, then the Banach algebra $\mathcal{X}$ is commutative. For more details see [7] and [8].

Motived by these results, our aim through this work is, on the one hand, to prove that all continuous map on a Banach algebra witch's zero-power valued on non-empty open subset is necessarily nilpotent, and on the other hand, to characterize the center of a prime Banach algebra which has a continuous homoderivation satisfying some specifc algebraic identities on non-empty open subsets. Our topological approach is based on Baire's category theorem and some properties of functional analysis. Among our results, we have shown that the center of a prime Banach algebra $\mathcal{X}$ contains the set $\{h(x) + x \mid x \in \mathcal{X}\}$, knowing that h is a continuous homoderivation of $\mathcal{X}$ fulfilling some conditions.

Throughout this article, we shall make some use of the following well-known results without explicit mention.

Remark. Let $\mathcal{R}$ be a prime ring.

1. If $x \in Z(\mathcal{R})$ and $xy \in Z(\mathcal{R})$, then $x = 0$ or $y \in Z(\mathcal{R})$.

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2. The nonzero elements from $Z(\mathcal{R})$ are not zero divisors.

2. General results

In all that follows, $\mathcal{X}$ will denote a real or complex Banach algebra with center $Z(\mathcal{X})$. The following lemma, due to F. F. Bonsall and J. Duncan [5], is crucial for developing the proof of our main results.

Lemma 2.1 *Let $\mathcal{X}$ be a real or complex Banach algebra and $P(t) = \sum_{i=0}^{n} a_i t^i$ a polynomial in the real variable t with coefficients in $\mathcal{X}$. If for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{X}$, then every a_i lies in M.*

Now the first main result we want to prove in this paper is the following.

Theorem 2.1 *Let f be a continuous additive operator from the Banach algebra $\mathcal{X}$ into itself and $\mathcal{H}$ a non-void open subset of $\mathcal{X}$. The following assertions are equivalent:*

i) f is zero-power on $\mathcal{H}$.

ii) f is nilpotent.

Proof:

The implication $(ii) \Rightarrow (i)$ is immediate.
If (i) is satisfied. For all $p \in \mathbb{N}^*$ let us define the following sets:

$$O_p = \{x \in \mathcal{X} \mid f^p(x) \neq 0\}$$

and

$$F_p = \{x \in \mathcal{X} \mid f^p(x) = 0\}.$$

We claim that each O_p is open in $\mathcal{X}$ or equivalenty its complement F_p is closed. For this, we consider a sequence $(x_k)_{k\in\mathbb{N}} \subset F_p$ converging to $x \in \mathcal{X}$, since $(x_k)_{k\in\mathbb{N}} \subset F_p$, so

$$f^p(x_k) = 0 \ \ \forall k \in \mathbb{N}.$$

Using the continuity of f^p the sequence $(f^p(x_k))_{k\in\mathbb{N}}$ converging to $f^p(x)$ and the fact that $\{0\}$ is closed implies $f^p(x) = 0$. Therefore $x \in F_p$ so that F_p is closed that is O_p is open.
In light of Baire category theorem, if every O_p is dense their intersection is also dense, which contradicts the existence of $\mathcal{H}$. So, there is strictly positive integer n such that O_n is not a dense set in $\mathcal{X}$ which forces existence of a nonvoid open subset O in F_n such that

$$f^n(x) = 0 \ \forall x \in O.$$

Let $x \in O$ and $y \in \mathcal{X}$, then $x + ty \in O$ for all sufficiently small real t, and we have

$$f^n(x + ty) = 0.$$

That is

$$f^n(x) + tf^n(y) = 0.$$

This shows that

$$f^n(y) = 0 \ \forall y \in \mathcal{X}$$

we conclude that f is nilpotent.

□

As a frst application of Theorem 2.1, the following corollary.

Corollary 2.1 *Let h be a continuous homoderivation of the Banach algebra $\mathcal{X}$. If h is zero-power valued on $\mathcal{X}$, then h is nilpotent.*

Theorem 2.2 *Let h be a continuous non-injective homoderivation on a prime Banach algebra $\mathcal{X}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are two non-void open subsets of $\mathcal{X}$. The following assertions are equivalent.*
1. $(\forall(x,y)\in\mathcal{H}_1\times\mathcal{H}_2)(\exists(n,m)\in(\mathbb{N}^*)^2)$ such that $h(x^ny^m)+x^ny^m\in Z(\mathcal{X})$.
2. $\{h(x)+x \mid x\in\mathcal{X}\}\subset Z(\mathcal{X})$.

Proof:
If (1) is satisfied. For all $p,q\in\mathbb{N}^*$ let us defne the following sets :

$$O_{p,q}=\{x,y\in\mathcal{X}\mid h(x^py^q)+x^py^q\notin Z(\mathcal{X})\}$$

and

$$F_{p,q}=\{x,y\in\mathcal{X}\mid h(x^py^q)+x^py^q\in Z(\mathcal{X})\}.$$

By using the same appoach as Theorem 2.1 and the fact that $Z(\mathcal{X})$ is closed, we conclude that, there are two strictly positive integers n,m such that $O_{n,m}$ is not dense in $\mathcal{X}\times\mathcal{X}$ which forces existence of a nonvoid open subset $O\times O'$ in $F_{n,m}$ such that :

$$h(x^ny^m)+x^ny^m\in Z(\mathcal{X})\ \forall x\in O,\forall y\in O^{'}.$$

Fix $y\in O^{'}$. Let $x\in O$ and $z\in\mathcal{X}$, then $x+tz\in O$ for all sufficiently small real t. Therefore

$$P(t)=h((x+tz)^ny^m)+(x+tz)^ny^m\in Z(\mathcal{X}).$$

Since $P(t)$ may be written as

$$P(t)=A_{n,0}(x,z,y)+A_{n-1,1}(x,z,y)t+A_{n-2,2}(x,z,y)t^2+...+A_{0,n}(x,z,y)t^n.$$

Then Lemma 2.1 yields

$$A_{0,n}(x,z,y)=h(z^ny^m)+z^ny^m\in Z(\mathcal{X}).$$

Consequently, given $y\in O^{'}$ we get

$$h(x^ny^m)+x^ny^m\in Z(\mathcal{X})\text{ for all }x\in\mathcal{X}.$$

Using a similar approach, we arrive at

$$h(x^ny^m)+x^ny^m\in Z(\mathcal{X})\text{ for all }(x,y)\in\mathcal{X}^2.$$

Now we claim that the restriction of h on $Z(\mathcal{X})$ is non-injective. Indeed, assume that the restriction of h on $Z(\mathcal{X})$ is injective and consider an element $a\in\mathcal{X}$ such that $h(a)=0$. For a non-zero element b in $Z(\mathcal{X})$, we have $h((ab+tb)^ny^m)+(ab+tb)^ny^m\in Z(\mathcal{X})$ for all $t\in\mathcal{R}$. Since $b^k\in Z(\mathcal{X})$ for all $k\in\mathbb{N}^*$, we can write

$$h((ab+tb)^ny^m)+(ab+tb)^ny^m=\sum_{k=0}^{n}\binom{n}{k}(h((ab)^{n-k}b^k)y^m+(ab)^{n-k}b^ky^m)t^k\in Z(\mathcal{X}).$$

Then invoking Lemma 2.1 it follows that $\binom{n}{k}(h((ab)^{n-k}b^k)y^m+(ab)^{n-k}b^ky^m)t^k\in Z(\mathcal{X})$ for all $0\le k\le n$. In particular, for all $y\in\mathcal{X}$

$$h((ab)b^{n-1}y^m)+(ab)b^{n-1}y^m\in Z(\mathcal{X})$$

which leads to

$$h(a.b^ny^m)+a.b^ny^m\in Z(\mathcal{X}).$$

That is

$$a.(h(b^ny^m)+b^ny^m)\in Z(\mathcal{X}).$$

Suppose that $h \neq -Id$. Hence there exists $y \in \mathcal{X}$ such that $h(b^n y^m) + b^n y^m$ is a non-null element of $Z(\mathcal{X})$. Accordingly, the last expression forces $a \in Z(\mathcal{X})$ and the fact that $h(a) = 0$ implies $a = 0$ that is h is injective, a contradiction. In conclusion, the restriction of h on $Z(\mathcal{X})$ is non injective and there exists a nonzero element $a \in Z(\mathcal{X})$ such that $h(a) = 0$. Since $a^k \in Z(\mathcal{X})$ for all $k \in \mathbb{N}^*$, we can writer

$$h((x+ta)^n y^m) + (x+ta)^n y^m = \sum_{k=0}^{n} \binom{n}{k} (h(x^{n-k} a^k y^m) + x^{n-k} a^k y^m) t^k \in Z(\mathcal{X}).$$

The coefficient of t^{n-1} in above polynomial is just $h(xa^{n-1}y^m) + xa^{n-1}y^m$, by Lemma 2.1 we obtain $h(xa^{n-1}y^m) + xa^{n-1}y^m \in Z(\mathcal{X})$ $(\forall x, y \in \mathcal{X})$, we find that

$$h(a^{n-1})h(xy^m) + h(a^{n-1})xy^m + a^{n-1}h(xy^m) + a^{n-1}xy^m \in Z(\mathcal{X}) \ (\forall x, y \in \mathcal{X}).$$

We also have

$$a^{n-1}h(xy^m) + a^{n-1}xy^m \in Z(\mathcal{X}) \ \forall x, y \in \mathcal{X}.$$

So that

$$a^{n-1}(h(xy^m) + xy^m) \in Z(\mathcal{X}) \ \forall x, y \in \mathcal{X}.$$

By the primeness of $\mathcal{X}$ assures that $h(xy^m) + xy^m \in Z(\mathcal{X})$ for all $x, y \in \mathcal{X}$.
Using a similar approach, we arrive at

$$h(xy) + xy \in Z(\mathcal{X}) \ (\forall x, y \in \mathcal{X}).$$

Now let a be a non-zero element of $Z(\mathcal{X})$ such that $h(a) = 0$, so we get $h(ax) + ax \in Z(\mathcal{X})$. Thus

$$\begin{aligned} h(ax) + ax &= h(a)h(x) + h(a)x + ah(x) + ax \\ &= (h(a) + a)(h(x) + x) \\ &= a(h(x) + x) \in Z(\mathcal{X}) \end{aligned}$$

By Remark 1, we conclude that

$$h(x) + x \in Z(\mathcal{X}) \ \forall x \in \mathcal{X}.$$

The implication (2) $\implies$ (1) is immediate.

□

As an application of Theorem 2.2, we get the following result.

Theorem 2.3 *Let h be a zero-power valued continuous homoderivation on a prime Banach algebra $\mathcal{X}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are two non-void open subsets of $\mathcal{X}$. If the following statement hold:*

$$(\forall (x, y) \in \mathcal{H}_1 \times \mathcal{H}_2)(\exists (n, m) \in (\mathbb{N}^*)^2) \ \textit{such that} \ h(x^n y^m) + x^n y^m \in Z(\mathcal{X}).$$

Then $\mathcal{X}$ must be commutative

Proof:

If $h = 0$, then our hypothesis reduces to $x^n y^m \in Z(\mathcal{X})$ for all $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ so the required result follows from [2]. Hence we may suppose that $h \neq 0$. First we need to show that h is non injective. Indeed Suppose that $h(x) \neq 0$ for all non-zero element $x \in \mathcal{R}$. By recurrence we have $h^n(x) \neq 0$ for all $x \in \mathcal{R} \setminus \{0\}$ and for all $n \in \mathbb{N}^*$. Since h is zero-power valued on $\mathcal{R}$, for each $x \in \mathcal{R}$, there exists a strictly positive integer $n(x)$ such that $h^{n(x)}(x) = 0$, it follows that for $b = h^{n(x)-1}(x) \neq 0$, $f(b) = h^{n(x)}(x) = 0$, which is a contradiction. Hence there is $a \in \mathcal{R} \setminus \{0\}$ such that $h(a) = 0$. That is h is non-injective ($h(0) = 0$). According to Theorem 2.2 we conclude that $h(x) + x \in Z(\mathcal{X})$ for all $x \in \mathcal{X}$. Using Lemma 3.1 [9] we obtain the assertion of the theorem.

□

Theorem 2.4 *Let $\mathcal{X}$ be a prime Banach algebra and $\mathcal{H}_1$ be a nonvoid open subset of $\mathcal{X}$. If $\mathcal{X}$ admits a non-zero and non-injective continuous homoderivation h satisfying: $(\forall x \in \mathcal{H}_1)(\exists n \in \mathbb{N}^*)$ such that $h(x^n) \in Z(\mathcal{X})$, then $\mathcal{X}$ is commutative.*

Proof: For any $p \in \mathbb{N}^*$, we define the following parts:

$$O_p = \{x \in \mathcal{X} \mid h(x^p) \notin Z(\mathcal{X})\}$$

and

$$F_p = \{x \in \mathcal{X} \mid h(x^p) \in Z(\mathcal{X})\}.$$

Using similar techniques as in the proof of Theorem 2.1, we can prove existence a strictly positive integer n such that

$$h(x^n) \in Z(\mathcal{X}) \ \ \forall x \in \mathcal{X}.$$

And we can prove existence of a non-nul element $a \in Z(\mathcal{X})$ such that $h(a) = 0$.
Consequently, given $t \in \mathbb{R}$ we have

$$h((a + tx)^n) \in Z(\mathcal{X}).$$

Thereby obtaining

$$h((a + tx)^n) = \sum_{k=0}^{n} \binom{n}{k} h(a^{n-k}x^k)t^k \in Z(\mathcal{X}).$$

By using Lemma 2.1 we concluded $h(a^{n-k}x^k) \in Z(\mathcal{X})$ for all $0 \leq k \leq n$. In particular for $k = 1$ we have $h(a^{n-1}x) \in Z(\mathcal{X})$ and hence $h(x)h(a^{n-1}) + h(a^{n-1})x + a^{n-1}h(x) \in Z(\mathcal{X})$ then $a^{n-1}h(x) \in Z(\mathcal{X})$ $(h(a^{n-1}) = 0)$, by Remark 1 we have $h(x) \in Z(\mathcal{X})$, according to Theorem 3.7 [1] the Banach algebra $\mathcal{X}$ is commutative. □

We close this article with the following corollary

Corollary 2.2 *Let $\mathcal{X}$ be a prime Banach algebra and D a part dense in $\mathcal{X}$. If $\mathcal{X}$ admits a continuous non-injective non-zero homoderivation h such that*

$$\exists n \in \mathbb{N}^* : \ h(x^n) \in Z(\mathcal{X}) \ \ (\forall x \in D).$$

Then $\mathcal{X}$ is commutative.

Proof:
Assume that $h \neq 0$. Let $x \in \mathcal{X}$, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in D converging to x. Since $(x_k)_{k\in\mathbb{N}} \subset D$, then

$$h((x_k)^n) \in Z(\mathcal{X}) \text{ for all } k \in \mathbb{N}.$$

Using the fact that h is continuous, the sequence $(h((x_k)^n)_{k\in\mathbb{N}}$ converges to $h(x^n)$, knowing that $Z(\mathcal{X})$ is closed, then $h(x^n) \in Z(\mathcal{X})$. We conclude that:

$$\exists n \in \mathbb{N}^* \ : \quad h(x^n) \in Z(\mathcal{X}) \ \ \forall x \in \mathcal{X}.$$

By Theorem 2.4 we get the desired conclusion.

□

3. Examples and Application

The following example shows that h is "zero-power valued on $\mathcal{X}$ " cannot be omitted in the hypothesis of Theorem 2.3.

Example 3.1 Let $\mathbb{R}$ be the field of real numbers. Let $\mathcal{X} = \mathcal{M}_2(\mathbb{R})$ the set of 2×2 matrix with matrix addition and matrix multiplication. We claim that $\mathcal{X}$ is a prime unital Banach algebra under the norm defined by

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_1 = |a| + |b| + |c| + |d|.$$

Consider $h = -id$, then it is clear that h is a not "zero-power valued homoderivation on $\mathcal{X}$ " which preserve $\mathcal{X}$ and satisfy the following condition for all
$n, m \in \mathbb{N}^*$ and for all $X, Y \in \mathcal{X}$:

$$h(X^n Y^m) + X^n Y^m \in Z(\mathcal{X}).$$

But $\mathcal{X}$ is not commutative.

The following example shows that the conclusion of Theorem 2.4 is not true if we replace $\mathbb{R}$ or $\mathbb{C}$ by $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

Example 3.2 Let $\mathcal{X} = \mathcal{M}_2(\mathbb{F}_3)$ and I_2 its unitary matrix. Clearly, $\mathcal{X}$ is 2-torsion free prime Banach algebra over $\mathbb{F}_3$ with norm $\|.\|_1$ defined by $\|A\|_1 = \sum_{1 \leq i,j \leq 2} |a_{i,j}|$ for $A = (a_{i,j})_{1 \leqslant i,j \leqslant 2} \in \mathcal{X}$ where $|.|$ is the norm defined on $\mathbb{F}_3$ by

$$|\overline{0}| = 0, \ |\overline{1}| = 1 \text{ and } |\overline{2}| = 2.$$

In fact, the set of invertible elements of $\mathcal{X}$, $Inv(\mathcal{X})$, is a finite non empty open subset of $\mathcal{X}$.
It is obvious to verify that the homoderivation defined on $\mathcal{X}$ by
$h\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} -a+c+d & -c-b \\ -a-b+d & a-c-d \end{pmatrix}$ is a continuous non-injective homoderivation. Furthermore, for any $A \in Inv(\mathcal{X})$ we have

$$h(A^{card(Inv(\mathcal{X}))}) = h(I_2) = 0 \in Z(\mathcal{X}).$$

This means that h satisfies the hypothesis of Theorem 2.4 but $\mathcal{X}$ is not commutative.

As an application of Theorem 2.1

Application 3.1 *Let $\mathbb{R}$ be the field of real numbers. Consider $\mathcal{A} = \mathbb{R}[X]$ with usual's addition "+" and multiplication ".". It is straightforward to check that $\mathcal{A}$ endowed with the norm $\|P\| = \sum_{k=0}^{n} |a_k|$ where $P = \sum_{k=0}^{n} a_k X^k$, is a real Banach space. Define $d : \mathcal{A} \longrightarrow \mathcal{A}$ by $d(P) = P'$ with P' denotes the usual derivative of P. Obviously, d is a non zero derivation and for all $P \in \mathcal{A}$ there is $n \in \mathbb{N}^*$ such that $d^n(P) = 0$. According to Theorem 2.1 we conclude that d is a not continuous derivation.*

Conclusion

In this article, we studied the effects of topology on the zero-power valued maps defined on a non-empty open set of a Banach algebra.

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