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Analyticity for the Fractional Navier-Stokes Equations in Critical Fourier-Besov-Morrey Spaces with Variable Exponents

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abstract: In this paper, by using the Littlewood-Paley theory and the Fourier localization argument, we obtained the analyticity of the solution to the fractional Navier-Stokes equations in variable exponents Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$, when the initial data are small.

Key Words: Fourier-Besov-Morrey spaces with variable exponents, fractional Navier-Stokes equations, analyticity.

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1. Introduction

The incompressible fractional Navier-Stokes equations are given by

$$\begin{cases} u_t + (-\Delta)^{\alpha}u + (u.\nabla)u + \nabla p = 0 & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \nabla \cdot u = 0, & \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Where $\frac{1}{2} < \alpha \leq 1$, $(-\Delta)^{\alpha}$ is the Fourier multiplier with symbol $|\xi|^{2\alpha}$, the scalar function p denotes the pressure, $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ denotes the unknown velocity vector and $u_0(x)$ is a divergence free vector.

Regular solutions of several equations, such as magneto-hydrodynamics equations, Generalized Porous Medium equations and Navier-Stokes equations are analytic; see, e.g., [2,11,28,30]. An important physical interpretation has been given to the space analyticity radius in fluids dynamics. Foias and Temam [18] proposed an effective method for estimating the analyticity radius for the Navier-Stokes equations by using Gevrey norms. The method is applicable to many other equations. While, this method is not suitable for L^p initial data due to the use of Fourier series expansions. In [35] Zoran and Igor introduced a new method which bridges this difficulty and offers a simple estimate of the analyticity radius in terms of the L^p norm of the initial data. Recently, Azanzal, Allalou and Abbassi [9] obtained the existence and uniqueness of analytic solution for the generalized Navier-Stokes equations in critical Fourier-Besov-Morrey spaces. Inspired by Xiao [32] in the classical case $\alpha = 1$, Li and Zhai [27] studied the problem (1.1) in various critical $Q-$type sapces for $\alpha \in (\frac{1}{2}, 1)$, in addition Yu and Zhai [33] obtained the well-posedness for (1.1) in the critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. Deng and Yao [16] established the well-posedness of (1.1) in Triebel-Lizorkin space $F^{-\alpha}_{3/(\alpha-1),2}$ and ill-posedness in $F^{-\alpha}_{3/(\alpha-1),q}$ $(q > 2)$ when $\alpha \in (1, \frac{5}{4})$. The Cauchy problem (1.1) has been studied by El Baraka and Tomlilin in critical Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\kappa}{p}}_{p,\kappa,q}(\mathbb{R}^n)$ and they obtained the global well-posedness when the initial data are small, recently, M. Z. Abidin and J. Chen [3] studied the problem (1.1) in Fourier-Besov-Morrey spaces with

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variable exponents $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$ and they established the global well-posedness result for (1.1) with small initial data belonging to $\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$.

For the classical case ($\alpha = 1$), the existence of mild solutions and the regularity were established locally in time and global for small initial data in various functional spaces. Leray [26] proved the existence of global weak solution. Fujita and Kato [22] transformed the classical incompressible Navier-Stokes equations into an integral equation and proved that it is locally well-posed in $H^s(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$ and globally well-posed in $H^{\frac{1}{2}}(\mathbb{R}^3)$ with small initial data. Cannone [15] studied the local well-posedness in $\dot{\mathcal{B}}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$. Koch and Tataru [24] established the well-posedness in BMO^{-1}. Cui [10] obtained the global existence in Besov type space $\mathcal{B}^{-1,\tau}_{\infty,q}(\mathbb{R}^3)$ for $1 \leq q \leq \infty$ and $\tau \geq 1 - min\left\{\frac{1}{q'}, \frac{1}{q}\right\}$. The study of classical Navier-stokes equations was continued by many authors in Fourier-Besov, Fourier-Besov-Morrey and different spaces, see [19,25,13,23]. On the other hand, the ill-posedness has been proved by Bourgain and Pavlovi [12] in the Besov space $\dot{\mathcal{B}}^{-1}_{\infty,\infty}$.

Before stating the main result of this paper, we first recall the definitions of Morrey spaces, Besov-Morrey spaces and Fourier-Besov-Morrey spaces and present some properties about these spaces. Our result on Gevrey class regularity are stated in Section 3 and in the same section we obtain the needed linear and nonlinear estimates and we prove the analyticity result. Throughout the paper, C will denote constants which can be different at different places. The notation $x \lesssim y$ means that there exists a constant $c > 0$ such that $x \leq cy$ and p' is the conjugate of p satisfiying $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

2. Preliminaries

In this section, we introduce some basic properties of the Littlewood-Paley theory and variable exponents Fourier-Besov-Morrey spaces and we recall an abstract fixed point lemma which will be useful to prove our main result.
Consider $\varphi \in S(\mathbb{R}^n)$ a radial positive function such that $0 \leq \varphi \leq 1$, $\operatorname{supp}(\varphi) \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \varphi\left(2^{-j}\xi\right) = 1, \qquad \text{for all } \xi \neq 0.$$

We denote

$$\varphi_j(\xi) = \varphi\left(2^{-j}\xi\right), \quad \psi_j(\xi) = \sum_{k\leq j-1} \varphi_k(\xi),$$

$$h(x) = \mathcal{F}^{-1}\varphi(x), \quad g(x) = \mathcal{F}^{-1}\psi(x),$$

$$\Delta_j f := \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}(f)\right) = 2^{nj}\int_{\mathbb{R}^n} h\left(2^j y\right) f(x-y)dy$$

and

$$S_j f := \sum_{k\leq j-1} \Delta_k f = \mathcal{F}^{-1}\left(\psi_j \mathcal{F}(f)\right) = 2^{nj}\int_{\mathbb{R}^n} g\left(2^j y\right) f(x-y)dy,$$

where $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.

First, we present the definition of Lebesgue space with variable exponent.

Definition 2.1. *([6]) Let $\mathcal{P}_0 = \mathcal{P}_0(\mathbb{R}^n)$ denotes the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that*

$$0 < p_- = \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x)\ ,\ \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x)\ = p_+ < \infty.$$

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{f : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^n} |f(x)|^{p(x)}dx < \infty\right\}.$$

It is a Banach space, equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\mu > 0 : \int_{\mathbb{R}^n} (\frac{|f(x)|}{\mu})^{p(x)} dx \leq 1\right\}.$$

Since, the space $L^{p(\cdot)}$ and L^p does not have the same properties. So, we assume the following standard conditions to ensure that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

Definition 2.2. *[6] Let $g : \mathbb{R}^n \to \mathbb{R}$.*
(1) We say that g is locally log-Hölder continuous, $g \in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists a constant $c_{\log} > 0$ with

$$|g(x) - g(y)| \leq \frac{c_{\log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \textit{for all} \quad x, y \in \mathbb{R}^n \textit{and } x \neq y.$$

(2) We say that g is globally log-Hölder continuous, $g \in C^{\log}(\mathbb{R}^n)$, if $g \in C^{\log}_{loc}(\mathbb{R}^n)$ and there exists a $g_\infty \in \mathbb{R}$ and a constant $c_\infty > 0$ with

$$|g(x) - g_\infty| \leq \frac{c_\infty}{\log(e + |x|)} \quad for \;\; all \;\; x \in \mathbb{R}^n.$$

(3) We write $g \in \mathcal{P}^{\log}_0(\mathbb{R}^n)$ if $0 < g^- \leq g(x) \leq g^+ \leq \infty$ with $1/g \in C^{\log}(\mathbb{R}^n)$.

We now define Morrey spaces $\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}$ with variable exponents.

Definition 2.3. *([5]) Let $p(\cdot)$, $\kappa(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < p_- \leq p(x) \leq \kappa(x) \leq \infty$, the Morrey space with variable exponents $\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)} := \mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions on $\mathbb{R}^n$ such that*

$$\|f\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r>0} \|r^{\frac{n}{\kappa(x)} - \frac{n}{p(x)}} f \chi_{B(x_0,r)}\|_{L^{p(\cdot)}} < \infty,$$

where $B(x_0, r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 with radius $r > 0$.
According to the definition of the $L^{p(\cdot)}$-norm, $\|f\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}}$ also has the following form

$$\|f\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r>0} \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(r^{\frac{n}{\kappa(x)} - \frac{n}{p(x)}} \frac{f}{\lambda} \chi_{B(x_0,r)}) \leq 1\right\}.$$

The following lemma will help us to obtain the spatial analyticity to Eq. (1.1).

Lemma 2.4. *([30]) Let X be a Banach space with norm $\|\cdot\|$ and $B : X \longrightarrow X$ a bilinear operator, such that for any $x_1, x_2 \in X$, $\|B(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|$, then for any $y \in X$ such that $\|y\| < \dfrac{1}{4\eta}$ the equation $x = y + B(x, x)$ has a solution $x \in X$. In particular, the solution is such that $\|x\| \leq 2\|y\|$ and it is the only one such that $\|x\| < \dfrac{1}{2\eta}$.*

Now, we define the mixed Morrey-squence spaces.

Definition 2.5. *([5]) Let $p(\cdot)$, $q(\cdot)$, $\kappa(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $p(\cdot) \leq \kappa(\cdot)$, the mixed Morrey-sequence space $l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})$ includes all sequences $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that*
$\rho_{l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})}(\mu\{f_j\}_{j\in\mathbb{Z}}) < \infty$ for some $\mu > 0$. For $\{f_j\}_j \in l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})$, we define

$$\|\{f_j\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})} := \inf\left\{\mu > 0, \; \rho_{l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})}\left(\{\frac{f_j}{\mu}\}_{j\in\mathbb{Z}}\right) \leq 1\right\} < \infty,$$

where $\rho_{l^{q(\cdot)}(\mathcal{M}_{p(\cdot)}^{\kappa(\cdot)})}(\{f_j\}_{j\in\mathbb{Z}}) := \sum_{j\in Z} \inf\left\{\gamma > 0,\ \int_{\mathbb{R}^n}\left(\frac{\left|r^{\frac{n}{\kappa(x)}-\frac{n}{p(x)}} f_j\chi_{B(x_0,r)}\right|}{\gamma^{\frac{1}{q(x)}}}\right)^{p(x)} dx \le 1\right\}.$

Notice that if $q_+ < \infty$ *or* $q_+ < \infty$ *and* $p(x) \ge q(x)$ *, then*

$$\rho_{l^{q(\cdot)}(\mathcal{M}_{p(\cdot)}^{\kappa(\cdot)})}(\{f_i\}_{i\in\mathbb{N}_0}) = \sum_{i\in\mathbb{N}_0} \sup_{x_0\in\mathbb{R}^n, r>0} \|(|r^{\frac{n}{\kappa(x)}-\frac{n}{p(x)}} f_i|\chi_{B(x_0,r)})^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Now, we recall the definition of Fourier-Besov spaces with variable exponents.

Definition 2.6. *([4]) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $p(\cdot),\ q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ *with* $0 < p_- \le p(\cdot) \le \infty$. *The homogeneous Fourier-Besov space with variable exponent* $\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ *is defined by the set of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathcal{F}\dot{B}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j\hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ *is the dual space of*

$$\mathcal{Z}(\mathbb{R}^n) = \left\{f \in S(\mathbb{R}^n) : (D^\beta f)(0) = 0, \forall\beta \in \mathbb{N}^n\right\}.$$

Definition 2.7. *([5]) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $p(\cdot),\ q(\cdot),\ \kappa(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)\cap C^{log}(\mathbb{R}^n)$ *with* $0 < p_- \le p(x) \le \kappa(x) \le \infty$. *The homogeneous Besov-Morrey space with variable exponent* $\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q(\cdot)}$ *is defined by the set of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\Delta_j f\}_{j\in\mathbb{Z}}\|_{l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})} < \infty.$$

Below, we present the Fourier-Besov-Morrey spaces with variable exponents.

Definition 2.8. *([3]) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $p(\cdot),\ q(\cdot),\ \kappa(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)\cap C^{log}(\mathbb{R}^n)$ *with* $0 < p_- \le p(\cdot) \le \kappa(\cdot) \le \infty$. *The homogeneous Fourier-Besov-Morrey space with variable exponent* $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q(\cdot)}$ *is defined by the set of all* $f \in \mathcal{Z}'(\mathbb{R}^n)$ *such that*

$$\|f\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\varphi_j\hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})} < \infty.$$

Definition 2.9. *([3]) Let* $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot),\ q(\cdot),\ \kappa(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$, *such that* $p(\cdot) \le \kappa(\cdot)$, $T \in [0,\infty)$ *and* $1 \le q,\ \rho \le \infty$. *We define the Chemin-Lerner type homogeneous Fourier-Besov-Morrey space with variable exponents* $\mathcal{L}^\rho([0,T); \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q})$ *by*

$$\mathcal{L}^\rho\ ([0,T); \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q}) = \left\{f \in \mathcal{Z}'(\mathbb{R}^n);\ \|f\|_{\mathcal{L}^\rho([0,T);\cdot\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q})} < \infty\right\},$$

with the norm

$$\|f\|_{\mathcal{L}^\rho([0,T);\cdot\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{p(\cdot),\kappa(\cdot),q})} = \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\varphi_j\hat{f}\|^q_{L^\rho([0,T);\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)})}\right)^{\frac{1}{q}}.$$

Proposition 2.10. *([3]) For Morrey spaces with variable exponents, the following inclusions are established.*

(1) (Hölder inequality)([3]) Let $p(\cdot),\ p_1(\cdot),\ p_2(\cdot),\ \kappa(\cdot),\ \kappa_1(\cdot),\ \kappa_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, *such that* $p(x) \le \kappa(x)$, $p_1(x) \le \kappa_1(x)$, $p_2(x) \le \kappa_2(x)$, $\dfrac{1}{p(x)} = \dfrac{1}{p_1(x)} + \dfrac{1}{p_2(x)}$ *and* $\dfrac{1}{\kappa(x)} = \dfrac{1}{\kappa_1(x)} + \dfrac{1}{\kappa_2(x)}$. *Then there exists a constant* C *depending only on* p_- *and* p_+ *such that*

$$\|fg\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}} \le C\|f\|_{\mathcal{M}^{\kappa_1(\cdot)}_{p_1(\cdot)}}\|g\|_{\mathcal{M}^{\kappa_2(\cdot)}_{p_2(\cdot)}},$$

holds for every $f \in \mathcal{M}^{\kappa_1(\cdot)}_{p_1(\cdot)}$ *and* $g \in \mathcal{M}^{\kappa_2(\cdot)}_{p_2(\cdot)}$.

(2) ([3]) Let $p_0(\cdot),\ p_1(\cdot),\ \kappa_0(\cdot),\ \kappa_1(\cdot),\ q(\cdot) \in \mathcal{P}_0,$ *and* $s_0(\cdot),\ s_1(\cdot) \in L^\infty \cap C^{log}(\mathbb{R}^n)$ *with* $s_0(\cdot) \geq s_1(\cdot)$. *If* $\frac{1}{q}$ *and* $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$ *are locally log-Hölder continuous, then*

$$\mathcal{N}^{s_0(\cdot)}_{p_0(\cdot),\kappa_0(\cdot),q} \hookrightarrow \mathcal{N}^{s_1(\cdot)}_{p_1(\cdot),\kappa_1(\cdot),q}.$$

(3) ([5]) For $p(\cdot) \in C^{log}(\mathbb{R}^n)$ *and* $\theta \in L^1(\mathbb{R}^n)$, *assume* $\Psi(x) = \sup_{y \notin B(0,|x|)} |\theta(y)|$ *is integrable. Then*

$$\|f * \theta_\epsilon\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} \|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f \in \mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, *where* $\theta_\epsilon = \frac{1}{\epsilon^n}\theta(\frac{1}{\epsilon})$ *and* C *depends only on* n.

We wil use the following proposition to prove the main result.

Proposition 2.11. *([3]) Let* $I = (0,T],\ s > 0,\ 1 \leq \gamma,\ \rho,\ \rho_1,\ \rho_2,\ q \leq \infty,\quad p(\cdot),\ \kappa(\cdot),\ \kappa(\cdot) \in C^{\log} \cap \mathcal{P}_0(\mathbb{R}^n),\ \frac{1}{\kappa(\cdot)} = \frac{1}{\kappa_1(\cdot)} + \frac{1}{\kappa_2(\cdot)},\ \frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$ *and* $\frac{1}{\rho} = \frac{1}{r(\cdot)} + \frac{1}{p(\cdot)}$. *Then we have*

$$\begin{aligned}\|ab\|_{\mathcal{L}^\gamma(I,\dot{\mathcal{N}}^s_{\rho,\kappa(\cdot),q})} \lesssim &\|a\|_{\mathcal{L}^{\gamma_1}(I,\mathcal{M}^{\kappa_1(\cdot)}_{r(\cdot)})}\|b\|_{\mathcal{L}^{\gamma_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),\kappa_2(\cdot),q})}\\ &+ \|b\|_{\mathcal{L}^{\gamma_1}(I,\mathcal{M}^{\kappa_1(\cdot)}_{r(\cdot)})}\|a\|_{\mathcal{L}^{\gamma_2}(I,\dot{\mathcal{N}}^s_{p(\cdot),\kappa_2(\cdot),q})}.\end{aligned}$$

Our purpose is to prove the analyticity of the solution obtained in the following proposition.

Proposition 2.12. *([3]) Let* $p(\cdot), \kappa(\cdot) \in C^{log}(\mathbb{R}^n) \cap P_0(\mathbb{R}^n)$ *such that* $p(\cdot) \leq \kappa(\cdot) < \infty$, $\frac{1}{2} < \alpha \leq 1$, $2 \leq p(\cdot) \leq \dfrac{6}{5-4\alpha}$, $1 \leq \gamma < \infty$ *and* $1 \leq q < \dfrac{3}{2\alpha - 1}$. *There exists a small* ε *such that if* $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$ *satisfying* $\nabla u_0 = 0$ *with* $\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)} < \varepsilon$, *then the problem (1.1) admits a unique small global solution* u *in the class*

$$u(t) \in \mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),\kappa(\cdot),q}) \cap \mathcal{L}^\gamma([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q}).$$

Moreover, let $p_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $s_1(\cdot) = \frac{2}{\gamma} + \frac{3}{p_1(\cdot)} + 4 - 2\alpha$ *and* $s_1(\cdot) \in C^{\log}(\mathbb{R}^n)$, *if there exists* $c > 0$ *such that* $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, *then we obtain that* $u \in \mathcal{L}^\gamma([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})$.

3. Gevrey class regularity

The analyticity of the solution is also an important subject studied by a lot of researchers, particularly with regard to the Navier-Stokes equations. In this section, we will prove the analyticity for (1.1) in the critical Fourier-Besov-Morrey spaces with variable exponents and the following theorem is our main result.

Theorem 3.1. *Let* $p(\cdot), \kappa(\cdot) \in C^{log}(\mathbb{R}^n) \cap P_0(\mathbb{R}^n)$ *such that* $p(\cdot) \leq \kappa(\cdot) < \infty$, $\frac{1}{2} < \alpha \leq 1$, $2 \leq p(\cdot) \leq \dfrac{6}{5-4\alpha}$, $1 \leq \gamma < \infty$, $1 \leq q < \dfrac{3}{2\alpha-1}$. *There exists a small* ε_0 *such that if* $u_0 \in \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$ *satisfying* $\nabla u_0 = 0$ *with* $\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)} < \varepsilon_0$, *then the solution obtained in Proposition 2.12 is analytic in the sense that*

$$\|e^{\sqrt{t}|D|^\alpha}u\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}) \cap \mathcal{L}^\gamma([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})} \leq \|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}.$$

Moreover, let $p_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $s_1(\cdot) = \frac{2}{\gamma} - \frac{3}{p_1(\cdot)} + 4 - 2\alpha$ *and* $s_1(\cdot) \in C^{\log}(\mathbb{R}^n)$, *if there exists* $c > 0$ *such that* $2 \leq p_1(\cdot) \leq c \leq p(\cdot)$, *then we obtain that* $e^{\sqrt{t}|D|^\alpha}u(t) \in \mathcal{L}^\gamma([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})$, *where* $e^{\sqrt{t}|D|^\alpha}$ *is a Fourier multiplier and* $e^{\sqrt{t}|\xi|^\alpha}$ *defines its symbol.*

Remark 3.2. *It is noted that the variable exponent Fourier-Besov-Morrey space* $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)$ *is invariant under the scaling of (1.1). In fact, if* $u(t,x)$ *is the solution of (1.1), then*

$$u_\lambda(t,x) = \lambda^{2\alpha-1}u(\lambda 2^\alpha t, \lambda x)$$

is also a solution of the same problem and

$$\|u(0,x)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)} \approx \|u_\lambda(0,x)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}(\mathbb{R}^3)}.$$

Proof. **(Proof of Theorem 3.1)** To prove the analyticity of the solution, we will use Lemma 2.4. We consider the Banach space

$$Y = \left\{\mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}) \cap \mathcal{L}^\gamma([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q}) \cap \mathcal{L}^\infty([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})\right\}.$$

We start with the integral equation

$$\begin{aligned} u(t,x) &= e^{-t(-\Delta)^\alpha}u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\nabla\cdot(u\otimes u)d\tau \\ &= H_\alpha(t)u_0 + B(u,u), \end{aligned} \tag{3.1}$$

where $\mathbb{P} = Id - \nabla\Delta^{-1}div$ is the Leray-Hopf projector, which is a pseudo differential operator of order 0. Put $U(t,x) = e^{\sqrt{t}|D|^\alpha}u(t,x)$. Using the integral equation (3.1), we get

$$\begin{aligned} U(t,x) &= e^{\sqrt{t}|D|^\alpha}H_\alpha(t)u_0 + e^{\sqrt{t}|D|^\alpha}B(u,u) \\ &= \tilde{H}_\alpha(t)u_0 + \tilde{B}(u,u). \end{aligned} \tag{3.2}$$

Our goal is to show that the mapping $\phi : \tilde{H}_\alpha(t)u_0 + \tilde{B}(u,u)$ admits a fixed point.
For the linear estimate, using the Fourier transform, multiplying by $2^{js_1}\varphi_j$ and taking $L^\gamma([0,\infty), M^{\kappa(\cdot)}_{p_1(\cdot)})$ -norm, we obtain

$$\begin{aligned} &\left\|2^{js_1(\cdot)}\varphi_j\widehat{H_\alpha(t)u_0}\right\|_{L^\gamma([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})} \\ &\lesssim \left\|2^{js_1(\cdot)}\varphi_j e^{-\frac{t}{2}|\xi|^{2\alpha}}e^{\sqrt{t}|\xi|^\alpha-\frac{t}{2}|\xi|^{2\alpha}}\hat{u}_0\right\|_{L^\gamma([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})} \\ &\lesssim \left\|2^{js_1(\cdot)}\varphi_j e^{-\frac{t}{2}|\xi|^{2\alpha}}\hat{u}_0\right\|_{L^\gamma([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}. \end{aligned}$$

Where we used the inequality $e^{\sqrt{t}|\xi|^\alpha-\frac{t}{2}|\xi|^{2\alpha}} = e^{-\frac{1}{2}}(\sqrt{t}\mid\xi\mid^\alpha-1)^2 + \frac{1}{2} \leq e^{\frac{1}{2}}$. Taking the ℓ^q-norm, and for $p_1(\cdot) \leq c \leq p(\cdot)$, we have

$$\begin{aligned} \|\tilde{H}_\alpha(t)u_0\|_{\mathcal{L}^\gamma([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})} &\leq \left\|\left\|2^{js_1(\cdot)}\varphi_j e^{-\frac{t}{2}|\xi|^{2\alpha}}\hat{u}_0\right\|_{\mathcal{L}^\gamma\left([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)}\right)}\right\|_{\ell^q} \\ &\lesssim \|\sum_{k=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{c})}\varphi_j\hat{u}_0\|_{\mathcal{M}^{\kappa(\cdot)}_c}\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2\alpha}{\gamma}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+k} \\ &\qquad e^{-\frac{t}{2}2^{2\alpha(j+k)}}\|_{L^\gamma([0,\infty),L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}})}\|_{\ell^q} \\ &\lesssim \left\|\sum_{k=0,\pm1}\|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\varphi_j\hat{u}_0\|_{\mathcal{M}^{\kappa(\cdot)}_{p(\cdot)}}\right\|_{\ell^q} \\ &\lesssim \|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha-\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}, \end{aligned}$$

where

$$\begin{aligned}&\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j(\frac{2}{\rho}+\frac{3}{c}-\frac{3}{p_1(\cdot)})}\varphi_{j+k}e^{-\frac{t}{2}2^{2\alpha(j+k)}}\right\|_{L^\gamma\left([0,\infty),L^{\frac{cp_1(\cdot)}{c-p_1(\cdot)}}\right)}\\&=\left\|r^{\frac{-3(c-p_1(\cdot))}{cp_1(\cdot)}}2^{j\frac{2}{\rho}}e^{-\frac{t}{2}2^{2\alpha(j+k)}}\right\|_{L^\gamma([0,\infty))}\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\mid\frac{\varphi_{j+k}2^{j(\frac{3}{c}-\frac{3}{p_1(x)})}}{\lambda}\mid^{\frac{cp_1(x)}{c-p_1(x)}}dx\le 1\right\}\\&\le\inf\left\{\lambda>0:\int_{\mathbb{R}^3}\mid\frac{\varphi_{j+k}}{\lambda}\mid^{\frac{cp_1(x)}{c-p_1(x)}}2^{-3j}dx\le 1\right\}\\&<\infty.\end{aligned}$$

Therefore,

$$\left\|\tilde{H_\alpha}(t)u_0\right\|_{\mathcal{L}^\gamma([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}.$$

Also, if $\rho=\infty$ and $p_1(\cdot)=p(\cdot)$, we obtain

$$\left\|\tilde{H_\alpha}(t)u_0\right\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q})}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}},$$

similarly, we get

$$\left\|\tilde{H_\alpha}(t)u_0\right\|_{\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}$$

and

$$\left\|\tilde{H_\alpha}(t)u_0\right\|_{\mathcal{L}^\gamma([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa,q})}\lesssim\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}.$$

Then,

$$\left\|\tilde{H_\alpha}(t)u_0\right\|_Y\le C_1\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}.\tag{3.3}$$

Now, we recall an auxiliary lemma that will help us to estimate bilinear term.

Lemma 3.3. *[31] Let $0<s\le t<+\infty$ and $0\le\alpha\le 1$. Then, the following inequality holds*

$$t\mid x\mid^\alpha-\frac{1}{2}(t^2-s^2)\mid x\mid^{2\alpha}-s\mid x-y\mid^\alpha-s\mid y\mid^\alpha\le\frac{1}{2}$$

for any $x,y\in\mathbb{R}^3$.

We have

$$\tilde{B}(u,v)=-\int_0^t e^{\sqrt{t}|D|^\alpha}H_\alpha(t-\tau)\mathbb{P}\nabla\cdot(u\otimes v)d\tau,\tag{3.4}$$

we also notice that

$$\tilde{B}(U,V)=-\int_0^t e^{\sqrt{t}|D|^\alpha}H_\alpha(t-\tau)\mathbb{P}\nabla\cdot(e^{-\sqrt{\tau}|D|^\alpha}U\otimes e^{-\sqrt{\tau}|D|^\alpha}V)d\tau.\tag{3.5}$$

Using the Fourier transform, multiplying with $2^{js_1(\cdot)}\varphi_j$, taking the $L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})$-norm and using Lemma 3.3, one reaches

$$\begin{aligned}
&\left\|2^{js_1(\cdot)}\varphi_j\widehat{\tilde{B}(U,V)}\right\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}\\
&\lesssim \left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\sqrt{t}|\xi|^{\alpha}-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}(\widehat{U\otimes V})d\tau\right\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}\\
&\lesssim\| \; 2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-1}{2}(t-\tau)|\xi|^{2\alpha}}\\
&\qquad\int_{\mathbb{R}^3}e^{\frac{-1}{2}(t-\tau)|\xi|^{2\alpha}+\sqrt{t}|\xi|^{\alpha}-\sqrt{\tau}(|\xi-y|^{\alpha}+|y|^{\alpha})}(\hat{U}(\xi-y,\tau)\otimes\hat{V}(y,\tau))dyd\tau \;\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}\\
&\lesssim \left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-1}{2}(t-\tau)|\xi|^{2\alpha}}\int_{\mathbb{R}^3}\hat{U}(\xi-y,\tau)\otimes\hat{V}(y,\tau)dyd\tau\right\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}\\
&\lesssim \left\|2^{j(s_1(\cdot)+1)}\varphi_j\int_0^t e^{\frac{-1}{2}(t-\tau)|\xi|^{2\alpha}}(\widehat{U\otimes V})d\tau\right\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}.
\end{aligned}$$

Taking ℓ^q-norm, applying Proposition 2.10 and Proposition 2.11, one obtains

$$\begin{aligned}
&\left\|\left\|\int_0^t 2^{j(s_1(\cdot)+1)}\varphi_j e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}(\widehat{U\otimes V})d\tau\right\|_{L^{\gamma}([0,\infty),\mathcal{M}^{\kappa(\cdot)}_{p_1(\cdot)})}\right\|_{\ell^q}\\
&\lesssim \left\|\left\|\int_0^t\left\|2^{j(s_1(\cdot)+1)}\varphi_j e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}r^{\frac{-3(6-(5-4\alpha)p_1(\cdot))}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}\left\|(\widehat{U\otimes V})\right\|_{\mathcal{M}^{\kappa(\cdot)}_{\frac{6}{5-4\alpha}}}d\tau\right\|_{L^{\gamma}([0,\infty))}\right\|_{\ell^q}\\
&\lesssim \|\|\int_0^t 2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2})}\left\|\varphi_j e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}r^{\frac{-3(6-(5-4\alpha)p_1(\cdot))}{6p_1(\cdot)}}2^{-3j\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}\\
&\qquad\|U\otimes V\|_{\mathcal{M}^{k(\cdot)}_{\frac{6}{4\alpha+1}}}d\tau\;\|_{L^{\gamma}([0,\infty))}\|_{\ell^q}\\
&\lesssim \|\|\int_0^t 2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2})}e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}\left\|\varphi_j r^{\frac{-3(6-(5-4\alpha)p_1(\cdot))}{6p_1(\cdot)}}2^{-3j\frac{6-(5-4\alpha)p_1(\cdot)}{6p_1(\cdot)}}\right\|_{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}\\
&\qquad\|\Delta_j(U\otimes V)\|_{\mathcal{M}^{\kappa(\cdot)}_{\frac{6}{4\alpha+1}}}d\tau\;\|_{L^{\gamma}([0,\infty))}\|_{\ell^q}\\
&\lesssim \left\|\left\|2^{j(\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha)}\|\Delta_j(U\otimes V)\|_{\mathcal{M}^{\kappa(\cdot)}_{\frac{6}{4\alpha+1}}}\right\|_{L^{\gamma}([0,\infty))}\left\|2^{2\alpha j}e^{-\frac{1}{2}t2^{2\alpha j}}\right\|_{L^1([0,\infty))}\right\|_{\ell^q}\\
&\lesssim \|U\|_{\mathcal{L}^{\gamma}([0,\infty),\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q}}\|V\|_{\mathcal{L}^{\infty}([0,\infty),L^{\frac{3}{2\alpha-1}})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q}}\|U\|_{\mathcal{L}^{\infty}([0,\infty),L^{\frac{3}{2\alpha-1}})}\\
&\lesssim \|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{B}^{\frac{5}{2}-2\alpha}_{2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{B}^{\frac{5}{2}-2\alpha}_{2,q})}\\
&\lesssim \|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}.
\end{aligned}$$

Then, it follows

$$\begin{aligned}
&\left\|\tilde{B}(U\otimes V)\right\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})}\lesssim\\
&\|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}.
\end{aligned}$$

Hence, if $p(\cdot)=p_1(\cdot)$ *and* $\gamma=\infty$, we get

$$\left\|\tilde{B}(U\otimes V)\right\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q})} \lesssim$$
$$\|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}.$$

Similarly, we get

$$\left\|\tilde{B}(U\otimes V)\quad\right\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})} \lesssim$$
$$\|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}$$

and

$$\left\|\tilde{B}(U\otimes V)\right\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})} \lesssim$$
$$\|U\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|V\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}+\|V\|_{\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})}\|U\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}.$$

Finally,

$$\left\|\tilde{B}(U\otimes V)\right\|_{Y}\leq C_2\|U\|_Y\|V\|_Y. \tag{3.6}$$

Then, by (3.3) and (3.6), one obtains

$$\begin{aligned}\|\phi(U)\|_Y&\leq\left\|\tilde{H_\alpha}(t)u_0\right\|_Y+\left\|\int_0^t\tilde{H_\alpha}(t-\tau)\mathbb{P}\nabla\cdot(U\otimes U)d\tau\right\|_Y\\&\leq\left\|\tilde{H_\alpha}(t)u_0\right\|_Y+\left\|\tilde{B}(V\otimes V)\right\|_Y\\&\leq C_1\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}+C_2\sigma^2.\end{aligned}$$

Taking $\sigma<\dfrac{1}{2max(C_1,C_2)}$ for any $u_0\in\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}$ with

$$\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}<\frac{\sigma}{2max(C_1,C_2)},$$

we get

$$\begin{aligned}\|\phi(U)\|_Y&<C_1\frac{\sigma}{2max(C_1,C_2)}+C_2\frac{\sigma}{2max(C_1,C_2)}\\&<\frac{\sigma}{2}+\frac{\sigma}{2}\\&<\sigma.\end{aligned}$$

Then by using Lemma 2.4, we can show that the unique solution u, is analytic in the sense that

$$\|e^{\sqrt{t}|D|^{\alpha}}u\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q})\cap\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})\cap\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q})}\leq\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q}}.$$

From the above and by using an analogous argument as in the case of the space Y, we obtain

$$e^{\sqrt{t}|D|^{\alpha}}u\in Z=$$
$$\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{s_1(\cdot)}_{p_1(\cdot),\kappa(\cdot),q})\cap\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'(\cdot)}}_{p(\cdot),\kappa(\cdot),q})\cap\mathcal{L}^{\gamma}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{2\alpha}{\gamma}+\frac{5}{2}-2\alpha}_{2,\kappa(\cdot),q})\cap\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\alpha}_{2,2,q}).$$

□

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