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On Algebraic Independence of Some Continued Fractions

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ABSTRACT: In the present paper, we prove the algebraic independence of a finite number of real continued fractions that have partial quotients that increase rapidly. We then use a general Liouville criteria to justify the algebraic independence of limits in some real series. We note that these results extend some work of Bundschuh, and we use a new and simple method.

Key Words: Continued fractions, algebraic independence, approximation of real series.

Contents

1. Introduction

Many results in transcendental number theory are proved by constructing a sequence of sufficiently good approximations (perhaps by rational numbers) or possibly a sequence of polynomials with integer coefficients that take small values at a number that we study.
So, in 1844, Liouville [8] has explicitly constructed the first example of transcendental numbers. We recall that the transcendence of the continued fractions having partial quotients that increase rapidly have been studied by several authors, such as W. W. Adams [1], P. Bundschuh [3], A. Durand [4], W. Lianxiang [7], G. Nettler [10], T. Okano [11].
We also note that the transcendence of some power series with rational or integer coefficients or p-adic numberss are given by some authors, see [6], [13].

Let $A = [a_0; a_1, a_2, \cdots, a_n, ...]$ and $B = [b_0; b_1, b_2, \cdots, b_n, ...]$ be two real continued fractions. In 1984, P. Bundschuh [3] proved that A and B are algebraically independent if there exists a reel number $r > 1$ such that

$$r^{-1} a_n \geq b_n \geq a_{n-1}^{n-1} \quad \text{for all} \quad n \geq 2.$$

In particular, the six numbers A, B, $A \pm B$, and $AB^{\pm 1}$ are transcendental.
Similarly, A. Kacha (see [5]) in 1993 has improved Bundschuh result by showing first that if α is a real constant > 3 such that for all $n \geq 2$

$$r^{-1} a_n > b_n > a_{n-1}^{\alpha}, \tag{1.1}$$

then for any non-constant polynomial $P \in \mathbb{Z}[X]$ of total degree $d < \dfrac{\alpha - 1}{2}$, $P(A, B)$ is a transcendental number.
To prove this result, he used the approximation theorem of Roth [12]. Then he has deduced the algebraic independence of A and B if in the relation (1.1) the exponent of a_{n-1} is an increasing sequence of real numbers α_{n-1} which tends to infinity. He also proved the transcendence of the six numbers A, B, $A \pm B$, and $AB^{\pm 1}$ in his recent paper [2].

A first aim in the present note is to prove the algebraic independence of a finite family of real numbers which are defined by their continued fraction expansions. Our work also generalizes a work of Bundschuh in [3] from two numbers to an arbitrary number of real numbers.
We use a general Liouville type algebraic independence criteria due to Adams [1]. We notice that our

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method is new and simple. The following theorem is a criterion of the algebraic independence over $\mathbb{Q}$ that will be used for the case of continued fractions.

Theorem 1.1. [1]. *Let $\theta_1, \theta_2, ..., \theta_m$ be m real numbers. Assume that we are given integers $P_{n,j}; Q_{n,j}$ for $j = 1, .., m$ such that $\lim_{n\to+\infty} Q_{n,j} = +\infty$. Assume that for $2 \leq j \leq m$,*

$$\lim_{n\to+\infty} \frac{|\theta_{j-1} - P_{n,j-1}/Q_{n,j-1}|}{|A_j - P_{n,j}/Q_{n,j}|} = 0. \tag{1.2}$$

Further assume that for all j, $1 \leq j \leq m$ and all positive integers d there is an $n_0 = n_0(d)$ such that for all $n \geq n_0$,

$$0 < |\theta_j - P_{n,j}/Q_{n,j}| < (Q_{n,1}Q_{n,2}...Q_{n,j})^{-d}. \tag{1.3}$$

Then, $\theta_1, \theta_2, ..., \theta_m$ are algebraically independent.

A second aim of this paper is also to prove the algebraic independence of limits of some real series.

2. Algebraic independence of a family of continued fractions

We study the algebraic independence over $\mathbb{Q}$ of real numbers by using their expansions as continued fractions. Let $A_1, A_2, ..., A_m$ be m real numbers ($m \geq 2$) which are defined by the simple continued fractions

$$A_j = [a_{0,j}; a_{1,j}, a_{2,j}, ...].$$

For all $1 \leq j \leq m$ and $n \geq 0$, denote the convergents of A_j by $\frac{{}^a p_{n,j}}{{}^a q_{n,j}}$ and

$$A_{n,j} = [a_{0,j}; a_{1,j}, \cdots, a_{n,j}] = \frac{{}^a p_{n,j}}{{}^a q_{n,j}}.$$

Our first main result is.

Theorem 2.1. *Let $(a_{n,j})$ be as before, we suppose that there exist a real number $r > 1$, (β_n) a real sequence > 1 which tends to $+\infty$ such that*

$$\begin{cases} r^{-1}a_{n,j-1} > a_{n,j}, & 2 \leq j \leq m, \\ a_{n+1,m} > a_{n,1}^{\beta_n} & \textit{for all } \ n \geq 1. \end{cases} \tag{2.1}$$

Then $A_1, A_2, ...A_m$ are algebraically independent over $\mathbb{Q}$.

Remark 2.2. *For the proof of Theorem 2.1, we can assume that the sequence β_n is increasing. Because if it is not, we can take $\gamma_n = inf(\beta_p,\ p \geq n)$ which is increasing since β_n tends to infinity. Further, the hypothesis $b_n > a_{n-1}^{\gamma_{n-1}}$ implies that $b_n > a_{n-1}^{\beta_{n-1}}$.*

In order to prove Theorem 2.1, we will first require a preliminary Lemma.

Lemma 2.3. *(i) For all $1 \leq j \leq m$, we have*

$$\frac{1}{2q_{n,j}q_{n+1,j}} < |A_j - A_{n,j}| < \frac{1}{q_{n,j}q_{n+1,j}}.$$

(ii) If $r^{-1}a_{n,j-1} > a_{n,j}$ for all $j = 2, ..., m$ and $n \geq 1$, then we get

$$q_{n,j-1} > r^{\frac{n}{2}} q_{n,j} > q_{n,j}. \tag{2.2}$$

(iii) If $a_{n,j} > a_{n-1,j}^{\beta_{n-1}}$ for all $n \geq 2$, then for any $\varepsilon > 0$ we obtain

$$q_{n,j} < \ a_{n,j}^{\frac{\beta_1}{\beta_1 - 1} + \varepsilon},$$

for all sufficiently large n.

Proof. (i) See [10].

(ii) We prove it as in Lemma 2 of Bundschuh [3].

(iii) We have

$$\begin{aligned} q_{n,j} &= a_{n,j}\ q_{n-1,j} + \ q_{n-2,j} < (a_{n,j} + 1)\ \ q_{n-1,j} \\ &< \prod_{k=1}^{n} (a_k + 1) . \end{aligned}$$

Which becomes

$$q_{n,j} < \ \prod_{k=1}^{n} \left(1 + \frac{1}{a_{k,j}}\right) \prod_{k=1}^{n} a_{k,j}.$$

However, for all $k \geq 1$ we get

$$a_{k+1,j} > a_{k,j}^{\beta_1} \ \text{ and } \ a_{1,j} \geq 2 \ \text{ then } \ a_{k,j} > 2^{\beta_1^{k-1}}.$$

Hence, there exists a positive real constant $C(\beta_1)$ such that

$$\prod_{k=n_1}^{n} \left(1 + \frac{1}{a_{k,j}}\right) < \prod_{k=1}^{n} \left(1 + \frac{1}{2^{\beta_1^{k-1}}}\right) < C\left(\beta_1\right).$$

Finally, we obtain

$$\begin{aligned} {}^{a}q_{n,j} &< \ C\left(\beta_1\right)\ \prod_{k=1}^{n} a_{k,j} \\ &< \ C\left(\beta_1\right)\ a_{n,j}^{1+\frac{1}{\beta_1}+\frac{1}{\beta_1^2}+...+\frac{1}{\beta_1^{n-1}}}. \end{aligned}$$

Then, for any $\varepsilon > 0$ we get

$${}^{a}q_{n,j} < C\left(\beta_1\right)\ {}^{a}a_{n,j}^{\frac{1}{1-\frac{1}{\beta_1}}} < a_{n,j}^{\frac{\beta_1}{\beta_1 - 1} + \varepsilon}$$

for all sufficiently large n. □

Proof of Theorem 2.1. (i) We will note bellow $\beta(n) = \beta_n$. For the proof of Theorem 2.1, we need only verify the hypotheses of Theorem 1.1. We have

$$|A_{j-1} - A_{n,j-1}| < \frac{1}{q_{n,j-1}q_{n+1,j-1}}.$$

By using (i) and (ii) of Lemma 2.1, $q_{n,j-1} > r^{n/2} q_{n,j}$ and $\frac{1}{2q_{n,j}q_{n+1,j}} < |A_j - A_{n,j}|$, the inequality above becomes

$$|A_{j-1} - A_{n,j-1}| < \frac{2}{r^{n/2} q_{n,j} q_{n+1,j}} < \frac{2}{r^{n/2}} |A_j - A_{n,j}|.$$

So, we obtain

$$\frac{|A_{j-1} - P_{n,j-1}/Q_{n,j-1}|}{|A_j - P_{n,j}/Q_{n,j}|} < \frac{2}{r^{n/2}},$$

which tends to zero by hypothesis. To verify (1.3) of Theorem 1.1, we use

$$|A_j - A_{n,j}| < \frac{1}{q_{n,j} q_{n+1,j}} < \frac{1}{a_{n+1,j}}.$$

Remark 2.4. *We recall that the hypotheses $a_{n+1,m} > a_{n,1}^{\beta_n}$ and $a_{n,j-1} > a_{n,j}$ for all $2 \leq j \leq m$, one gets*

$$a_{n+1,j} > a_{n,l}^{\beta_n}, \quad \textit{for all} \ \ 1 \leq l \leq m.$$

From this remark, we deduce that

$$|A_j - A_{n,j}| < \frac{1}{a_{n,l}^{\beta_n}} < \frac{1}{\prod_{l=1}^{j} a_{n,l}^{\beta_n/m}}.$$

By using (iii) of Lemma 2.1, for any $\varepsilon > 0$ one find

$$|A_j - A_{n,j}| < \frac{1}{\prod_{l=1}^{j} q_{n,l}^{\frac{\beta_1 - 1}{m(\beta_1 + \varepsilon(\beta_1 - 1))}\beta_n}},$$

for all sufficiently large n.

We then tend ε to zero in the last inequality. On the other hand, one has $\lim_{n\to+\infty} \frac{\beta_1-1}{m\beta_1}\beta_n = +\infty$, then for all positive integers d there is an $n_0 = n_0(d)$ such that for all $n > n_0$ we have $\frac{\beta_1-1}{m\beta_1}\beta_n > d$. So, we get

$$0 < |A_j - P_{n,j}/Q_{n,j}| < (Q_{n,1}...Q_{n,j})^{-d}.$$

which completes the proof of Theorem 2.1.

Example. Let

$$\left\{ \begin{array}{c} a_0 = b_0 = 0, \ a_1 = b_1 = 1, \\ a_2 = 9, \ b_2 = 3, \\ r = 1,5 \end{array} \right. \qquad \left\{ \begin{array}{c} a_n = 3^{\frac{(2n)!}{2^n}}, \ n \geq 3 \\ \\ b_n = 3^{\frac{(2n)!}{2^{n+1}}}, n \geq 3 \\ \\ \beta_n = (2n-1)(n+1), \ n \geq 1. \end{array} \right.$$

By applying Theorem 2.1, we deduce that the real numbers $A_1, A_2, ..., A_m$ are algebraically independent.

3. Algebraic independence of some series

Theorem 3.1. *Let $g_1, ..., g_m$ be m distinct integers ≥ 2, such that $g_{j-1} > 2g_j$ for all $2 \leq j \leq m$, δ a real number > 0. Let*

$$\theta_j = \sum_{n=1}^{+\infty} g_j^{-a_n}$$

for all $1 \leq j \leq m$, where $a_{n+1} = a_n^{1+\delta}$ for $n \geq 1$ and $a_1 = 3$. Then, the real numbers $\theta_1, \theta_2, ..., \theta_m$ are algebraically independent.

Proof. From the definition of θ_j, it is clear that these series are convergent. In order to prove Theorem 3.1, it suffices to verify the hypotheses of Theorem 1.1. To this end, consider $(\frac{p_{n,j}}{q_{n,j}}) = (\sum_{k=1}^{n} g_j^{-a_k})$ a sequence of rational approximations of θ_j which is expressed in reduced form. So we obtain $q_{n,j} = g_j^{a_n}$ and

$$\left|\theta_{j-1} - \frac{p_{n,j-1}}{q_{n,j-1}}\right| = \frac{1}{g_{j-1}^{a_{n+1}}}\left(1 + \sum_{k=n+1}^{+\infty} \frac{1}{g_{j-1}^{a_{k+1}-a_{n+1}}}\right).$$

Which yields

$$\frac{1}{g_{j-1}^{a_{n+1}}} < \left|\theta_{j-1} - \frac{p_{n,j-1}}{q_{n,j-1}}\right| < \frac{2}{g_{j-1}^{a_{n+1}}}. \tag{3.1}$$

We deduce from (2.3) that θ_j is a Liouville number, so it is a transcendental number. Therefore, (3.1) and $g_{j-1} > 2g_j$ imply that

$$\left|\theta_{j-1} - \frac{p_{n,j-1}}{q_{n,j-1}}\right| < \frac{2}{(2g_j)^{a_{n+1}}} < \frac{1}{2^{a_{n+1}-1}}\left|\theta_j - \frac{p_{n,j}}{q_{n,j}}\right|.$$

It follows that

$$\left|\theta_{j-1} - \frac{P_{n,j-1}}{Q_{n,j-1}}\right| / \left|\theta_j - \frac{P_{n,j}}{Q_{n,j}}\right| < \frac{1}{2^{a_{n+1}-1}}$$

which tends to zero. Since we have,

$$\left|\theta_j - \frac{p_{n,j}}{q_{n,j}}\right| < \frac{2}{g_j^{a_{n+1}}},$$

to verify (1.3) of Theorem 1.1, it suffices to prove that for all $n \geq n_0(d)$,
$g_j^{a_{n+1}} > (g_1^{a_n} ... g_j^{a_n})^d$. We can see that

$$\lim_{n \to +\infty} \frac{g_j^{a_{n+1}}}{(g_1 ... g_j)^{a_n}} = +\infty.$$

This equality is true since one has

$$\frac{a_{n+1} \ln g_j}{a_n \ln(g_1 ... g_j)} = a_n^{\delta} \frac{\ln g_j}{\ln(g_1 ... g_j)} = 3^{\delta(1+\delta)^{n-1}} \frac{\ln g_j}{\ln(g_1 ... g_j)},$$

which tends to infinity. Hence, for any positive integer d there exists an $n_0 = n_0(d)$ such that, for all $n \geq n_0$

$$\ln(g_j^{a_{n+1}}) > d \ln(g_1 ... g_j)^{a_n} = \ln(g_1^{a_n} ... g_j^{a_n})^d.$$

Which yields

$$g_j^{a_{n+1}} > (g_1^{a_n} ... g_j^{a_n})^d.$$

Finally, we obtain

$$\left|\theta_j - \frac{p_{n,j}}{q_{n,j}}\right| < \frac{2}{(g_1^{a_n} ... g_j^{a_n})^d}.$$

Which completes the proof of Theorem 3.1. □

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