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Approximating an Advanced Multi-dimensional Reciprocal-quadratic Mapping via a Fixed Point Approach*

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ABSTRACT: There are many results on stability of various forms of functional equations available in the theory of functional equations. The intention of this paper is to introduce an advanced and a new multi-dimensional reciprocal-quadratic functional equation involving $p > 1$ variables. It is interesting to note that it has two different solutions, namely, quadratic and multiplicative inverse quadratic functions. We solve its various stability problems in the setting of non-zero real numbers and non-Archimedean fields via fixed point approach.

Key Words: Reciprocal functional equation, quadratic functional equation, Hyers-Ulam-Rassias stability, non-Archimedean field.

Contents

1. Introduction

A remarkable and significant question in the theory of functional equations is under what conditions is it true that a mapping which satisfies a functional equation $\mathcal{E}$ near to an exact solution of $\mathcal{E}$? From this question, the theory of stability of functional equation is originated. The question raised by Ulam [28] concerning the stability problem of group homomorphims forms the foundation for investigating the stability of functional equations. Hyers [15] was the foremost mathematician to provide a noteworthy response to Ulam's question. The result of Hyers motivated many researchers to develop the stability results in different versions [1,8,12]. Hyers method is very simple and easy for other researchers to deal with several functional equations since an approximate solution near to the exact solution is obtained directly from the assumed functional inequality. Hence the method adopted by Hyers is celebrated as Direct method. Later, T. M. Rassias [22], J. M. Rassias [21], and Gavruta [14] further generalized the stability results by considering different upper bounds. For the past four decades, there are many interesting, new, motivating and important and remarkable results pertinent to various forms of functional equations, one can refer to [4,9,13,18,19,20,25].

There is another method of solving stability problems of functional equations, known as fixed point method. The following fixed point theorem is useful to obtain the stability of functional equation.

Theorem 1.1. [10] *Let (Ω, d) be a complete generalized metric space and $J : \Omega \longrightarrow \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\beta \in \Omega$, either $d\left(J^n\beta, J^{n+1}\beta\right) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

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(i) $d\left(J^n\beta, J^{n+1}\beta\right) < \infty$ *for all* $n \geq n_0$;

(ii) *the sequence* $\{J^n\beta\}$ *is convergent to a fixed point* $\beta^\star$ *of* J;

(iii) $\beta^\star$ *is the unique fixed point of* J *in the set* $\Lambda = \{\beta \in \Omega : d(J^{n_0}\beta, \beta) < \infty\}$;

(iv) $d(\beta, \beta^\star) \leq \dfrac{1}{1-L} d(\beta, J\beta)$ *for all* $\beta \in \Lambda$.

In this method, an approximate solution near to the exact solution of functional equation exists by employing fixed point alternative theorem. Isac and T. M. Rassias [16] attempted first time to solve stability problem of functional equations using fixed point technique. Cadariu [5,6] applied fixed point method to obtain stabilities of Cauchy and Jensen's functional equations. The stability results of quadratic functional equations obtained through fixed point method are discussed in [7].

On the other hand, the research on stability of multiplicative inverse functional equations is attracted by many researchers. There are many such functional equations that can be interpreted in electrical engineering with the reciprocal formula for the resistances connected in parallel, related with harmonic and arithmetic means of several values, used in studying different physical properties of objects, found in combination of different lenses in optics and in electromagnetism. The pioneering multiplicative inverse functional equation

$$f(\alpha+\beta) = \frac{f(\alpha)f(\beta)}{f(\alpha)+f(\beta)} \tag{1.1}$$

is the foremost equation dealt in [24] to study its application, interpretation and stability results. It is easy to note that the multiplicative inverse function $f(\alpha) = \frac{c}{\alpha}$ is a solution of (1.1). The results associated with (1.1) motivated many mathematicians to introduce and investigate various multiplicative inverse functional equations for their applications in other domains. The detailed information regarding the applications, interpretations and stabilities of various multiplicative inverse functional equations are available in [11,17,23,26]. The reciprocal-quadratic functional equations were introduced for the first time in [2] and studied as a generalized version in [3].

Quite recently, the following functional equation

$$f\left(\sum_{j=1}^{m} y_j\right) + f\left(\frac{\prod_{j=1}^{m} y_j}{\sum_{j=1}^{m} \prod_{\substack{k=1,\\ k\neq j}}^{m} y_k}\right) = \frac{\prod_{j=1}^{m} f(y_j)}{\sum_{j=1}^{m} \prod_{\substack{k=1,\\ k\neq j}}^{m} f(y_k)} + \sum_{j=1}^{m} f(y_j) \tag{1.2}$$

is considered in [27] to investigate its hyperstability. It is interesting to note that (1.2) has two different solutions, namely additive function $f(y) = y$ and multiplicative inverse function $f(y) = \frac{1}{y}$.

Inspired by the remarkable role of multiplicative inverse functional equations in various other fields, in this article, we introduce a new multi-dimensional reciprocal-quadratic functional equation involving

$p > 1$ variables of the form

$$r\left(\sum_{k=1}^{p}\alpha_k\right)+r\left(\frac{\prod_{k=1}^{p}\alpha_k}{\sum_{k=1}^{p}\left(\prod_{\substack{m=1,\\ m\neq k}}^{p}\alpha_m\right)}\right)=\sum_{k=1}^{p}r\left(\alpha_k\right)+2\sum_{k=1}^{p}\sum_{\substack{m=1,\\ m\neq k}}^{p}\sqrt{r(\alpha_m)}$$
$$+\frac{\prod_{k=1}^{p}r(\alpha_k)}{\sum_{k=1}^{p}\left(\prod_{\substack{m=1,\\ m\neq k}}^{p}r\left(\alpha_m\right)\right)+\sum_{k=1}^{p}\sqrt{r\left(\alpha_k\right)}\left(\sum_{\substack{m=1,\\ m\neq k}}^{p}\sqrt{r\left(\alpha_m\right)}\left(\prod_{\substack{n=1,\\ n\neq k,m}}^{p}r\left(\alpha_n\right)\right)\right)}. \tag{1.3}$$

It can be easily verified that the quadratic function $r(\alpha) = \alpha^2$ and multiplicative inverse quadratic function $r(\alpha) = \frac{1}{\alpha^2}$ are solutions of (1.3). We solve the stability problems of (1.3) in the setting of non-zero real numbers and non-Archimedean fields using fixed point technique.

Remark 1.2. *When $p = 2$, equation (1.3) reduces to*

$$r(\alpha_1+\alpha_2)+r\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)$$
$$=r(\alpha_1)+r(\alpha_2)+2\sqrt{r(\alpha_1)r(\alpha_2)}+\frac{r(\alpha_1)r(\alpha_2)}{r(\alpha_1)+r(\alpha_2)+2\sqrt{r(\alpha_1)r(\alpha_2)}}.$$

In order to avoid singularties in our main results, we assume that $\sum_{k=1}^{p}\alpha_k \neq 0$, $\sum_{m=1}^{p}\prod_{\substack{m=1,\\ m\neq k}}^{p}\alpha_m \neq 0$, $\sum_{k=1}^{p}\prod_{\substack{m=1,\\ m\neq k}}^{p}r\left(\alpha_m\right)+\sum_{k=1}^{p}\sqrt{r\left(\alpha_k\right)}\left(\sum_{\substack{m=1,\\ m\neq k}}^{p}\sqrt{r\left(\alpha_m\right)}\left(\prod_{\substack{n=1,\\ n\neq k,m}}^{p}r\left(\alpha_n\right)\right)\right) \neq 0$, and $r(\alpha_k) \neq 0$, for all $k = 1, 2, \ldots, p$. For the sake of obtaining the stability results easily, let us define the difference operator D_p as follows.

$$D_p r(\alpha_1,\alpha_2,\ldots,\alpha_p)=r\left(\sum_{k=1}^{p}\alpha_k\right)+r\left(\frac{\prod_{k=1}^{p}\alpha_k}{\sum_{k=1}^{p}\left(\prod_{\substack{m=1,\\ m\neq k}}^{p}\alpha_m\right)}\right)-\sum_{k=1}^{p}r\left(\alpha_k\right)-2\sum_{k=1}^{p}\sum_{\substack{m=1,\\ m\neq k}}^{p}\sqrt{r(\alpha_m)}$$
$$-\frac{\prod_{k=1}^{p}r(\alpha_k)}{\sum_{k=1}^{p}\left(\prod_{\substack{m=1,\\ m\neq k}}^{p}r\left(\alpha_m\right)\right)+\sum_{k=1}^{p}\sqrt{r\left(\alpha_k\right)}\left(\sum_{\substack{m=1,\\ m\neq k}}^{p}\sqrt{r\left(\alpha_m\right)}\left(\prod_{\substack{n=1,\\ n\neq k,m}}^{p}r\left(\alpha_n\right)\right)\right)}.$$

A few basic notions of non-Archimedean field which play major role to obtain our main results, are available in [6,11,17]. Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^\star = \mathbb{R}\backslash\{0\}$.

2. Key result pertinent to equation (1.3)

We present the following definition which plays a vital role in obtain a key result pertinent to equation (1.3).

Definition 2.1. *A mapping $r : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is said to be a reciprocal-quadratic if it satisfies the following general functional equation:*

$$r(p\alpha) + r\left(\frac{\alpha}{p}\right) = \frac{m^2+1}{m} r(\alpha) \tag{2.1}$$

for all $\alpha \in \mathbb{R}^\star$ and any integer $m = 3p - 2$.

Remark 2.2. *From the above definition, it can be easily verified that (1.3) satisfies (2.1) by substituting $\alpha_k = \alpha$, for $k = 1, 2, \ldots, p$ in (1.3). Hence (1.3) is said to be reciprocal-quadratic functional equation.*

Theorem 2.3. *If a mapping $r : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfies (1.3), then it also satisfies*

$$r\left(p^q\alpha\right) + r\left(\frac{\alpha}{p^q}\right) = \frac{m^{2q}+1}{m^q} r(\alpha) \tag{2.2}$$

for all $\alpha \in \mathbb{R}^\star$, where $m = 3p - 2$ and $q > 0$ is an integer.

Proof. Let us put $\alpha_k = \alpha$ for $k = 1, 2, \ldots, p$ in (1.3), we obtain

$$r(p\alpha) + r\left(\frac{\alpha}{p}\right) = \frac{m^2+1}{m} r(\alpha) \tag{2.3}$$

for all $\alpha \in \mathbb{R}^\star$. Now, replacing α by $p\alpha$ and $\frac{\alpha}{p}$ in (2.3), and then summing the resulting equations, we get

$$r(p^2\alpha) + r\left(\frac{\alpha}{p^2}\right) = \frac{m^4+1}{m^2} r(\alpha) \tag{2.4}$$

for all $\alpha \in \mathbb{R}^\star$. Again, replacing α by $p\alpha$ and $\frac{\alpha}{p}$ in (2.4), and then adding the resultants, we get

$$r(p^3\alpha) + r\left(\frac{\alpha}{p^3}\right) = \frac{m^6+1}{m^3} r(\alpha)$$

for all $\alpha \in \mathbb{R}^\star$. Let us assume that (2.2) is true for any integer $n > 0$. Hence, we have

$$r(p^n\alpha) + r\left(\frac{\alpha}{p^n}\right) = \frac{m^{2n}+1}{m^n} r(\alpha) \tag{2.5}$$

for all $\alpha \in \mathbb{R}^\star$. Now, plugging $p\alpha$ into α and $\frac{\alpha}{p}$ in (2.5), and then adding the consequent equations and by virtue of (2.5), we obtain

$$\begin{aligned} r(p^{n+1}\alpha) + r\left(\frac{\alpha}{p^{n+1}}\right) &= \frac{m^{2n}+1}{m^n}\left(r(p\alpha) + r\left(\frac{\alpha}{p}\right)\right) - \left(r\left(p^{n-1}\alpha\right) + r\left(\frac{\alpha}{p^{n-1}}\right)\right) \\ &= \frac{m^{2n}+1}{m^n}\left(\frac{m^2+1}{m} r(\alpha)\right) - \frac{m^{2(n-1)}+1}{m^{n-1}} r(\alpha) \\ &= \frac{m^{2(n+1)}+1}{m^{n+1}} r(\alpha) \end{aligned}$$

for all $\alpha \in \mathbb{R}^\star$. Hence by mathematical induction, one can find for any integer $q > 0$,

$$r(p^q\alpha) + r\left(\frac{\alpha}{p^q}\right) = \frac{m^{2q}+1}{m^q} r(\alpha)$$

for all $\alpha \in \mathbb{R}^\star$, which completes the proof. □

3. Stabilities of equation (1.3) in the setting of non-zero real numbers

Applying the idea of fixed point theorem proved in [10], we obtain the solutions to stability problems of equation (1.3) in the setting of non-zero real numbers.

Theorem 3.1. *Let $r : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a mapping satisfying the following inequality*

$$|D_p r(\alpha_1, \alpha_2, \dots, \alpha_p)| \leq \phi(\alpha_1, \alpha_2, \dots, \alpha_p) \tag{3.1}$$

for all $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}^\star$, where $\phi : \underbrace{\mathbb{R}^\star \times \cdots \times \mathbb{R}^\star}_{p-times} \longrightarrow [0, \infty)$ is a given function. If there exists a constant $0 < L < 1$ such that the mapping $\alpha \mapsto \Phi(\alpha) = \phi(\underbrace{\alpha, \dots, \alpha}_{p-times})$ possess the property

$$\Phi\left(\frac{\alpha}{p}\right) \leq \frac{m}{m^2+1} L\Phi(\alpha) \tag{3.2}$$

for all $\alpha \in \mathbb{R}^\star$. Then there exists a unique mapping $G : \mathbb{R}^\star \longrightarrow \mathbb{R}$ such that

$$|r(\alpha) - G(\alpha)| \leq \frac{mL}{(m^2+1)(1-L)} \Phi(\alpha) \tag{3.3}$$

for all $\alpha \in \mathbb{R}^\star$, where $m = 3p - 2$.

Proof. Let us define T as a set of all mappings $\psi : \mathbb{R}^\star \longrightarrow \mathbb{R}$. Let us also define a generalized metric d on the set T as follows.

$$d(\psi, \chi) = \inf \left\{K \in \mathbb{R}_+ : |\psi(\alpha) - \chi(\alpha)| \leq K\Phi(\alpha), \text{ for all } \alpha \in \mathbb{R}^\star \right\}. \tag{3.4}$$

Using the definition of d, one can easily prove that (T, d) is a generalized complete metric space. Now, let us define a mapping $\lambda : T \longrightarrow T$ by

$$\lambda(\psi(\alpha)) = \frac{m^2+1}{m} r\left(\frac{\alpha}{p}\right) - r\left(\frac{\alpha}{p^2}\right) \tag{3.5}$$

for all $\alpha \in \mathbb{R}^\star$. Next, let us prove that λ is a strictly contractive mapping on the set T. For given $\psi, \chi \in T$, suppose $0 \leq \mu_{\psi\chi} \leq \infty$ is an arbitrary constant with $d(\psi, \chi) \leq \mu_{\psi\chi}$. Hence for all $\alpha \in \mathbb{R}^\star$, we have

$$\begin{aligned}
&d(\psi, \chi) \leq \mu_{\psi\chi} \\
&\quad \Longrightarrow |\psi(\alpha) - \chi(\alpha)| \leq \mu_{\psi\chi}\Phi(\alpha) \\
&\quad \Longrightarrow \left|\frac{m^2+1}{m}\psi\left(\frac{\alpha}{p}\right) - \psi\left(\frac{\alpha}{p^2}\right) - \frac{m^2+1}{m}\chi\left(\frac{\alpha}{p}\right) + \chi\left(\frac{\alpha}{p^2}\right)\right| \leq \frac{m^2+1}{m}\mu_{\psi\chi}\Phi\left(\frac{\alpha}{p}\right) \\
&\quad \Longrightarrow \left|\frac{m^2+1}{m}\psi\left(\frac{\alpha}{p}\right) - \psi\left(\frac{\alpha}{p^2}\right) - \frac{m^2+1}{m}\chi\left(\frac{\alpha}{p}\right) + \chi\left(\frac{\alpha}{p^2}\right)\right| \leq L\mu_{\psi\chi}\Phi(\alpha) \\
&\quad \Longrightarrow d(\lambda\psi, \lambda\chi) \leq L\mu_{\psi\chi}.
\end{aligned}$$

From the above inequality, we observe that $d(\lambda\psi, \lambda\chi) \leq Ld(\psi, \chi)$ for all $\psi, \chi \in T$. This implies that λ is a strictly contractive mapping of T, with the Lipschitz constant L. Now, replacing $\alpha_k = \frac{\alpha}{p}$, for $k = 1, 2, \dots, p$, we obtain that

$$\left|r(\alpha) - \left\{\frac{m^2+1}{m} r\left(\frac{\alpha}{p}\right) - r\left(\frac{\alpha}{p^2}\right)\right\}\right| \leq \Phi\left(\frac{\alpha}{p}\right)$$

for all $\alpha \in \mathbb{R}^\star$. Using (3.4), we find that $d(\lambda r, r) \leq 1$. Hence, by the application of fixed point alternative theorem, there exists a mapping $G : \mathbb{R}^\star \longrightarrow \mathbb{R}$ satisfying the following:

(1) G is a fixed point of d, that is,

$$\frac{m^2+1}{m}G\left(\frac{\alpha}{p}\right)-G\left(\frac{\alpha}{p^2}\right)=G(\alpha) \tag{3.6}$$

for all $\alpha \in \mathbb{R}^\star$. The mapping G is the unique fixed point of λ in the set

$$\Lambda=\{r\in T : d(G,r)<\infty\}.$$

This confirms that G is the unique mapping satisfying (3.6) such that there exists $0<\mu<\infty$ satisfying

$$|G(\alpha)-r(\alpha)|\leq \mu\Phi(\alpha)$$

for all $\alpha \in \mathbb{R}^\star$.

(2) $d(\lambda^n r, G)\to 0$ as $n\to\infty$. Hence, we obtain

$$\lim_{n\to\infty}\left\{\frac{(m^2+1)^{n-2}(m^4+m^2+1)}{m^n}r\left(\frac{\alpha}{p^n}\right)-\frac{(m^2+1)^{n-1}}{m^{n-1}}r\left(\frac{\alpha}{p^{n+1}}\right)\right.$$
$$\left.-\sum_{k=2}^{n-1}\left(\frac{m^2+1}{m}\right)^{k-2}r\left(\frac{\alpha}{p^k}\right)\right\}=G(\alpha) \tag{3.7}$$

for all $\alpha \in \mathbb{R}^\star$.

(3) $d(G,r)\leq \frac{1}{1-L}d(G,\lambda r)$, which results in $d(G,r)\leq \frac{1}{1-L}$.

Hence, the inequality (3.3) holds. On the other hand, from (3.2) and (3.7), we have

$$|D_pG(\alpha_1,\ldots,\alpha_p)|\leq \lim_{n\to\infty}L^n\phi(\alpha_1,\ldots,\alpha_p)=0$$

for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$. This confirms that G is a solution of equation (1.3). □

The following results are pertinent to the stabilities of equation (1.3) involving a positive constant and sum of powers of norms as upper bounds. The proofs are directly obtained from Theorem 3.1. In the sequel, let $r:\mathbb{R}^\star\longrightarrow\mathbb{R}$ be a mapping.

Corollary 3.2. *Suppose a constant $\mu\geq 0$ exists such that*

$$|D_pr(\alpha_1,\alpha_2,\ldots,\alpha_p)|\leq\frac{\mu}{m}$$

holds for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$. Then there exists a solution $G:\mathbb{R}^\star\longrightarrow\mathbb{R}$ of (1.3) which is unique and

$$|r(\alpha)-G(\alpha)|\leq\frac{\mu}{(m^2+1)(m-1)}$$

for all $\alpha\in\mathbb{R}^\star$.

Proof. Replacing $\phi(\alpha_1,\ldots,\alpha_p)=\frac{\mu}{m}$ and $L=\frac{1}{m}$ in Theorem 3.1, the desired result is attained. □

Corollary 3.3. *Let there exist a real number $a>-2$ or $a>2$ and $\delta_1\geq 0$ such that*

$$|D_pr(\alpha_1,\alpha_2,\ldots,\alpha_p)|\leq\delta_1\sum_{k=1}^{p}|\alpha_k|^a$$

for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$. Then, there exists a solution $G:\mathbb{R}^\star\longrightarrow\mathbb{R}$ of (1.3) which is unique and

$$|r(\alpha)-G(\alpha)|\leq\begin{cases}\frac{mp\delta_1}{(m^2+1)(m^{a+2}-1)}\,|\alpha|^a, & a>-2\\ \frac{mp\delta_1}{(m^2+1)(m^{a-2}-1)}\,|\alpha|^a, & a>2\end{cases}$$

for all $\alpha\in\mathbb{R}^\star$.

Proof. The required result is obtained by the application of Theorem 3.1 with

$$\phi(\alpha_1,\ldots,\alpha_p)=\delta_1\sum_{k=1}^{p}|\alpha_k|^a$$

for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$ and then $L=\frac{1}{m^{a+2}}$ and $L=\frac{1}{m^{a-2}}$, respectively. □

Corollary 3.4. *Let there exist a real number $a>-2$ or $a>2$ and $\delta_2\geq 0$ such that*

$$|D_p r(\alpha_1,\alpha_2,\ldots,\alpha_p)|\leq \delta_2\prod_{k=1}^{p}|\alpha_k|^{a/p}$$

for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$. Then, there exists a solution $G:\mathbb{R}^\star\longrightarrow\mathbb{R}$ of (1.3) which is unique and

$$|r(\alpha)-G(\alpha)|\leq\begin{cases}\frac{m\delta_2}{(m^2+1)(m^{a+2}-1)}|\alpha|^a, & a>-2\\ \frac{m\delta_2}{(m^2+1)(m^{a-2}-1)}|\alpha|^a, & a>2\end{cases}$$

for all $\alpha\in\mathbb{R}^\star$.

Proof. The desired result is acquired by the similar arguments as in Theorem 3.1 with $\phi(\alpha_1,\ldots,\alpha_p)=\delta_2\prod_{k=1}^{p}|\alpha_k|^{a/p}$, for all $\alpha_1,\ldots,\alpha_p\in\mathbb{R}^\star$ and then $L=\frac{1}{m^{a+2}}$ and $L=\frac{1}{m^{a-2}}$, respectively. □

4. Non-Archimedean stabilities of equation (1.3)

In this section, we assume that $\mathbb{U}$ and $\mathbb{V}$ are a non-Archimedean field and a complete non-Archimedean field, respectively. Let us denote $\mathbb{U}^*$ to represent $\mathbb{U}\backslash\{0\}$. We obtain the non-Archimedean stabilities of equation (1.3) associated with different upper bounds as a general control function, a small positive constant, a sum of powers of norms and a product of powers of norms.

Theorem 4.1. *Let $\phi:\underbrace{\mathbb{U}^\star\times\mathbb{U}^\star\times\cdots\times\mathbb{U}^\star}_{p-times}\longrightarrow\mathbb{V}$ be a given function. Suppose a mapping $r:\mathbb{U}^\star\longrightarrow\mathbb{V}$ satisfies the ensuing functional inequality*

$$|D_p r(\alpha_1,\ldots,\alpha_p)|\leq\phi(\alpha_1,\ldots,\alpha_p) \tag{4.1}$$

for all $\alpha_k\in\mathbb{U}^\star$, $k=1,\ldots,p$. If $0<L<1$ and the mapping φ has the property that

$$\varphi\left(\frac{\alpha_1}{p},\ldots,\frac{\alpha_p}{p}\right)\leq\frac{m}{m^2+1}L\varphi(\alpha_1,\ldots,\alpha_p) \tag{4.2}$$

for all $\alpha_k\in\mathbb{U}^\star$, $k=1,\ldots,p$, then there exists a unique mapping $G:\mathbb{U}^\star\longrightarrow\mathbb{V}$ satisfying equation (1.3) and

$$|r(\alpha)-G(\alpha)|\leq\frac{mL}{(m^2+1)(1-L)}\Phi(\alpha) \tag{4.3}$$

where $\Phi(\alpha)=\phi(\underbrace{\alpha,\ldots,\alpha}_{p-times})$ for all $\alpha\in\mathbb{U}^\star$, where $m=3p-2$.

Proof. Let us define a set $\mathcal{D}$ as follows: $\mathcal{D}=\{u|u:\mathbb{U}^*\longrightarrow\mathbb{V}\}$. Let η be a metric equipped with $\mathcal{D}$ and define

$$\eta(u,v)=\inf\{\beta>0:|u(\alpha)-v(\alpha)|\leq\beta\varphi(\underbrace{\alpha,\ldots,\alpha}_{p-\text{times}}),u,v\in\mathcal{D},\ \text{for all }\alpha\in\mathbb{U}^\star\}.$$

From the above definite of η, we notice that η is a a complete generalized non-Archimedean complete metric on $\mathcal{D}$. Let $\Omega : \mathcal{D} \longrightarrow \mathcal{D}$ be a mapping defined by

$$\Omega(u_k)(\alpha) = \frac{m^2+1}{m} u\left(\frac{\alpha}{p}\right) - u\left(\frac{\alpha}{p^2}\right)$$

for all $\alpha \in \mathbb{U}^\star$ and $u \in \mathcal{D}$. Then Ω is strictly contractive on $\mathcal{D}$, in fact if

$$|u(\alpha) - v(\alpha)| \leq \beta\varphi(\underbrace{\alpha, \ldots, \alpha}_{p-\text{times}}),$$

where $\alpha \in \mathbb{U}^\star$. Then using (4.2), we obtain

$$\begin{aligned}
|\Omega(u)(\alpha) - \Omega(v)(\alpha)| &= \left|\frac{m^2+1}{m} u\left(\frac{\alpha}{p}\right) - u\left(\frac{\alpha}{p^2}\right) - \frac{m^2+1}{m} v\left(\frac{\alpha}{p}\right) + v\left(\frac{\alpha}{p^2}\right)\right| \\
&\leq \frac{m^2+1}{m} \beta\varphi\left(\underbrace{\frac{\alpha}{p}, \ldots, \frac{\alpha}{p}}_{p-\text{times}}\right) \\
&\leq \beta L\varphi(\underbrace{\alpha, \ldots, \alpha}_{(m-1)\ \text{times}}) \quad (\alpha \in \mathbb{U}^\star).
\end{aligned}$$

From the above inequality, we find that $(\Omega(u), \Omega(v)) \leq L\eta(u, v)$, where $u, v \in \mathcal{D}$. As a consequence, we notice that the mapping η is strictly contractive with Lipschitiz constant L. Now, putting $\alpha_k = \frac{\alpha}{p}$, for all $k = 1, \ldots, p$, we have

$$\begin{aligned}
|\Omega(r)(\alpha) - r(\alpha)| &= \left|\frac{m^2+1}{m} r\left(\frac{\alpha}{p}\right) - r\left(\frac{\alpha}{p^2}\right) - r(\alpha)\right| \\
&\leq \varphi\left(\underbrace{\frac{\alpha}{p}, \ldots, \frac{\alpha}{p}}_{p-\text{times}}\right) \leq \left|\frac{m}{m^2+1}\right| L\varphi(\underbrace{\alpha, \ldots, \alpha}_{p-\text{times}}) \quad (\alpha \in \mathbb{U}^\star).
\end{aligned}$$

This shows that $\eta(\Omega(r), r) \leq L\left|\frac{m}{m^2+1}\right|$. In view of alternative fixed point theorem in non-Archimedean space, Ω has a unique fixed point $G : \mathbb{U}^\star \longrightarrow \mathbb{V}$ in the set $H = \{h \in \mathcal{D} : \eta(r, h) < \infty\}$ and for each $\alpha \in \mathbb{U}^\star$,

$$\begin{aligned}
G(\alpha) &= \lim_{n\to\infty} \Omega^n r(\alpha) \\
&= \lim_{n\to\infty} \left\{ \frac{(m^2+1)^{n-2}(m^4+m^2+1)}{m^n} r\left(\frac{\alpha}{p^n}\right) - \frac{(m^2+1)^{n-1}}{m^{n-1}} r\left(\frac{\alpha}{p^{n+1}}\right) \right. \\
&\qquad \left. - \sum_{k=2}^{n-1} \left(\frac{m^2+1}{m}\right)^{k-2} r\left(\frac{\alpha}{p^k}\right) \right\} (\alpha \in \mathbb{U}^\star).
\end{aligned}$$

Therefore, for all $\alpha_k \in \mathbb{U}^*$, $k = 1, \dots, p$,

$$\begin{aligned}|D_pG(\alpha_1, \dots, \alpha_p)| &= \lim_{n\to\infty} \frac{(m^2+1)^{n-2}(m^4+m^2+1)}{m^n}\left|D_p\left(\frac{\alpha_1}{p^n}, \dots, \frac{\alpha_p}{p^n}\right)\right| \\ &\quad - \lim_{n\to\infty} \frac{(m^2+1)^{n-1}}{m^{n-1}}\left|D_p\left(\frac{\alpha_1}{p^{n+1}}, \dots, \frac{\alpha_p}{p^{n+1}}\right)\right| \\ &\quad - \lim_{n\to\infty}\left\{\sum_{k=2}^{n-1}\left(\frac{m^2+1}{m}\right)^{k-2}\right\}\left|D_p\left(\frac{\alpha_1}{p^k}, \dots, \frac{\alpha_p}{p^k}\right)\right| \\ &\leq \lim_{n\to\infty}\left(\frac{m^2+1}{m}\right)^n \varphi\left(\frac{\alpha_1}{p^n}, \dots, \frac{\alpha_p}{p^n}\right) \\ &\leq \lim_{n\to\infty} L^n\varphi(\alpha_1, \dots, \alpha_p) = 0.\end{aligned}$$

Hence the mapping G is a solution of equation (1.3). Utilizing the result of fixed point theorem, we obtain that $\eta(r, G) \leq \eta(r, \Omega(r))$, which implies that,

$$|r(\alpha) - G(\alpha)| \leq \left|\frac{m}{m^2+1}\right| |L\varphi(\underbrace{\alpha, \dots, \alpha}_{p-\text{ times}}),\ (\alpha \in \mathbb{U}^\star).$$

Let $G' : \mathbb{U}^* \longrightarrow \mathbb{V}$ be another mapping which satisfies (4.3), then G' is a fixed point of Ω in $\mathfrak{D}$. Hence, by fixed point theorem, Ω has only one fixed point in H. This completes the proof. □

The following corollaries are direct consequences of Theorem 4.1. In the following corollaries, we assume that $\left|\frac{1}{m}\right| < 1$ for a non-Archimedean field $\mathbb{U}$.

Corollary 4.2. *Let μ (independent of $\alpha_1, \dots, \alpha_p$)$\geq 0$ be a constant, exists for a mapping $r : \mathbb{U}^\star \longrightarrow \mathbb{V}$ such that the inequality $|D_pr(\alpha_1, \dots, \alpha_p)| \leq \frac{\mu}{m}$, for all $\alpha_k \in \mathbb{U}^\star$, $k = 1, \dots, p$. Then there exists a unique solution $G : \mathbb{U}^\star \longrightarrow \mathbb{V}$ of equation (1.3) and $|r(\alpha) - G(\alpha)| \leq \left|\frac{m}{(m^2+1)(m-1)}\right| \mu$, for all $\alpha \in \mathbb{U}^\star$.*

Proof. Considering $\varphi(\alpha_1, \dots, \alpha_p) = \frac{\mu}{m}$ and then choosing $L = \frac{1}{m}$ in Theorem 4.1, we get the desired result. □

Corollary 4.3. *Let $a > -2$ or $a > 2$ and $\delta_1 \geq 0$ be real numbers, exist for a mapping $r : \mathbb{U}^\star \longrightarrow \mathbb{V}$ such that $|D_pr(\alpha_1, \dots, \alpha_p)| \leq \delta_1 \sum_{k=1}^p |\alpha_k|^a$, for all $\alpha_1, \dots \alpha_p \in \mathbb{U}^\star$. Then there exists a unique solution $G : \mathbb{U}^\star \longrightarrow \mathbb{V}$ of equation (1.3) and*

$$|r(\alpha) - G(\alpha)| \leq \begin{cases} \left|\frac{mp\delta_1}{(m^2+1)(m^{a+2}-1)}\right| |\alpha|^a, & a > -2 \\ \left|\frac{mp\delta_1}{(m^2+1)(m^{a-2}-1)}\right| |\alpha|^a, & a > 2 \end{cases}$$

for all $\alpha \in \mathbb{U}^\star$.

Proof. The proof follows by assigning $\varphi(\alpha_1, \dots, \alpha_p) = \delta_1 \sum_{k=1}^p |\alpha_k|^a$ in Theorem 4.1 and then taking $L = \frac{1}{m^{a+2}}$, $a > -2$ and $L = \frac{1}{m^{a-2}}$, $a < 2$, respectively. □

Corollary 4.4. *Let there exist a real number $a > -2$ or $a > 2$ and $\delta_2 \geq 0$ such that*

$$|D_pr(\alpha_1, \alpha_2, \dots, \alpha_p)| \leq \delta_2 \prod_{k=1}^p |\alpha_k|^{a/p}$$

for all $\alpha_1, \dots, \alpha_p \in \mathbb{R}^\star$. Then, there exists a solution $G : \mathbb{R}^\star \longrightarrow \mathbb{R}$ of equation (1.3) which is unique and

$$|r(\alpha) - G(\alpha)| \leq \begin{cases} \frac{m\delta_2}{(m^2+1)(m^{a+2}-1)} |\alpha|^a, & a > -2 \\ \frac{m\delta_2}{(m^2+1)(m^{a-2}-1)} |\alpha|^a, & a > 2 \end{cases}$$

for all $\alpha \in \mathbb{R}^\star$.

Proof. The desired result is acquired by the similar arguments as in Theorem 4.1 with $\phi(\alpha_1, \ldots, \alpha_p) = \delta_2 \prod_{k=1}^{p} |\alpha_k|^{a/p}$, for all $\alpha_1, \ldots, \alpha_p \in \mathbb{R}^\star$ and then $L = \frac{1}{m^{a+2}}$ and $L = \frac{1}{m^{a-2}}$, respectively. □

5. Counter-examples to invalidate the stability results of equation (1.3)

Motivated through the excellent illustration proved in [13], in this section, we demonstrate an apt example to prove the failure of stability of equation (1.3) for the singularity when $a = 2$ in Corollary 3.3.

Theorem 5.1. *Consider a mapping $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ defined as follows*

$$\phi(\alpha) = \begin{cases} k\alpha^2, & \text{if } 0 < \alpha < 1 \\ k, & \text{otherwise} \end{cases}$$

where $k > 0$ is a constant and a mapping $r : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$r(\alpha) = \sum_{n=0}^{\infty} 4^{-n} \phi(2^n \alpha) \qquad \text{for all} \quad \alpha \in \mathbb{R}.$$

Then the mapping r satisfies the following inequality

$$|D_p r(\alpha_1, \alpha_2, \ldots, \alpha_p)| \leq 32k \sum_{k=1}^{p} |\alpha_k|^2 \tag{5.1}$$

for every $\alpha_1, \alpha_2, \ldots, \alpha_p \in \mathbb{R}$. Then, a quadratic mapping $G : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\rho > 0$ do not exist so that

$$|r(\alpha) - G(\alpha)| \leq \rho |\alpha|^2 \qquad \text{for every} \quad \alpha \in \mathbb{R}. \tag{5.2}$$

Proof. Firstly, let us show that r is bounded. By virtue of the mapping r's definition, we obtain $|r(\alpha)| = \sum_{n=0}^{\infty} 4^{-n}\phi(2^n\alpha) \leq \sum_{n=0}^{\infty} \frac{k}{4^n} = \frac{4k}{3}$ which indicates that 4 is bounded. Let us show that r satisfies the inequality (5.1). Suppose $\sum_{k=1}^{p} |\alpha_k|^2 \geq 1$. Then the left hand side of (5.1) is less than $32k$. On the contrary, suppose that $0 < \sum_{k=1}^{p} |\alpha_k|^2 < 1$. Then there exists a positive integer m such that

$$\frac{1}{16^{m+1}} \leq \sum_{k=1}^{p} |\alpha_k|^2 < \frac{1}{16^m} \tag{5.3}$$

which in turn gives rise to $16^m \alpha_k^2 < 1$, for $k = 1, 2, \ldots, p$ and hence, we have

$$4^{m-1}(\alpha_k) < \frac{1}{4} < 1$$

for $k = 1, 2, \ldots, p$. Hence, for every $n = 0, 1, \ldots, m-1$, we find

$$4^n \alpha_k < 1$$

for $k = 1, 2, \ldots, p$ and $D_p r(4^n\alpha_1, 4^n\alpha_2, \ldots, 4^n\alpha_p) = 0$ for $n = 0, 1, 2, \ldots, m-1$. Applying the definition of r and (5.3), we obtain

$$\begin{aligned} |D_p r(\alpha_1, \alpha_2, \ldots, \alpha_p)| &\leq \sum_{n=m}^{\infty} \frac{k}{4^n} + \sum_{n=m}^{\infty} \frac{k}{4^n} + \sum_{n=m}^{\infty} \frac{k}{4^n} + 2\sum_{n=m}^{\infty} \frac{k}{4^n} + \sum_{n=m}^{\infty} \frac{k}{4^n} \\ &\leq 6k \sum_{n=m}^{\infty} \frac{1}{4^n} \leq \frac{8k}{4^m} = \frac{32k}{4^{m+1}} \leq 32k \sum_{k=1}^{p} |\alpha_k|^2 \end{aligned}$$

for every $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$, which indicates that r satisfies (5.1) for all $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$. Next is to prove that (1.3) fails to be stable for $a = 2$ in Corollary 3.3. Suppose a quadratic mapping $G : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $\rho > 0$ satisfy (5.2). Since r is bounded and using the result of Corollary 3.3, the mapping $G(\alpha)$ must be a quadratic mapping and $G(\alpha) = c\alpha^2$ for any $\alpha \in \mathbb{R}$. Therefore, we arrive at

$$|r(\alpha)| \leq (\rho + |c|)\, |\alpha|^2 . \tag{5.4}$$

At the same time, we can choose a positive integer m with $m\mu > \rho + |c|$. Suppose $0 < \alpha < 4^{n-1}$, then $0 < 4^{-n}\alpha < 1$ for every $n = 0, 1, \dots, m-1$. Hence, for this α, we obtain

$$r(\alpha) = \sum_{n=0}^{\infty} 4^{-n}\phi(2^n\alpha) \geq \sum_{n=0}^{m-1} 4^{-n}(2^n\alpha)^2 = m\mu\alpha^2 > (\rho + |c|)\, \alpha^2$$

which is a contradiction to (5.4). Hence, (1.3) fails to be stable for $a = 2$ in Corollary 3.3. □

Similar counter-examples can be illustrated for the non-stability of equation (1.3) for the case $a = 2$ in Corollary 4.3 and for the case $a = -2$ in Corollaries 3.3 and 4.3.

6. Conclusion

So far various forms of quadratic and multiplicative inverse quadratic functional equations are dealt in this research field to investigate their stability results. This is our foremost attempt to focus on an advanced functional equation with quadratic function and multiplicative inverse quadratic as solutions to investigate its stability results using fixed point method in the setting of non-zero real numbers and non-Archimedean fields. At the end of the study, suitable counter-examples are presented to disprove the stability results of equation (1.3) when singular cases arise.

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