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Recombination of Stable Sampling Sets and Stable Interpolation Sets in Functional Quasinormed Spaces *

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ABSTRACT: We obtain new results in combining stable sampling sets (respectively, stable interpolation sets) for a given quasinormed space in order to construct other new ones. We apply these results to Paley-Wiener spaces. In addition, we study the problem of obtaining a generator system of a given quasinormed space, and obtain conditions for a finite product of subsets of a given quasinormed space to be a generator system, using the interpolation and sampling theory for quasinormed spaces of functions.

Key Words: Quasinormed spaces, Banach spaces, p-stable sampling set, p-stable interpolation set, p-complete interpolation set, uniqueness set, Paley-Wiener spaces.

Contents

1. Introduction

There are many contributions on the problems of existence and construction of exponential Riesz bases in the Hilbert spaces $L^2(D)$, with $D \subseteq \mathbb{R}^n$ bounded, Lebesgue measurable and disconnected. G. Kozma and S. Nitzan have proved the existence of exponential bases on the finite union of disjoint intervals of $\mathbb{R}$ and disjoint rectangles in $\mathbb{R}^n$ (see [10] and [11], respectively). Besides, G. Kozma and N. Lev constructed in [9] an exponential Riesz basis for the space $L^2(D)$ where $D \subseteq \mathbb{R}$ is the union of finitely many disjoint intervals whose lengths belong to $\mathbb{Z} + \alpha\,\mathbb{Z}$, where α is a given irrational number. For this construction they used the theory of quasicrystals by Yves Meyer (see [15] and [16]). See [7] for a construction of an exponential Riesz basis for $L^2(J)$, with $J = [0,\, \beta) \cup [\beta + r, L + r)$, $0 < \beta < r$, $r > 0$ (problem of the *broken interval*).

On the existence of exponential bases on finite union of disjoint cubes see, for example, [14]; and on the finite union of two disjoint dimensional trapezoids see [6].

On the other hand, several authors have used the idea of combining exponential Riesz bases on individual disjoint two by two rectangles in order to construct a basis for their union (for example, see [5], [10], [11] and [13]).

Our contribution is aimed at obtaining new results in combining stable sampling sets (respectively, stable interpolation sets) of a given quasinormed space in order to obtain new ones. We also apply these results to Paley-Wiener spaces.

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Namely, given a quasinormed functional space E, and several vector subspaces whose topological direct sum is E, and also given a stable sampling set (respectively, stable interpolation set) chosen for each of these subspaces in such a way that those sampling (respectively, interpolation) sets are disjoint two by two, we want combine them in order to form a new stable sampling set (respectively, stable interpolation set). That is, we will directly study sampling and interpolation sets in a general context.

A motivation is the following result, which is well known in sampling and interpolation in the Hilbert spaces E_S^2, classical Paley-Wiener spaces, being $S \subseteq \mathbb{R}^n$ a bounded and Lebesgue measurable set. See [18], Proposition 2.8, p. 16; and Proposition 4.6, p. 36. Also see [23], chapter 4.

Theorem 1.1 (See Definition 1.7 and Definition 4.1)**.** *Let $S \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set, and $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. For each $\lambda \in \Lambda$ consider the function $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases} .$$

We also define the set $E(\Lambda) := \{\phi_\lambda\}_{\lambda \in \Lambda} \subseteq L^2(S)$. Then:

1. *Λ is a 2-SS for E_S^2 if and only if $E(\Lambda)$ is a frame for $L^2(S)$.*
2. *Λ is a 2-SIS for E_S^2 if and only if $E(\Lambda)$ is a Riesz sequence for $L^2(S)$.*
3. *Λ is a 2-SCIS for E_S^2 if and only if $E(\Lambda)$ is a Riesz basis for $L^2(S)$.*

Hence the results on 2-SCIS (respectively, 2-SS, 2-SIS) for the classical Paley-Wiener space E_S^2 can be translated into results on exponential Riesz bases (respectively, frames, Riesz sequences) in $L^2(S)$, and viceversa.

Furthermore, the research in frame theory, Riesz sequences and bases theory in L^p spaces has allowed to obtain advances in stable sampling and interpolation theory. For example, in 1964 M. I. Kadec proved his celebrated theorem:

Theorem 1.2 (Kadec-1/4 Theorem)**.** *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers. Suppose that*

$$|\lambda_n - n| \le L < \frac{1}{4} \text{ for every } n \in \mathbb{Z}.$$

Then the set of exponential functions $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2((-\pi,\,\pi))$.

This result says, in terms of sampling and interpolation theory, that $\mathbb{Z}$ is a complete interpolation set (this is, both sampling and interpolation set) for the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$, and that every L-perturbation of $\mathbb{Z}$ also verifies it whenever $L < \frac{1}{4}$.

The bound 1/4 is sharp, and Theorem 1.2 improves a previous very important result by R. Paley and N. Wiener, where the bound is $\frac{1}{\pi^2}$ ([19], page 113). In 1974 S. A. Avdonin obtained a generalization of Theorem 1.2 using a certain type of mean of the values λ_n's ([3]).

Kadec-1/4 Theorem has been generalized in several ways to L^p spaces and to sequences $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, obtaining $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$, as a result, the property of completeness (see for example [12], [20], [22] and [21]). In addition, the Riesz basis problem in the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$ has been proposed for non-exponential basis. In this sense several important results analogous to Kadec-1/4 Theorem have been obtained for sets of sinc functions, involving the Lamb-Oseen constant (see [1] and [2]).

Finally, we investigate the generator systems of quasi-Banach spaces, being these generator systems the result of multiplying a finite number of subsets of a given quasi-Banach space, and we apply the sampling and interpolation theory in order to obtain such generator systems.

1.1. Definitions

Remind that, given $\Omega \subseteq \mathbb{R}^n$, the vector space $\mathfrak{F}(\Omega, \mathbb{C})$ of the complex functions defined in Ω is a commutative $\mathbb{C}$-algebra with the usual product of functions. Given $\Omega \subseteq \mathbb{R}^n$, $A, B \subseteq \mathfrak{F}(\Omega, \mathbb{C})$, we denote by $A \cdot B$ the set

$$A \cdot B := \{f \cdot g \mid f \in A,\ g \in B\},$$

as usual.

Definition 1.3 (Uniformly discrete set)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be infinite countable. We say that Λ is uniformly discrete (briefly u.d.) if*

$$\delta(\Lambda) := \inf_{\lambda,\ \lambda' \in \Lambda,\ \lambda \neq \lambda'} \|\lambda - \lambda'\| > 0.$$

The constant $\delta(\Lambda)$ is called the separation constant of Λ.

Definition 1.4 (Uniqueness set)**.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. We say that Λ is a uniqueness or complete set (briefly, US) for E if for every $f \in E$ we have that*

$$(\forall \lambda \in \Lambda\ f(\lambda) = 0) \Rightarrow f = 0.$$

Definition 1.5 (Sequence space $l^p(\Lambda)$)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be u.d.*

1. *Let $p \in (0, +\infty)$. We define the set*

$$l^p(\Lambda) := \left\{ (a_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda \mid \sum_{\lambda \in \Lambda} |a_\lambda|^p < \infty \right\}.$$

The mapping $\| \|_p : l^p(\Lambda) \to \mathbb{R}$ given by $\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_p := \left(\sum_{\lambda\in\Lambda} |a_\lambda|^p\right)^{\frac{1}{p}}$, is a quasinorm for $l^p(\Lambda)$, which is a norm if $p \geq 1$. With this quasinorm $l^p(\Lambda)$ is a complete space.

2. $$l^\infty(\Lambda) := \left\{ (a_\lambda)_{\lambda \in \Lambda} \in \mathbb{C}^\Lambda \mid \sup_{\lambda \in \Lambda} |a_\lambda| < \infty \right\}.$$

The mapping $\| \|_\infty : l^\infty(\Lambda) \to \mathbb{R}$ defined by $\left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_\infty := \sup_{\lambda\in\Lambda} |a_\lambda|$ is a norm for $l^\infty(\Lambda)$ which make this space a Banach space.

Definition 1.6. *Let $\Lambda \subseteq \mathbb{R}^n$ be u.d.*

1. *We define the product operation in $l^\infty(\Lambda)$ in the following way: Let $a = (a_\lambda)_{\lambda\in\Lambda}$, $b = (b_\lambda)_{\lambda\in\Lambda} \in l^\infty(\Lambda)$. We define the object $a \cdot b$ as the element of $l^\infty(\Lambda)$ given by*

$$a \cdot b := (c_\lambda := a_\lambda \cdot b_\lambda)_{\lambda\in\Lambda} \in l^\infty(\Lambda).$$

2. *This product operation in $l^\infty(\Lambda)$ is an associative binary operation which makes the vector space $l^\infty(\Lambda)$ a complex commutative algebra with unity. The unity is, obviously, the element of $l^\infty(\Lambda)$ whose all components are equal to 1.*

3. *Let $A, B \subseteq l^\infty(\Lambda)$ be non empty sets. We define*

$$A \cdot B := \{a \cdot b \mid a \in A,\ b \in B\}.$$

Definition 1.7. *Let $(E, \| \|)$ be a quasinormed space, verifying $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $p \in (0, +\infty]$ and $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set. Assume that*

$$(f(\lambda))_{\lambda\in\Lambda} \in l^p(\Lambda) \text{ for all } f \in E.$$

- *The $\mathbb{C}$-linear mapping $S : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$ given by $f \to (f(\lambda))_{\lambda\in\Lambda}$ is called the p-sampling operator of $(E, \| \|)$ with respect to Λ.*

- *We say that Λ verifies the p-Plancherel-Polya condition (briefly, p-P.P.C.) for $(E, \| \|)$ if S is continuous, this is, if there exists a constant $C > 0$ such that*

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq C \, \|f\| \text{ for all } f \in E.$$

- *Λ is said to be a p-interpolation set (in short, p-IS) for E if S is surjective. Given $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ and $f \in E$, we say that f interpolates c (over Λ) if $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.*

- *We say that Λ a p-stable interpolation set (briefly, p-SIS) for $(E, \| \|)$ if S is continuous, surjective and has a continuous inverse by right.*

- *Λ is said to be a p-stable sampling set (briefly, p-SS) for $(E, \| \|)$ if S is a topological isomorphism over its image, this is, if there exist constants c, $C > 0$, $c \leq C$, such that*

$$c \, \|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq \|f\| \leq C \, \|(f(\lambda))_{\lambda\in\Lambda}\|_p$$

  for every $f \in E$. That is, if S is continuous, injective and has a continuous inverse by left.

- *We say that Λ a p-complete interpolation set (briefly, p-CIS) for E if S is bijective.*

- *Λ is called a p-stable complete interpolation set (briefly, p-SCIS) for $(E, \| \|)$ if S is a topological isomorphism.*

Observe that Λ is a uniqueness set for E if and only if S is injective. In addition notice that every p-SS for $(E, \| \|)$ is also a US for E.

Remark 1.8. *Λ is a p-SS for $(E, \| \|)$ if and only if there exists a vector subspace of $(l^p(\Lambda), \| \ \|_p)$ topologically isomorphic to $(E, \| \|)$ through the sampling mapping, and this is a representation of $(E, \| \|)$ as a subspace of $(l^p(\Lambda), \| \|_p)$.*

Indeed, Λ is a p-SS for $(E, \| \|)$ if and only if the function

$$\| \|_{\Lambda,\, p} : E \to \mathbb{R}$$

defined by $\|f\|_{\Lambda,\, p} := \|(f(\lambda))_{\lambda\in\Lambda}\|_p$ for all $f \in E$, is a quasinorm in E equivalent to $\| \|$.

Remark 1.9. *Observe that the following statements are equivalent between themselves:*

1. *Λ is a p-SCIS for $(E, \| \|)$.*
2. *Λ is both a p-SS and a p-IS for $(E, \| \|)$.*
3. *Λ is both a p-SIS and a uniqueness set for $(E, \| \|)$.*

As consequence of the Banach open mapping theorem, we have the following observation.

Remark 1.10. *Let $(E, \| \|)$ be a quasi-Banach space, verifying $E \subseteq \mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $p \in (0, +\infty]$ and $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set. Assume that the sampling operator $S : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$ is continuous. Then*

1. *Λ is a p-IS for E if and only if Λ is a p-SIS for $(E, \| \|)$.*
2. *Λ is a p-CIS for E if and only if Λ is a p-SCIS for $(E, \| \|)$.*

In the rest of this article we will omit the quasinorm of E, except if necessary, and will refer to the quasinormed space $(E, \| \|)$ simply as E.

1.2. Main results

Our main results are the following ones.

Theorem 1.11. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(X, \| \|)$ be a quasinormed space with $X \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $E, F \subseteq X$ be vector subspaces. Let $m \in \mathbb{Z}^+$, and $E_1, ..., E_m \subseteq E$ be vector subspaces. Let $p \in (0, +\infty]$, and $\Lambda_1, ..., \Lambda_m \subseteq \Omega$ be uniformly discrete sets and disjoint two by two. Define $\Lambda := \bigcup_{j=1}^m \Lambda_j$, and assume that*

i) Λ_j is a p-IS for E_j for every $j \in \{1, ..., m\}$.

ii) Λ is an ∞-IS for F.

iii) $E_j \cdot F \subseteq E$ for each $j \in \{1, ..., m\}$.

Then Λ is a p-IS for E.

Theorem 1.12. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(E, \| \|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $m \in \mathbb{Z}^+$, and let $E_1, ..., E_m \subseteq E$ be closed vector subspaces of E such that*

$$E = \oplus_{j=1}^m E_j$$

is an algebraic direct sum. Let $p \in (0, +\infty]$, and $\Lambda_1, ..., \Lambda_m \subseteq \Omega$ be u.d. and disjoint two by two. Define $\Lambda := \bigcup_{j=1}^m \Lambda_j$. Suppose that

1. Λ_j is a p-SS for E_j for every $j \in \{1, ..., m\}$.

2. Λ is a uniqueness set for E.

Then Λ is a p-SS for E.

As we will see in the proof of Theorem 1.12, if Λ verifies the p-P.P.C. for E, then Theorem 1.12 is also true for any countable set of Λ_j's and any countable set of vector subspaces $E_j's$ that split E into a topological direct sum.

The structure of this paper is as follows. Section 1 contains the introduction with the main results, Theorem 1.11 and Theorem 1.12. In section 2 we prove Theorem 1.11 and apply it to Paley-Wiener spaces and to Lebesgue spaces $L^2(S)$. Section 3 is devoted to the proof of Theorem 1.12. In section 4 we apply Theorem 1.12 to Paley-Wiener spaces, obtaining Corollary 4.3. Finally, in section 4 we apply the sampling and interpolation theory in order to obtain several results which allow us in certain cases to construct some generator systems of quasi-Banach spaces, where these generator systems are product of a finite number of subsets of the given quasi-Banach space.

2. Proof of Theorem 1.11

Proof of Theorem 1.11. It is obvious that Λ is u.d. We will prove that Λ is a p-IS for E.

Let $a := (a_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$. Let us see that there exists $f_a \in E$ such that $f_a(\lambda) = a_\lambda$ for every $\lambda \in \Lambda$.

Define

$$a^j := (a_\lambda)_{\lambda \in \Lambda_j} \in l^p(\Lambda_j) \text{ for every } j \in \{1, ..., m\}.$$

Let $j \in \{1, ..., m\}$. By the assumption i) we have that there exists $f_{a^j} \in E_j$ such that

$$f_{a^j}(\lambda) = a_\lambda \text{ for each } \lambda \in \Lambda_j.$$

Now we consider $b^j := \left(b_\lambda^j\right)_{\lambda \in \Lambda}$ defined by

$$b_\lambda^j := \begin{cases} 1, & \text{if } \lambda \in \Lambda_j \\ 0, & \text{if } \lambda \notin \Lambda_j, \end{cases}, \text{ for all } \lambda \in \Lambda.$$

It is clear that $b^j \in l^\infty\left(\bigcup_{j=1}^m \Lambda_j\right) = l^\infty(\Lambda)$. By the hypothesis ii) we have that there exists $g_j \in F$ such that

$$g_j(\lambda) = b^j_\lambda = \begin{cases} 1, & \text{if } \lambda \in \Lambda_j \\ 0, & \text{if } \lambda \notin \Lambda_j \end{cases}, \text{ for every } \lambda \in \Lambda.$$

Notice that for each $j \in \{1, ..., m\}$ the function g_j does not depend on a. Consider now the function

$$f_a := \sum_{j=1}^m f_{a^j} \cdot g_j.$$

By the assumption iii) and since E is closed by sums, then $f_a \in E$.

Let $\lambda \in \Lambda$. Let us see that $f_a(\lambda) = a_\lambda \in \mathbb{C}$.

Since there exists a unique $j_\lambda \in \{1, ..., m\}$ such that $\lambda \in \Lambda_{j_\lambda}$, then

$$f_a(\lambda) = \sum_{j=1}^m f_{a^j}(\lambda) \cdot g_j(\lambda) = f_{a^{j_\lambda}}(\lambda) = a_\lambda.$$

Conclusion: Λ is a p-IS for E. □

Now we will apply Theorem 1.11 to Paley-Wiener spaces and to Lebesgue spaces $L^2(S)$.

Corollary 2.1. *Let S, $K \subseteq \mathbb{R}^n$ be compact sets with positive measure, and let $L \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable such that $S \subseteq L$. Let $m \in \mathbb{Z}^+$, and $S_1, ..., S_m \subseteq \operatorname{Int}(S)$ be compact sets with positive measure. Let $p \in (0, +\infty]$, and $\Lambda_1, ..., \Lambda_m \subseteq \mathbb{R}^n$ be uniformly discrete and disjoint two by two. We define $\Lambda := \bigcup_{j=1}^m \Lambda_j$, and suppose that*

i) Λ_j is a p-IS for $E^p_{S_j}$ for every $j \in \{1, ..., m\}$.

ii) Λ is an ∞-IS for E^∞_K.

iii) $S_j + K \subseteq S$ for each $j \in \{1, ..., m\}$.

Then Λ is a p-IS for E^p_S.

Proof. This result is an immediate consequence of Theorem 1.11 taking

$$X := E^p_L,\ E := E^p_S,\ F := E^p_K, \text{ and}$$

$$E_j := E^p_{S_j} \text{ for every } j \in \{1, ..., m\}.$$

□

In order to obtain a consequence for Riesz sequences in the Lebesgue spaces we will use the following theorem by A. Olevskii and A. Ulanovskii; see [17, Theorem 2.1].

Theorem 2.2 (Olevskii-Ulanovskii Transitivity theorem)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be u.d., $S \subseteq \mathbb{R}^n$ be a compact set and $\varepsilon > 0$.*

1. *If Λ is an IS for E^2_S, then Λ is an IS for $E^\infty_{S+[-\varepsilon,\, \varepsilon]^n}$.*
2. *If Λ is an IS for E^∞_S, then Λ is an IS for $E^2_{S+[-\varepsilon,\, \varepsilon]^n}$.*
3. *If Λ is a SS for $E^\infty_{S+[-\varepsilon,\, \varepsilon]^n}$, then Λ is a SS for E^2_S.*
4. *If Λ is a SS for $E^2_{S+[-\varepsilon,\, \varepsilon]^n}$, then Λ is a SS for E^∞_S.*

Corollary 2.3. *Let S, $T \subseteq \mathbb{R}^n$ be compact sets with positive measure and let $L \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable such that $S \subseteq L$. Let $m \in \mathbb{Z}^+$, and $S_1, ..., S_m \subseteq \operatorname{Int}(S)$ be compact sets with positive measure. Let $p \in (0, +\infty]$, and $\Lambda_1, ..., \Lambda_m \subseteq \mathbb{R}^n$ be uniformly discrete and disjoint two by two. We define $\Lambda := \bigcup_{j=1}^m \Lambda_j$, and suppose that*

i) Λ_j is a 2-IS for $E^2_{S_j}$ for every $j \in \{1, ..., m\}$.

ii) Λ is a 2-IS for E^2_T.

iii) $S_j + T \subseteq \operatorname{Int}(S)$ for each $j \in \{1, ..., m\}$.

Then Λ is a 2-IS for E^2_S.

Proof. It is an immediate consequence of both Corollary 2.1 and Theorem 2.2. □

Finally we obtain the version of Corollary 2.3 for Riesz sequences in the Lebesgue spaces $L^2(S)$.

Corollary 2.4. *Let S, $T \subseteq \mathbb{R}^n$ be compact sets with positive measure and let $L \subseteq \mathbb{R}^n$ be bounded and Lebesgue measurable such that $S \subseteq L$. Let $m \in \mathbb{Z}^+$, and $S_1, ..., S_m \subseteq \operatorname{Int}(S)$ be compact sets with positive measure. Let $p \in (0, +\infty]$, and $\Lambda_1, ..., \Lambda_m \subseteq \mathbb{R}^n$ be uniformly discrete and disjoint two by two. We define $\Lambda := \bigcup_{j=1}^m \Lambda_j$.*

For every $\lambda \in \Lambda$ consider the function $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

Define the set $E(\Lambda) := \{\phi_\lambda\}_{\lambda \in \Lambda} \subseteq L^2(S)$. Then:

1. *$E(\Lambda_j) := \left\{\phi_\lambda|_{S_j}\right\}_{\lambda \in \Lambda_j} \subseteq L^2(S_j)$ is a Riesz sequence in $L^2(S_j)$ for every $j \in \{1, ..., m\}$.*
2. *$E(\Lambda)|_T := \{\phi_\lambda|_T\}_{\lambda \in \Lambda} \subseteq L^2(T)$ is a Riesz sequence in $L^2(T)$.*
3. *$S_j + T \subseteq \operatorname{Int}(S)$ for each $j \in \{1, ..., m\}$.*

Then $E(\Lambda)$ is a Riesz sequence in $L^2(S)$.

3. Proof of Theorem 1.12

Proof of Theorem 1.12. We have to prove that the sampling operator $S : (E, \| \|) \to (l^p(\Lambda), \| \|_p)$ is a topological isomorphism over its image. By the assumption 2) we have that S is injective.

In one hand, observe that

$$l^p(\Lambda) = \oplus_{j=1}^m l^p(\Lambda_j)$$

is a topological direct sum. In particular $l^p(\Lambda_j)$ is a closed vector subspace of $l^p(\Lambda)$ for each $j \in \{1, ..., m\}$.

On the other hand, since E is a quasi-Banach space and $E_1, ..., E_m \subseteq E$ are closed vector subspaces of E, then the algebraic direct sum $E = \oplus_{j=1}^m E_j$ is, in fact, a topological direct sum.

The result is an immediate consequence of both Banach Homorphism Theorem and the fact consisting that the finite topological direct sum of complete vector subspaces, these are, the images of the sampling operators

$$S_j := S|_{E_j} : (E_j, \| \|) \to (l^p(\Lambda_j), \| \|_p) \hookrightarrow (l^p(\Lambda), \| \|_p),\ j \in \{1, ..., m\},$$

is complete.

Indeed, S is continuous because both $E = \oplus_{j=1}^m E_j$ is a topological direct sum and by assumption 1) the sampling operators S_j, $j \in \{1, ..., m\}$, are continuous. By Banach Homorphism Theorem we have only to prove that $S(E)$ is closed in $l^p(\Lambda)$.

Let $j \in \{1, ..., m\}$. Again, by assumption 1) and since E_j is complete we obtain that $S_j(E_j)$ is complete in $l^p(\Lambda_j)$, and consequently $S_j(E_j)$ is closed in $l^p(\Lambda_j)$. As $l^p(\Lambda_j)$ is closed in $l^p(\Lambda)$, then $S(E_j) = S_j(E_j)$ is closed in $l^p(\Lambda)$, and hence $S(E_j) \subseteq l^p(\Lambda)$ is complete.

So that $S(E) = S\left(\oplus_{j=1}^m E_j\right) = \oplus_{j=1}^m S(E_j) \subseteq l^p(\Lambda)$ is a direct topological sum of complete subspaces of $l^p(\Lambda)$, and therefore $S(E)$ is complete. By Banach Homorphism Theorem we finally conclude that Λ is a p-SS for E. □

4. Application to Paley-Wiener spaces

We establish some notation. Given a Lebesgue measurable set $K \subseteq \mathbb{R}^n$, we denote by $m_n(K)$ its Lebesgue measure and by $\overline{K}$ its closure. Given a function $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \text{ for all } \xi \in \mathbb{R}^n.$$

Definition 4.1 (Paley-Wiener spaces)**.** *Let $A \subseteq \mathbb{R}^n$ be a Lebesgue measurable and bounded set and $p \in (0, +\infty]$. We define*

$$E_A^p := \{f \in \mathcal{S}'(\mathbb{R}^n) \, : \, \mathrm{supp}(\hat{f}) \subseteq A \text{ and } \|f\|_p < \infty\}.$$

It is a closed vector subspace of $(L^p(\mathbb{R}^n), \| \|_p)$, and it is called (p, A)-Paley-Wiener space.

When $p = 2$, the Hilbert space E_A^2 is called classical Paley-Wiener space *of spectrum contained in A. The Fourier transform $\mathcal{F} : \left(E_A^2, \| \|_2\right) \to \left(L^2(A), \| \|_2\right)$, is an isometrical and topological isomorphism. When $p = \infty$ we define $B_A := (E_A^\infty, \| \|_\infty)$, which is called (classical) Bernstein space of spectrum contained in A.*

It is well known that, if $\Lambda \subseteq \mathbb{R}^n$ is u.d., then Λ verifies the P.P.C. for E_S^p for all $p \in (0, +\infty]$ and all compact $S \subseteq \mathbb{R}^n$ (see Plancherel-Polya theorem, [4], p. 101).

Lemma 4.2. *Let $n \in \mathbb{Z}^+$, $p \in (0, +\infty]$, and $S \subseteq \mathbb{R}^n$ be a compact set with positive measure. Let $m \in \mathbb{Z}^+$, $m \geq 2$, and $S_1, ..., S_m \subseteq S$ be compact sets with positive measure such that $\{S_1, ..., S_m\}$ is a partition of S. Suppose that the characteristic function of S, $\chi_S : \mathbb{R}^n \to \mathbb{R}$, is a Fourier multiplier for $\mathcal{F}L^p$. Also assume that χ_{S_j} is a Fourier multiplier for $\mathcal{F}L^p$ for every $j \in \{1, ..., m-1\}$.*

Then χ_{S_m} is a Fourier multiplier for $\mathcal{F}L^p$, and the algebraic direct sum

$$E_S^p = \oplus_{j=1}^m E_{S_j}^p$$

is topological.

Proof. As

$$\chi_S = \sum_{j=1}^m \chi_{S_j},$$

then $\chi_{S_m} = \chi_S - \sum_{j=1}^{m-1} \chi_{S_j}$, and consequently χ_{S_m} is a Fourier multiplier for $\mathcal{F}L^p$. Now we will prove that the algebraic direct sum $E_S^p = \oplus_{j=1}^m E_{S_j}^p$ is topological.

Consider the projection in the i-th component

$$P_i : E_S^p = \oplus_{j=1}^m E_{S_j}^p \to E_{S_i}^p,$$

for every $i \in \{1, ..., m\}$. We will show that all these projections are continuous, finishing the proof.

Indeed, let $F \in E_S^p$. Define $f := \hat{F}$. Then $\mathrm{supp}\,(f) \subseteq S$, what is equivalent to

$$f = f \cdot \chi_S.$$

Since $\{S_1, ..., S_m\}$ is a partition of S, then $\chi_S = \sum_{j=1}^m \chi_{S_j}$. Therefore

$$f = f \cdot \chi_S = \sum_{j=1}^m f \cdot \chi_{S_j}.$$

Let $j \in \{1, ..., m-1\}$. Obviously $\mathrm{supp}\left(f \cdot \chi_{S_j}\right) \subseteq S_j$. As by one of our hypotheses χ_{S_j} is a Fourier multiplier for $\mathcal{F}L^p$, then there exists $G_j \in L^p(\mathbb{R}^n)$ such that $f_j := \widehat{G_j} = f \cdot \chi_{S_j}$. Thus $G_j \in E_{S_j}^p$.

In addition, since χ_{S_j} is a Fourier multiplier for $\mathcal{F}L^p$, then there exists a constant $C_{j,\,p} = C\left(p,\ S_j\right) > 0$ independent of f (and of F) verifying that

$$\|G_j\|_p \leq C_{j,\,p}\,\|F\|_p.$$

On the other hand

$$\hat{F} = f = \sum_{i=1}^{m} f\cdot\chi_{S_i} = \sum_{i=1}^{m}\widehat{G_i} = \widehat{\sum_{i=1}^{m} G_i} = \hat{G},$$

where we have defined $G := \sum_{i=1}^{m} G_i \in L^p\left(\mathbb{R}^n\right)$.

As $G_i \in E^p_{S_i} \subseteq E^p_S$ for each $i \in \{1, ..., m\}$, then we obtain that $G \in E^p_S$.

Hence, $F,\ G \in E^p_S$ and $\hat{F} = \hat{G}$. This implies that

$$F = G = \sum_{i=1}^{m} G_i,$$

and we know that $G_i \in E^p_{S_i} \subseteq E^p_S$ for each $i \in \{1, ..., m\}$.

Conclusion: The projection in the j-th component,

$$P_j : E^p_S = \oplus_{i=1}^{m} E^p_{S_i} \to E^p_{S_j},$$

is continuous. □

The following result is a immediate consequence of both Theorem 1.12 and Lemma 4.2 for Paley-Wiener spaces.

Corollary 4.3. *Let $n \in \mathbb{Z}^+$, $p \in (0,\ +\infty]$, and $S \subseteq \mathbb{R}^n$ be a compact set with positive measure. Let $m \in \mathbb{Z}^+$, $m \geq 2$, and $S_1, ..., S_m \subseteq S$ be compact sets with positive measure such that $\{S_1, ..., S_m\}$ is a partition of S. Suppose that the characteristic function of S, $\chi_S : \mathbb{R}^n \to \mathbb{R}$, is a Fourier multiplier for $\mathcal{F}L^p$. Also assume that χ_{S_j} is a Fourier multiplier for $\mathcal{F}L^p$ for every $j \in \{1, ..., m-1\}$. Let $\Lambda_1, ..., \Lambda_m \subseteq \mathbb{R}^n$ be u.d. and disjoint two by two. Define $\Lambda := \bigcup_{j=1}^{m}\Lambda_j$. Assume that*

1. *Λ_j is a p-SS for $E^p_{S_j}$ for every $j \in \{1, ..., m\}$.*
2. *Λ is a uniqueness set for E^p_S.*

Then Λ is a p-SS for E^p_S.

Notice that since Λ is u.d., then, by the Plancherel-Polya theorem, Λ verifies the p-P.P.C., and consequently the version of Corollary 4.3 for every countable set of disjoint two by two u.d. sets Λ_j's and every countable partition S_j's of a given bounded and Lebesgue measurable set S with positive measure, whose indicator functions are Fourier multipliers for $\mathcal{F}L^p$, is true.

In order to apply Corollary 4.3 to the Hilbert spaces $L^2\left(S\right)$, we need the next well known result.

Lemma 4.4. *Let $S \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set, and $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. For each $\lambda \in \Lambda$ consider the function $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by*

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

We also define the set $E\left(\Lambda\right) := \{\phi_\lambda\}_{\lambda\in\Lambda} \subseteq L^2(S)$. Then:

1. *$\left(F\left(\lambda\right) = 0 \text{ for all } \lambda \in \Lambda\right) \Leftrightarrow \widehat{F} \in E\left(-\Lambda\right)^{\perp}$.*
2. *$E\left(\Lambda\right)$ is total in $L^2(S)$ if and only if $E\left(-\Lambda\right)$ is total in $L^2(S)$ (see Definition 5.1 and Remark 5.2).*
3. *Λ is a US for E^2_S if and only if $E\left(\Lambda\right)$ is total in $L^2\left(S\right)$.*

The key step in Lemma 4.4 (namely in the second and third items) is that the space $L^2(S)$ is closed by conjugation.

As particular case of Corollary 4.3, also using Theorem 1.1 and the third item of Lemma 4.4, we have the following result for frames in spaces $L^2(S)$.

Corollary 4.5. *Let $n \in \mathbb{Z}^+$, and $S \subseteq \mathbb{R}^n$ be a compact set with positive measure. Let $m \in \mathbb{Z}^+$, $m \geq 2$, and $S_1, ..., S_m \subseteq S$ be compact subsets with positive measure such that $\{S_1, ..., S_m\}$ is a partition of S. Let $\Lambda_1, ..., \Lambda_m \subseteq \mathbb{R}^n$ be u.d. and disjoint two by two. We define $\Lambda := \bigcup_{j=1}^{m} \Lambda_j$.*

For every $\lambda \in \Lambda$ consider the function $\phi_\lambda : \mathbb{R}^n \to \mathbb{C}$ defined by

$$x \mapsto \phi_\lambda(x) := \begin{cases} 0, & \text{if } x \notin S \\ e^{i\lambda x}, & \text{if } x \in S \end{cases}.$$

We also define the set $E(\Lambda) := \{\phi_\lambda\}_{\lambda \in \Lambda} \subseteq L^2(S)$, and also analogously the set of restrictions

$$E(\Lambda_j) := \left\{\phi_\lambda|_{S_j}\right\}_{\lambda \in \Lambda_j} \subseteq L^2(S_j) \text{ for each } j \in \{1, ..., m\}.$$

Suppose that

1. *$E(\Lambda_j)$ is a frame for $L^2(S_j)$ for every $j \in \{1, ..., m\}$.*
2. *$E(\Lambda)$ is a total set for $L^2(S)$, that is,*

$$\overline{\operatorname{Span}(E(\Lambda))} = L^2(S).$$

Then $E(\Lambda)$ is a frame for $L^2(S)$.

5. Application of Sampling and Interpolation Theory in order to obtain product generator systems

Definition 5.1. *Let $(E, \| \|)$ be a quasinormed space, and let $S \subseteq E$. We say that S is a total set in $(E, \| \|)$, or also that S is a generator system of $(E, \| \|)$, if $\overline{\operatorname{Span}(S)} = E$.*

Remark 5.2. *Remind that given a Hilbert space E and a subset $S \subseteq E$, the following statements are equivalent:*

1. *S is a total set in E.*
2. *$S^\perp = \{0\}$.*

Lemma 5.3. *Let $f : (E, \| \|_E) \to (F, \| \|_F)$ be a topological isomorphism of quasinormed spaces. Let $C \subseteq F$, $S \subseteq E$ be, and suppose that $f(S) \supseteq C$. If C is a total set in $(F, \| \|_F)$, then S is a total set in $(E, \| \|_E)$.*

Proof. Assume that C is total in $(F, \| \|_F)$. Let us prove that S is total in $(E, \| \|_E)$.

As C is total in F, then, by definition, we have:

$$\overline{\operatorname{Span}(C)}^{\| \|_F} = F.$$

On the other hand, as f is a topological isomorphism, then f is a homeomorphism. Hence

$$F \supseteq f\left(\overline{\operatorname{Span}(S)}^{\| \|_E}\right) = \overline{f(\operatorname{Span}(C))}^{\| \|_F} =$$
$$= \overline{\operatorname{Span}(f(S))}^{\| \|_F} \supseteq \overline{\operatorname{Span}(C)}^{\| \|_F} = F.$$

Thus

$$f\left(\overline{\operatorname{Span}(S)}^{\| \|_E}\right) = F.$$

That is, the restriction $f|_{\overline{\mathrm{Span}(S)}^{\|\ \|_E}} : \overline{\mathrm{Span}\,(S)}^{\|\ \|_E} \to F$ is surjective, and consequently it is bijective. Since f is bijective, then

$$\overline{\mathrm{Span}\,(S)}^{\|\ \|_E} = E.$$

Conclusion: S is total in $(E,\ \|\ \|_E)$. $\square$

Corollary 5.4. *Let $(E,\ \|\ \|)$ be a quasi-Banach space, and let $S \subseteq E$, $p \in (0,\ +\infty)$, and $\Lambda \subseteq \mathbb{R}^n$ be u.d. For every $\lambda \in \Lambda$ we define*

$$e_\lambda := \left(\delta_{\lambda\,\mu}\right)_{\mu\in\Lambda} \in l^p\left(\Lambda\right),$$

where δ is the Kronecker's delta. Suppose that there exists a topological isomorphism $f : (E,\ \|\ \|) \to (l^p(\Lambda),\ \|\ \|_p)$ such that

$$e_\lambda \in f(S) \text{ for each } \lambda \in \Lambda.$$

Then S is total in E.

Proof. This result is an immediate consequence of Lemma 5.3 because both $C := \{e_\lambda\ :\ \lambda \in \Lambda\}$ is a total set in $(l^p(\Lambda),\ \|\ \|_p)$, and $f(S) \supseteq C$ by hypothesis. $\square$

Theorem 5.5. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(E,\ \|\ \|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}\,(\Omega,\ \mathbb{C})$. Let $\Lambda,\ \Gamma \subseteq \Omega$ be u.d. such that $\Lambda \subseteq \Gamma$. Let $p \in [1,\ +\infty)$, and let $q \in (1,\ +\infty]$ be the conjugate exponent of p (this is, $\frac{1}{p} + \frac{1}{q} = 1$). Let $r \in (0,\ +\infty]$. Consider*

$$l^r\left(\Lambda\right) := \left\{\left(a_\gamma\right)_{\gamma\in\Gamma} \in l^r\left(\Gamma\right) \mid a_\gamma = 0 \text{ for all } \gamma \in \Gamma \setminus \Lambda\right\} \subseteq l^r\left(\Gamma\right)$$

is a closed vector subspace of the quasi-Banach space $(l^r(\Gamma),\ \|\ \|_r)$. Suppose that Γ is an 1-SCIS for E. Let $E_1,\ E_2 \subseteq E$ be non empty sets such that $E_1 \cdot E_2 \subseteq E$. Suppose that $S\left(E_1\right) \subseteq l^p\left(\Lambda\right)$, $S\left(E_2\right) \subseteq l^q\left(\Lambda\right)$, and that Λ is both a p-IS for E_1 and a q-IS for E_2. Then Λ is an 1-SIS for $F := \overline{\mathrm{Span}\left(E_1 \cdot E_2\right)} \subseteq E$.

Proof. Let $r \in (0,\ +\infty]$. For every $\gamma \in \Gamma$ we define

$$e_\gamma := \left(\delta_{\lambda\,\mu}\right)_{\mu\in\Gamma} \in l^r\left(\Gamma\right),$$

where δ is the Kronecker's delta. It is obvious that

$$e_\lambda = \left(\delta_{\lambda\,\mu}\right)_{\mu\in\Gamma} \in l^r\left(\Lambda\right) \text{ for all } \lambda \in \Lambda.$$

Now consider the set $C := \{e_\lambda\ :\ \lambda \in \Lambda\} \subseteq l^1\left(\Lambda\right)$, which is a total set in $l^1(\Lambda)$; that is:

$$\overline{\mathrm{Span}\,(C)}^{\|\ \|_1} = l^1\left(\Lambda\right).$$

Since Γ is an 1-SCIS for E, then the sampling operator

$$S : (E,\ \|\ \|) \to \left(l^1(\Gamma), \|\ \|_1\right),$$

given by $f \to S(f) := (f(\gamma))_{\gamma\in\Gamma}$, is a topological isomorphism. In one hand we have that

$$S\left(E_1 \cdot E_2\right) \subseteq S\left(E_1\right) \cdot S\left(E_2\right) = l^p\left(\Lambda\right) \cdot l^q\left(\Lambda\right) \subseteq l^1\left(\Lambda\right),$$

where the last equality is due to Λ is both a p-IS for E_1 and a q-IS for E_2, and the inclusion is consequence of Hölder inequality.

On the other hand we have that

$$e_\lambda = e_\lambda \cdot e_\lambda \in l^p\left(\Lambda\right) \cdot l^q\left(\Lambda\right) \subseteq l^1\left(\Lambda\right) \text{ for each } \lambda \in \Lambda.$$

This is:

$$C \subseteq l^p\left(\Lambda\right) \cdot l^q\left(\Lambda\right) \subseteq l^1\left(\Lambda\right).$$

Therefore

$$\mathrm{Span}\,(C) \subseteq \mathrm{Span}\,(l^p(\Lambda) \cdot l^q(\Lambda)) \subseteq l^1(\Lambda),$$

and consequently

$$l^1(\Lambda) = \overline{\mathrm{Span}\,(C)}^{\|\,\|_1} \subseteq \overline{\mathrm{Span}\,(l^p(\Lambda) \cdot l^q(\Lambda))}^{\|\,\|_1} \subseteq l^1(\Lambda).$$

So that

$$\overline{\mathrm{Span}\,(l^p(\Lambda) \cdot l^q(\Lambda))}^{\|\,\|_1} = l^1(\Lambda).$$

Since the sampling operator S is an homeomorphism, we obtain:

$$S(F) = S\left(\overline{\mathrm{Span}\,(E_1 \cdot E_2)}^{\|\,\|_1}\right) = \overline{S\,(\mathrm{Span}\,(E_1 \cdot E_2))}^{\|\,\|_1} = \overline{\mathrm{Span}\,(S\,(E_1 \cdot E_2))}^{\|\,\|_1} =$$
$$= \overline{\mathrm{Span}\,(l^p(\Lambda) \cdot l^q(\Lambda))}^{\|\,\|_1} = l^1(\Lambda).$$

So that the restriction $S|_F : (F, \|\|) \to \left(l^1(\Lambda), \|\,\|_1\right)$, defined by $f \to (f(\lambda))_{\lambda\in\Lambda}$, is well defined, continuous and surjective.

As $(F, \|\,\|)$ and $\left(l^1(\Lambda), \|\,\|_1\right)$ are complete, then Λ is an 1-SIS for

$$F = \overline{\mathrm{Span}\,(E_1 \cdot E_2)},$$

by the Banach open mapping theorem. □

Remark 5.6. *Notice that in the proof of Theorem 5.5 we have proved that*

$$\overline{\mathrm{Span}\,(l^p(\Lambda) \cdot l^q(\Lambda))}^{\|\,\|_1} = l^1(\Lambda),$$

this is, $l^p(\Lambda) \cdot l^q(\Lambda)$ is total in $\left(l^1(\Lambda), \|\,\|_1\right)$.

Corollary 5.7. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(E, \|\,\|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be u.d. Let $p \in [1, +\infty)$, and let $q \in (1, +\infty]$ be the conjugate exponent of p. Assume that Λ is a p-SCIS for E. Let $E_1, E_2 \subseteq E$ be non empty sets such that $E_1 \cdot E_2 \subseteq E$. Suppose that $S(E_1) \subseteq l^p(\Lambda)$, $S(E_2) \subseteq l^q(\Lambda)$, and that Λ is both a p-IS for E_1 and a q-IS for E_2. Then*

$$E = \overline{\mathrm{Span}\,(E_1 \cdot E_2)}.$$

Proof. This result is an immediate consequence of Theorem 5.5 taking $\Gamma := \Lambda$. Indeed, the mappings $S : E \to l^1(\Lambda)$ and $S|_F : F \to l^1(\Lambda)$ are bijective, where $F := \overline{\mathrm{Span}\,(E_1 \cdot E_2)}$. Thus $E = F$, this is, $E_1 \cdot E_2$ is total in E. □

Theorem 5.8. *Let $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Int}(\Omega) \neq \emptyset$, and let $(E, \|\,\|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda, \Gamma \subseteq \Omega$ be u.d. such that $\Lambda \subseteq \Gamma$. Let $m \in \mathbb{Z}^+$, $r_1, ..., r_m \in (0, +\infty]$, with r_1 finite. Let $s \in (0, +\infty]$. Consider*

$$l^s(\Lambda) := \left\{(a_\gamma)_{\gamma\in\Gamma} \in l^s(\Gamma) \mid a_\gamma = 0 \text{ for all } \gamma \in \Gamma \setminus \Lambda\right\} \subseteq l^s(\Gamma),$$

which is a closed vector subspace of the quasi-Banach space $(l^s(\Gamma), \|\,\|_s)$. Assume that Γ is a r_1-SCIS for E. Let $E_1, ..., E_m \subseteq E$ be non empty sets such that $A := E_1 \cdot ... \cdot E_m \subseteq E$. Suppose that Λ is a r_j-IS for E_j for every $j \in \{1, ..., m\}$. Then Λ is a r_1-SIS for $F := \overline{\mathrm{Span}\,(A)} \subseteq E$.

Proof. As Γ is a r_1-SCIS for E, then the sampling operator

$$S : (E, \|\,\|) \to (l^{r_1}(\Gamma), \|\,\|_{r_1}),$$

given by $f \to S(f) := (f(\gamma))_{\gamma\in\Gamma}$, is a topological isomorphism.

It is clear that $l^{r_j}(\Lambda) \subseteq l^\infty(\Lambda)$ for every $j \in \{1, ..., m\}$. Thus

$$S(A) = S(E_1 \cdot ... \cdot E_m) = S(E_1) \cdot ... \cdot S(E_m) =$$
$$= l^{r_1}(\Lambda) \cdot ... \cdot l^{r_m}(\Lambda) \subseteq l^{r_1}(\Lambda),$$

where we have used that Λ is a r_j-IS for E_j for every $j \in \{1, ..., m\}$.

For each $\gamma \in \Gamma$ we define

$$e_\gamma := (\delta_{\gamma\,\mu})_{\mu \in \Gamma} \in \bigcap_{s \in (0,\,+\infty]} l^s(\Gamma).$$

It is clear that

$$e_\lambda := (\delta_{\lambda\,\mu})_{\mu \in \Gamma} \in \bigcap_{s \in (0,\,+\infty]} l^s(\Lambda) \text{ for all } \lambda \in \Lambda.$$

In fact we know that the set $C := \{e_\lambda\}_{\lambda \in \Lambda} \subseteq l^{r_1}(\Lambda)$ is total in $(l^{r_1}(\Lambda), \| \|_{r_1})$, as r_1 is finite. That is $\overline{\text{Span}(A)}^{\| \|_{r_1}} = l^{r_1}(\Lambda)$.

On the other hand we have that

$$e_\lambda = \prod_{j=1}^{m} e_\lambda \in \prod_{j=1}^{m} l^{r_j}(\Lambda) = S(A) \text{ for each } \lambda \in \Lambda.$$

That is, $C \subseteq S(A)$. Therefore $\text{Span}(C) \subseteq \text{Span}(S(A))$, and consequently

$$l^{r_1}(\Lambda) = \overline{\text{Span}(C)}^{\| \|_{r_1}} \subseteq \overline{\text{Span}(S(A))}^{\| \|_{r_1}} \subseteq l^{r_1}(\Lambda).$$

So that

$$\overline{\text{Span}(S(A))}^{\| \|_{r_1}} = l^{r_1}(\Lambda).$$

So that, as S is a homeomorphism, we obtain

$$S(F) = S\left(\overline{\text{Span}(A)}\right) = \overline{S(\text{Span}(A))}^{\| \|_{r_1}} =$$
$$= \overline{\text{Span}(S(A))}^{\| \|_{r_1}} = l^{r_1}(\Lambda).$$

Hence $S(F) = l^{r_1}(\Lambda)$, that is, the restriction $S|_F : (F, \| \|) \to (l^{r_1}(\Lambda), \| \|_{r_1})$, defined by $f \to (f(\lambda))_{\lambda \in \Lambda}$, is well defined, continuous and surjective.

As the quasinormed spaces $(F, \| \|)$ and $(l^{r_1}(\Lambda), \| \|_{r_1})$ are complete, then Λ is a r_1-SIS for $F = \overline{\text{Span}(A)}$, by the Banach open mapping theorem. □

Corollary 5.9. *Let $\Omega \subseteq \mathbb{R}^n$, $\text{Int}(\Omega) \neq \emptyset$, and let $(E, \| \|)$ be a quasi-Banach space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $\Lambda \subseteq \Omega$ be u.d. Let $m \in \mathbb{Z}^+$, $r_1, ..., r_m \in (0, +\infty]$, with r_1 finite. Assume that Λ is a r_1-SCIS for E. Let $E_1, ..., E_m \subseteq E$ be non empty sets such that $A := E_1 \cdot ... \cdot E_m \subseteq E$. Suppose that Λ is a r_j-IS for E_j for every $j \in \{1, ..., m\}$. Then $E = \overline{\text{Span}(A)}$.*

Proof. It is an immediate consequence of Theorem 5.8 taking $\Gamma := \Lambda$. Indeed, the mappings $S : E \to l^{r_1}(\Lambda)$ and $S|_F : F \to l^{r_1}(\Lambda)$ are bijective, where

$$F := \overline{\text{Span}(A)}.$$

Thus $E = F$, this is, A is total in E. □

Finally, we conclude with the following easy but useful result.

Lemma 5.10. *Let $\Omega \subseteq \mathbb{R}^n$, $\text{Int}(\Omega) \neq \emptyset$, and let $(E, \| \|)$ be a quasinormed space with $E \subseteq \mathfrak{F}(\Omega, \mathbb{C})$. Let $E_1, E_2 \subseteq E$ be non empty subsets such that $F := E_1 \cdot E_2 \subseteq E$. Let $p \in (0, +\infty]$, and $\Lambda \subseteq \Omega$ be uniformly discrete. Suppose that Λ is an ∞-IS for E_1 and a p-IS for E_2. Then Λ is a p-IS for F.*

Proof. Define $a := (a_\lambda)_{\lambda \in \Lambda}$, with $a_\lambda := 1$ for each $\lambda \in \Lambda$. Since $a \in l^\infty(\Lambda)$, then

$$l^\infty(\Lambda) \cdot l^p(\Lambda) = l^p(\Lambda),$$

whereby we obtain the result. □

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