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Alternative Planes and the Curves on Them

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ABSTRACT: In this study, the planes formed by Frenet elements are defined with a U vector chosen different from T, N and B, which are the elements of the Frenet frame, and the curves on these planes are also characterized. As it is known, the planes formed by Frenet elements between themselves have been defined and investigated many times. In this present article, the plane formed by an arbitrary chosen vector U with T, N and B is defined and the curves lying in this plane are characterized.

Key Words: Frenet frame, rectifying curve, normal curve, osculating curve.

Contents

1. Introduction

Since the theory of curves is one of the most remarkable subjects of differential geometry, this subject is studied in depth by many researchers. For a regular space curve, it is possible to construct a frame along the curve at each point of the curve. The creation of this framework and the expression of its derivatives according to this framework made independently by the French mathematicians Jean-Frederic and Joseph Alfred Serret in 1850, therefore this framework is called the Serret-Frenet framework. In the following studies, the planes formed by these frame elements separately from each other are emphasized and the properties of the curves lying on this plane are investigated. In particular, along with the applicability of different algebraic structures to differential geometry, some special curves play a decisive role in answering important problems. Some of these special curves are the rectifying, osculating and normal curves, which are characterized by the plane formed by the Frenet elements. In answer to the question in which cases the position vector of a spatial curve always lies in the rectifying plane of the ground vector, author defined the rectifying curve in 2003 [3]. Afterwards, various characterizations of this curve are studied and defined in different spaces [1,2,4,5]. On the other hand, the concept of osculating and normal curve are given, similar to the rectifying curve [7]. In addition, similar to the rectifying curve, these curves have been investigated by many researchers and their differences in different spaces have been obtained [6,8,9].

In this study, firstly, a U vector is determined, different from the Frenet vectors T, N and B vectors at a point of the curve. Afterwards, the TU plane, NU plane and BU planes are defined formed by this U vector with T, N and B vectors, respectively. Finally, the curves lying in these defined planes are characterized.

2. Preliminaries

Let $\alpha = \alpha(s)$ be a regular unit speed curve in the Euclidean 3−space where s measures its arc length. Also, let $T = \alpha'$ be its unit tangent vector, $N = \frac{T'}{\|T'\|}$ be its principal normal vector and $B = T \times N$ be

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its binormal vector. The triple $\{T, N, B\}$ be the Frenet frame of the curve α. Then the Frenet formula of the curve is given by

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.1}$$

where $\kappa(s) = \left\| \frac{d^2\alpha}{ds^2} \right\|$ and $\tau(s) = \left\langle \frac{dN}{ds},\ B \right\rangle$ are curvature and torsion of α respectively, [10].

Definition 2.1. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. The curve α is called rectifying curves for all $s \in I$, if the orthogonal complement of $N(s)$ contains a fixed point. Since the orthogonal complement of N is $N^{\perp} = \{v \in T_\alpha E^3 \mid \langle v,\ N\rangle = 0\}$. The position vector of rectifying curve α in E^3 can be written as*

$$\alpha(s) = \lambda T(s) + \mu B(s)$$

where λ and μ are differentiable function, [3].

Definition 2.2. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. It is said that α is an osculating curve, if its position vector is in the orthogonal complement of the binormal vector for all $s \in I$ and consequently*

$$\alpha(s) = \lambda T(s) + \mu N(s)$$

where λ, μ are differentiable function, [5].

Definition 2.3. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. It is said that α is a normal curve, if its position vector is in the orthogonal complement of the tangent vector for all $s \in I$ and consequently*

$$\alpha(s) = \lambda N(s) + \mu B(s)$$

where λ, μ are differentiable function, [5].

3. Main Results

In this section, by choosing U vector other than Frenet elements, the planes formed by Frenet elements separately with this selected vector are defined. In addition, the curves lying on these planes have been characterized.

3.1. TU plane and the curves lying on it

Definition 3.1. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As seen in Figure 1, the $Sp\{T, U\}$ plane is called the TU plane where the vector U is perpendicular to T which is the tangent vector of the curve α.*

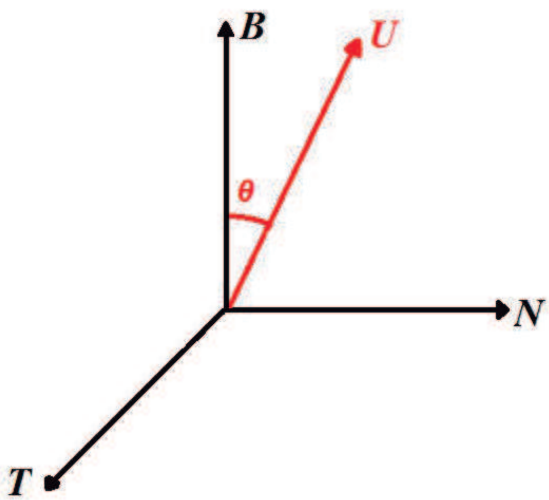

Figure 1. TU Plane.

Definition 3.2. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If a curve α lies in a TU plane, α is called TU curves. So, its position vector is expressed as follows for all $s \in I$*

$$\alpha(s) = \lambda U(s) + \mu T(s) \tag{3.1}$$

where λ , μ are differentiable function.

Corollary 3.3. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As can be seen from Figure 1, the following equation exist*

$$U = \cos\theta B + \sin\theta N. \tag{3.2}$$

where θ is the fixed angle between U and B.

Theorem 3.4. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane TU, the following equation exists*

$$\frac{\tau}{\kappa} = -\frac{\lambda' \cos\theta}{\mu' - 1}.$$

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the derivative of equation (3.1) is taken, we obtain

$$\alpha' = \lambda' U + \lambda U' + \mu' T + \mu T'.$$

If equation (3.2) and Frenet formulas are used in the above equation, following equation

$$\alpha' = (\mu' - \lambda\kappa \sin\theta)T + (\mu\kappa + \lambda' \sin\theta - \lambda\tau \cos\theta)N + (\lambda' \cos\theta + \lambda\tau \sin\theta)B$$

is obtained. From the reciprocal equations above, we have

$$\begin{aligned} \mu' - \lambda\kappa \sin\theta &= 1 \\ \lambda' \cos\theta + \lambda\tau \sin\theta &= 0. \end{aligned}$$

Finally, from the above two equations, the following result can be given

$$\frac{\tau}{\kappa} = \frac{\lambda' \cos\theta}{1 - \mu'}. \tag{3.3}$$

Corollary 3.5. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. Considering equation (3.3), if μ is a constant and λ is a linear function, then α is a helix.*

Theorem 3.6. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane TU, the following equations exists*

i. $\kappa = \frac{-\lambda'}{\mu \sin\theta}$,

ii. $\frac{\tau}{\kappa} = \frac{\mu}{\lambda} \cos\theta$,

iii. $\lambda = e^{-\int \tau \tan\theta}$.

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the derivative of equation (3.1) is taken ,we obtain

$$\alpha' = \lambda' U + \lambda U' + \mu' T + \mu T'.$$

If equation (3.2) and Frenet formulas are used in the above equation, following equation

$$\alpha' = (\mu' - \lambda\kappa \sin\theta)T + (\mu\kappa + \lambda' \sin\theta - \lambda\tau \cos\theta)N + (\lambda' \cos\theta + \lambda\tau \sin\theta)B$$

is obtained. From the reciprocal above equations, we have

$$\lambda' \cos\theta + \lambda\tau \sin\theta = 0, \tag{3.4}$$

$$\mu\kappa + \lambda' \sin\theta - \lambda\tau \cos\theta = 0. \tag{3.5}$$

The expression κ from the equations obtained above is as follows

$$\kappa = \frac{-\lambda'}{\mu \sin\theta}. \tag{3.6}$$

On the other hand, if the equation is summed up (3.4) and (3.5) side to side, we have

$$\frac{\tau}{\kappa} = \frac{\mu}{\lambda}\cos\theta. \tag{3.7}$$

Finally, considering the equations (3.6) and (3.7), we have

$$\lambda = e^{-\int \tau \tan\theta}.$$

Corollary 3.7. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. Considering equation (3.7), if $\frac{\mu}{\lambda}$ is a constant, then α is a helix.*

3.2. NU plane and the curves lying it

Definition 3.8. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As seen in Figure 2, the $Sp\{N, U\}$ plane is called the NU plane where the vector U is perpendicular to N which is the principal normal vector of the curve α.*

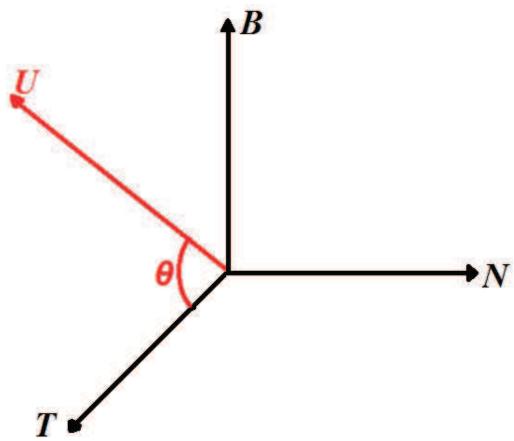

Figure 2. NU Plane.

Definition 3.9. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in a NU plane, α is called NU curves. Hence, its position vector can be written as*

$$\alpha(s) = \delta U(s) + \zeta N(s) \tag{3.8}$$

where δ , ζ are differentiable function.

Corollary 3.10. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As can be seen from Figure 2, the following equation exist*

$$U = \cos\theta T + \sin\theta B. \tag{3.9}$$

where θ is the fixed angle between U and T.

Theorem 3.11. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane NU, the following equation exists*

$$\frac{\tau}{\kappa} = -\frac{\delta' \sin\theta}{\delta' \cos\theta - 1}.$$

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If we take the derivative of equation (3.8) we can easily see that

$$\alpha' = \delta' U + \delta U' + \zeta' N + \zeta N'.$$

If equation (3.9) and Frenet formulas are used in the above equation, we obtain that

$$\alpha' = (\delta' \cos\theta - \zeta\kappa)T + (\delta\kappa\cos\theta - \delta\tau\sin\theta + \zeta')N + (\delta'\sin\theta + \zeta\tau)B.$$

From the reciprocal equations above, following equations can be written

$$\begin{aligned} \delta'\cos\theta - \zeta\kappa &= 1, \\ \delta'\sin\theta + \zeta\tau &= 0. \end{aligned}$$

Finally, from the above two equations, the following result can be given

$$\frac{\tau}{\kappa} = -\frac{\delta'\sin\theta}{\delta'\cos\theta - 1}. \tag{3.10}$$

Corollary 3.12. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. Considering equation (3.10), if δ is a constant, then α is a helix.*

Theorem 3.13. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane NU, the following equation exists*

$$\delta' = \frac{\tau}{\tau\cos\theta + \kappa\sin\theta}.$$

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If we take the derivative of equation (3.8) we can easily see that

$$\alpha' = \delta' U + \delta U' + \zeta' N + \zeta N'.$$

If equation (3.9) and Frenet formulas are used in the above equation, we obtain that

$$\alpha' = (\delta' \cos\theta - \zeta\kappa)T + (\delta\kappa\cos\theta - \delta\tau\sin\theta + \zeta')N + (\delta'\sin\theta + \zeta\tau)B.$$

From the reciprocal equations above, following equations can be written

$$\begin{aligned} \delta'\cos\theta - \zeta\kappa &= 1, \\ \delta'\sin\theta + \zeta\tau &= 0. \end{aligned}$$

Finally, from the above two equations, the following result can be given

$$\delta' = \frac{\tau}{\tau\cos\theta + \kappa\sin\theta}. \tag{3.11}$$

Corollary 3.14. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. Considering equation (3.11), if δ is a constant, then α is a planar curve.*

3.3. BU plane and the curves lying on it

Definition 3.15. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As seen in Figure 3, the $Sp\{B, U\}$ plane is called the BU plane where the vector U is perpendicular to B which is the principal normal vector of the curve α.*

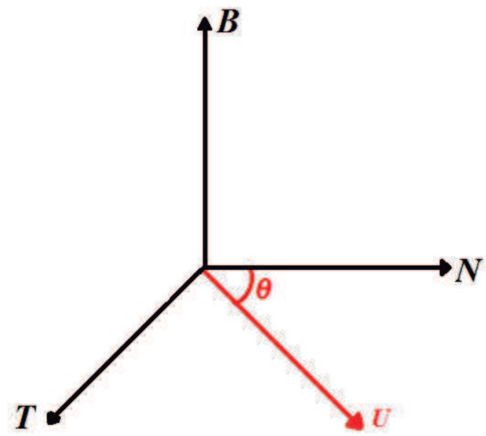

Figure 3. BU Plane.

Definition 3.16. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in a BU plane, α is called BU curves. Hence, its position vector can be written as*

$$\alpha(s) = \eta U(s) + \sigma B(s) \tag{3.12}$$

where η, σ are differentiable function.

Corollary 3.17. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve and $\{T, N, B\}$ be a Frenet vector fields. As can be seen from Figure 3, the following equation exist*

$$U = \sin\theta T + \cos\theta N \tag{3.13}$$

where θ is fixed angle between U and N.

Theorem 3.18. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane BU, the following equation exists*

$$\frac{\tau}{\kappa} = -\frac{\sigma'}{\eta' \sin\theta - 1}.$$

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the derivative of equation (3.12) is taken, we get

$$\alpha' = \eta' U + \eta U' + \sigma' B + \sigma B'.$$

If equation (3.13) and Frenet formulas are used in the above equation, we obtain following

$$\alpha' = (\eta' \sin\theta - \eta\kappa\cos\theta)T + (\eta' \cos\theta + \eta\kappa\sin\theta - \sigma\tau)N + (\eta\tau\cos\theta + \sigma')B$$

From the reciprocal equations above we have

$$\begin{aligned} \eta' \sin\theta - \eta\kappa\cos\theta &= 1, \\ \eta\tau\cos\theta + \sigma' &= 0. \end{aligned}$$

Finally, from the above two equations, the following result can be given

$$\frac{\tau}{\kappa} = -\frac{\sigma'}{\eta' \sin\theta - 1}. \tag{3.14}$$

Corollary 3.19. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. Considering equation (3.14), if σ is a constant or linear function, then α is a helix.*

Theorem 3.20. *Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the curve α lies in the plane BU, the following equations exists*
i. $\tau = \frac{\eta' - \sin\theta}{\sigma\cos\theta}$,
ii. $\kappa = \frac{\sigma\tau\sin\theta - \cos\theta}{\eta}$.

Proof: Let $\alpha : I \subset R \to E^3$ be arc length parametrized curve. If the derivative of equation (3.12) is taken, we get

$$\alpha' = \eta' U + \eta U' + \sigma' B + \sigma B'.$$

If equation (3.13) and Frenet formulas are used in the above equation, we obtain following

$$\alpha' = (\eta' \sin\theta - \eta\kappa\cos\theta)T + (\eta'\cos\theta + \eta\kappa\sin\theta - \sigma\tau)N + (\eta\tau\cos\theta + \sigma')B$$

From the reciprocal equations above we have

$$\begin{aligned} \eta'\sin\theta - \eta\kappa\cos\theta &= 1, \\ \eta'\cos\theta + \eta\kappa\sin\theta - \mu\tau &= 0. \end{aligned}$$

Finally, from the above two equations, the following result can be given

$$\tau = \frac{\eta' - \sin\theta}{\sigma\cos\theta},$$

$$\kappa = \frac{\sigma\tau\sin\theta - \cos\theta}{\eta}.$$

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