Approximation and Analysis Regarding the Structure of a Multiple Variable Mapping

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Abstract: The article introduces a several variables mapping as the multimixed quadratic-cubic mapping in order to characterize such mappings. It reduces a system of equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. It is shown that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multiquadratic-cubic. Further, the generalized Hyers-Ulam stability and hyperstability for multimixed quadratic-cubic functional equations in quasi-$\beta$-normed spaces have been investigated.

Key Words: Hyers-Ulam stability, multi-quadratic mapping, multi-cubic mapping, multiquadratic-cubic mapping, multimixed quadratic-cubic mapping.

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1. Introduction

The stability problem for functional equations, which Ulam [30] proposed for group homomorphisms, has been answered and explored for multiple variable mappings in recent decades. We recall that a functional equation $\Gamma$ is said to be stable if any function $f$ satisfying the equation $\Gamma$ approximately must be near to an exact solution. Moreover, $\Gamma$ is called hyperstable if any function $f$ satisfying the equation $\Gamma$ approximately (in some senses) is actually a solution for it; for some stability results in one variable mappings and functional equations see for instance the papers and books [13], [19], [24], [26], [29] and references therein.

We now state some basic notions and developments about the structure and the stability of several variables mappings. Let $V$ be a commutative group, $W$ be a linear space, and $n \geq 2$ be an integer. A mapping $f : V^n \rightarrow W$ is called

- multi-additive if it is additive (satisfies Cauchy’s functional equation $A(x + y) = A(x) + A(y)$) in each variable.
- multi-quadratic if it fulfills the quadratic functional equation

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$$

(1.1)

in each variable [11]. A lot of information about the structure of multi-additive mappings and their Ulam stabilities are available in [10], [12] and [20, Sections 13.4 and 17.2]. C.-G. Park was the first author who studied the stability of multi-quadratic in the setting of Banach algebras in [22]. After that, Ciepliński [11] studied the generalized Hyers-Ulam stability of multi-quadratic mappings in Banach spaces. Zhao et al. [32] described the structure of multi-quadratic mappings and in fact showed that a mapping $f : V^n \rightarrow W$ is multi-quadratic if and only if the equation

$$\sum_{s \in \{-1,1\}^n} f(sx_1 + sx_2) = 2^n \sum_{j_1, j_2, \ldots, j_n \in \{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n})$$

(1.2)

holds, where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. Various versions of multi-quadratic mappings and their stability can be found in [7] and [28]. For the structure of multi-additive-quadratic, we refer to [1].
Ghaemi et al. [15] introduced the multi-cubic mappings and then for a special case of such mappings have been studied in [8]. In fact, a mapping \( f : V^n \rightarrow W \) is called multi-cubic if it is cubic in each variable, i.e., satisfies the equation
\[
C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x)
\] (1.3)
in each variable [17]. In [8], the authors unified the system of functional equations defining a multi-cubic mapping to a single equation, namely, the multi-cubic functional equation. Furthermore, the general system of cubic functional equations which is defined in [15], characterized as a single equation in [14]. Other forms of cubic functional equations for instance are available in [3] and [23]. In [8], it is shown that every multi-cubic functional equation is stable and moreover such functional equations under some conditions can be hyperstable; for the miscellaneous versions of multi-cubic mappings and their stabilities in non-Archimedean normed and modular spaces, we refer to [14] and [21], respectively.

Chang and Jung [9] introduced the following mixed type quadratic and cubic functional equation
\[
6f(x + y) - 6f(x - y) + 4f(3y) = 3f(x + 2y) - 3f(x - 2y) + 9f(2y).
\] (1.4)
They established the general solution of the functional equation (1.4) and investigated the Hyers-Ulam stability of this equation; for a different form of mixed type quadratic-cubic functional equation, one can see [18].

The following mixed type quadratic-cubic functional was considered in [27] which is somewhat different from (1.4) as follows:
\[
f(x + 2y) - f(x - 2y) = 2[f(x + y) - f(x - y)] + 3f(2y) - 12f(y).
\] (1.5)
It is easily verified that the function \( f(x) = ax^2 + bx^3 \) is a solution of equations (1.4) and (1.5). Recently, the first author and Mitrović [6] have studied the structure of multimixed quadratic-cubic mappings and established \( \varepsilon \)-stability (Hyers' stability) of such mappings in Banach spaces setting by applying an alternative fixed point method.

Motivated by equation (1.5), in this paper, we define multimixed quadratic-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of \( n \) equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. We also show that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multi-quadratic-cubic. We also prove the generalized Hyers-Ulam stability and hyperstability for multimixed quadratic-cubic functional equations in quasi-\( \beta \)-normed spaces.

2. Characterization of the multimixed quadratic-cubic mappings

Throughout this paper, \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{Q} \) are the set of all positive integers, integers and rational numbers, respectively, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty) \). For any \( l \in \mathbb{N}_0, n \in \mathbb{N}, t = (t_1, \ldots, t_n) \in \{-1, 1\}^n \), and \( x = (x_1, \ldots, x_n) \in V^n \) we write \( lx := (lx_1, \ldots, lx_n) \) and \( tx := (t_1x_1, \ldots, t_nx_n) \), where \( lx \) stands, as usual, for the scalar product of \( l \) on \( x \) in the commutative group \( (V, +) \).

Let \( V \) and \( W \) be linear spaces, \( n \in \mathbb{N} \) and \( k \in \{0, \ldots, n\} \). Put
\[
n := \{1, \ldots, n\}. \tag{2.1}
\]
Each subset of \( n \) with \( m \) elements is denoted by \( (m)_n \). Recall from [5] that a mapping \( f : V^n \rightarrow W \) is called \( k \)-quadratic and \( n - k \)-cubic (briefly, multi-quadratic-cubic) if \( f \) satisfies the following functional equations system.
\[
\begin{align*}
f(v_1, \ldots, v_{i-1}, & v_i + v'_i, v_{i+1}, \ldots, v_n) + f(v_1, \ldots, v_{i-1}, v_i - v'_i, v_{i+1}, \ldots, v_n) \\
&= 2f(v_1, \ldots, v_n) + 2f(v_1, \ldots, v_i', \ldots, v_n), \quad i \in (k)_n,
\end{align*}
\]
\[
\begin{align*}
f(v_1, \ldots, v_{i-1}, & 2v_i + v'_i, v_{i+1}, \ldots, v_n) + f(v_1, \ldots, v_{i-1}, 2v_i - v'_i, v_{i+1}, \ldots, v_n) \\
&= 2f(v_1, \ldots, v_{i-1}, v_i + v'_i, v_{i+1}, \ldots, v_n) + 2f(v_1, \ldots, v_{i-1}, v_i - v'_i, v_{i+1}, \ldots, v_n) + 12f(v_1, \ldots, v_n), \quad i \in (n-k)_n.
\end{align*}
\]
Note that we can suppose for simplicity that \( f \) is quadratic in each of the first \( k \) variables, but one can obtain analogous results without this assumption. Let us note that for \( k = n \) (\( k = 0 \), the above definition leads to the so-called multi-quadratic (multi-cubic) mappings; some basic facts on such mappings can be found for instance in \([8]\) and \([32]\).

**Definition 2.1.** Let \( V \) and \( W \) be vector spaces over \( \mathbb{Q} \), \( n \in \mathbb{N} \). A several variables mapping \( f : V^n \rightarrow W \) is called \( n \)-mixed quadratic-cubic or briefly multimixed quadratic-cubic if \( f \) fulfills (1.5) in each of its \( n \) arguments, that is

\[
\begin{align*}
f(v_1, \ldots, v_{i-1}, v_i + 2v_i', v_i+1, \ldots, v_n) - f(v_1, \ldots, v_{i-1}, v_i, v_i+1, \ldots, v_n) \\
- 3f(v_1, \ldots, v_{i-1}, 2v_i', v_i+1, \ldots, v_n) \\
= 2[f(v_1, \ldots, v_{i-1}, v_i + v_i', v_i+1, \ldots, v_n) - f(v_1, \ldots, v_{i-1}, v_i - v_i', v_i+1, \ldots, v_n)] \\
- 12f(v_1, \ldots, v_{i-1}, v_i', v_i+1, \ldots, v_n).
\end{align*}
\]

Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( x^n_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n \), where \( i \in \{1, 2\} \). We will write \( x^n_i \) simply \( x_i \) when no confusion can arise. For \( x_1, x_2 \in V^n \), set

\[
M^n = \{ \mathcal{M}_n = (M_1, \ldots, M_n) | M_j \in \{ x_{1j} \pm 2x_{2j}, 2x_{2j} \} \},
\]

and

\[
N^n = \{ \mathcal{N}_n = (N_1, \ldots, N_n) | N_j \in \{ x_{1j} \pm x_{2j}, x_{2j} \} \},
\]

for all \( j \in \{1, \ldots, n\} \). For \( p_i, q_i \in \mathbb{N}_0 \) with \( 0 \leq p_i, q_i \leq n \), consider the subsets \( M^n_{(q_1, q_2)} \) and \( N^n_{(p_1, p_2)} \) of \( M^n \) and \( N^n \), respectively, as follows:

\[
M^n_{(q_1, q_2)} := \{ \mathcal{M}_n \in M^n | \text{Card}\{M_j : M_j = x_{1j} - 2x_{2j}\} = q_1, \text{Card}\{M_j : M_j = x_{2j}\} = q_2 \},
\]

\[
N^n_{(p_1, p_2)} := \{ \mathcal{N}_n \in N^n | \text{Card}\{N_j : N_j = x_{1j} - x_{2j}\} = p_1, \text{Card}\{N_j : N_j = x_{2j}\} = p_2 \}.
\]

Hereafter, for a multimixed quadratic-cubic mappings \( f \), we use the following notations:

\[
f\left(M^n_{(q_1, q_2)}\right) := \sum_{\mathcal{M}_n \in M^n_{(q_1, q_2)}} f(\mathcal{M}_n), \tag{2.2}
\]

\[
f\left(M^n_{(q_1, q_2)}, z\right) := \sum_{\mathcal{M}_n \in M^n_{(q_1, q_2)}} f(\mathcal{M}_n, z) \quad (z \in V),
\]

\[
f\left(N^n_{(p_1, p_2)}\right) := \sum_{\mathcal{N}_n \in N^n_{(p_1, p_2)}} f(\mathcal{N}_n), \tag{2.3}
\]

and

\[
f\left(N^n_{(p_1, p_2)}, z\right) := \sum_{\mathcal{N}_n \in N^n_{(p_1, p_2)}} f(\mathcal{N}_n, z) \quad (z \in V).
\]

For each \( x_1, x_2 \in V^n \), we consider the equation

\[
\sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{2q_1} (-3)^{q_2} f\left(M^n_{(q_1, q_2)}\right) = \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f\left(N^n_{(p_1, p_2)}\right), \tag{2.4}
\]

where \( f\left(M^n_{(q_1, q_2)}\right) \) and \( f\left(N^n_{(p_1, p_2)}\right) \) are defined in (2.2) and (2.3), respectively.

We recall that the binomial coefficient for all \( n, r \in \mathbb{N}_0 \) with \( n \geq r \) is defined and denoted by

\[
\binom{n}{r} := \frac{n!}{r!(n-r)!}.
\]
Definition 2.2. Let \( r \in \mathbb{N} \). We say the mapping \( f : V^n \rightarrow W \)

(i) satisfies (has) the \( r \)-power condition in the \( j \)th variable if
\[
f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),
\]
for all \( z_1, \ldots, z_n \in V^n \). In particular, 2-power and 3-power conditions are called quadratic and cubic condition, respectively.

(ii) has zero condition if \( f(x) = 0 \) for any \( x \in V^n \) with at least one component which is equal to zero.

(iii) is odd in the \( j \)th variable if
\[
f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = -f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).
\]

(iv) is even in the \( j \)th variable if
\[
f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).
\]

Here, we bring an elementary lemma from [4].

Lemma 2.3. Let \( n, k, p_l \in \mathbb{N}_0 \), such that \( k + \sum_{l=1}^{m} p_l \leq n \), where \( l \in \{1, \ldots, m\} \). Then
\[
\left( \begin{array}{c}
\frac{n-k}{n-k-\sum_{l=1}^{m} p_l} \\
\end{array} \right) \left( \begin{array}{c}
\sum_{l=1}^{m} p_l \\
\sum_{l=1}^{m-1} p_l \\
\end{array} \right) \ldots \left( \begin{array}{c}
p_1 + p_2 \\
p_1 \\
\end{array} \right) = \left( \begin{array}{c}
\frac{n-k}{p_1} \\
\frac{n-k-p_1}{p_2} \\
\vdots \\
\frac{n-k-\sum_{l=1}^{m-1} p_l}{p_m} \\
\end{array} \right).
\]

Consider \( n \) as in (2.1). For a subset \( T = \{j_1, \ldots, j_i\} \) of \( n \) with \( 1 \leq j_1 < \cdots < j_i \leq n \) and \( x = (x_1, \ldots, x_n) \in V^n \),
\[
t x := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_i}, 0, \ldots, 0) \in V^n
\]
denotes the vector which coincides with \( x \) in exactly those components, which are indexed by the elements of \( T \) and whose other components are set equal zero. Note that \( o x = 0 \), \( n x = x \). We use these notations in the proof of upcoming lemma.

Next, we reduce the system of \( n \) equations defining the multimixed quadratic-cubic mapping in obtaining the single functional equation (2.4). For doing this, we need the next lemma.

Lemma 2.4. If a mapping \( f : V^n \rightarrow W \) satisfies equation (2.4), then it has zero condition.

Proof. We argue by induction on \( k \) that \( f(kx) = 0 \), when \( 0 \leq k \leq n - 1 \). Putting \( x_1 = x_2 = o x \) in (2.4), we have
\[
\left( \begin{array}{c}
\sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} \\
\end{array} \right) \left( \begin{array}{c}
( n - q_1 - q_2) \\
(q_1 + q_2) \\
\end{array} \right) \left( \begin{array}{c}
(-1)^{q_1} (-3)^{q_2} \\
\end{array} \right) f(o x)
\]
\[
= \left( \begin{array}{c}
\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \\
\end{array} \right) \left( \begin{array}{c}
( n - p_1 - p_2) \\
(p_1 + p_2) \\
\end{array} \right) 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f(o x).
\]

Here we compute the left side of (2.5). Using Lemma 2.3 for \( k = 0 \), we have
\[
\sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} \left( \begin{array}{c}
( n - q_1 - q_2) \\
(q_1 + q_2) \\
\end{array} \right) \left( \begin{array}{c}
(-1)^{q_1} (-3)^{q_2} \\
\end{array} \right)
\]
\[
= \sum_{q_1=0}^{n} \left( \begin{array}{c}
( n - q_1) \\
(q_1) \\
\end{array} \right) \left( \begin{array}{c}
(-1)^{q_1} \sum_{q_2=0}^{n-q_1} \left( \begin{array}{c}
( n - q_1) \\
(q_2) \\
\end{array} \right) 1^{n-q_1-q_2} (-3)^{q_2} \\
\end{array} \right)
\]
\[
= \sum_{q_1=0}^{n} \left( \begin{array}{c}
( n - q_1) \\
(q_1) \\
\end{array} \right) (-1)^{q_1} (-2)^{n-q_1} = (-1 - 2)^n = (-3)^n.
\]
Similarly, one can show from Lemma 2.3 that

$$
\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \binom{n-p_1}{p_1} \binom{n-p_2}{p_2} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} = (-12)^n. \quad (2.7)
$$

It follows from relations (2.5), (2.6) and (2.7) that $f(0) = 0$. Assume that $f(kx) = 0$ for any $k \in \{1, \ldots, n-1\}$. We show that $f(kx) = 0$. Without loss of generality, we assume that the first $k$ variables are non-zero. By our assumption, replacing $(x_1, x_2)$ by $(kx_1, 0)$ in equation (2.4), we have

$$
\sum_{q_1=0}^{n-k} \sum_{q_2=0}^{n-k-q_1} \binom{n-k}{q_1} \binom{n-k-q_1}{q_2} (-1)^{q_1} (-3)^{q_2} f(kx) = \left[ \sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} \binom{n-k}{p_1} \binom{n-k-p_1}{p_2} 2^{n-k-p_1-p_2} (-2)^{p_1} (-12)^{p_2} \right] f(kx).
$$

Similar the above and by using lemma 2.3, we can obtain $(-3)^{n-k} f(kx) = (-12)^{n-k} f(kx)$, and this implies that $f(kx) = 0$. This finishes the proof. □

In the upcoming results which are our aim in this section, we unify the general system of quadratic-cubic functional equations defining a multimixed quadratic-cubic mapping to an equation and indeed this functional equation describe a multimixed quadratic-cubic mapping.

**Proposition 2.5.** If a mapping $f : V^n \rightarrow W$ is multimixed quadratic-cubic, then it satisfies equation (2.4).

**Proof.** We proceed the proof by induction on $n$, and in fact we show that equation (2.4) is valid for $f$. Clearly, $f$ satisfies equation (1.5) and this guarantees the assertion for $n = 1$. If (2.4) holds for some
positive integer \( n > 1 \), then

\[
\sum_{q_1=0}^{n+1} \sum_{q_2=0}^{n+1-q_1} (-1)^{q_1}(-3)^{q_2} f \left( M_{(q_1,q_2)}^{n+1} \right) = \sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1}(-3)^{q_2} f \left( M_{(q_1,q_2)}^{n}, x_{1,n+1} + 2x_{2,n+1} \right) 
\]

\[
- \sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1}(-3)^{q_2} f \left( M_{(q_1,q_2)}^{n}, x_{1,n+1} - 2x_{2,n+1} \right) - 3 \sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1}(-3)^{q_2} f \left( M_{(q_1,q_2)}^{n}, 2x_{2,n+1} \right) 
\]

\[
= \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, x_{1,n+1} + 2x_{2,n+1} \right) 
\]

\[
- \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, x_{1,n+1} - 2x_{2,n+1} \right) 
\]

\[
- 3 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, 2x_{2,n+1} \right) 
\]

\[
= 2 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, x_{1,n+1} + x_{2,n+1} \right) 
\]

\[
- 2 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, x_{1,n+1} - x_{2,n+1} \right) 
\]

\[
- 12 \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n}, x_{2,n+1} \right) 
\]

\[
= \sum_{p_1=0}^{n+1} \sum_{p_2=0}^{n+1-p_1} 2^{n+1-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f \left( N_{(p_1,p_2)}^{n+1} \right) 
\]

This means that (2.4) holds for \( n + 1 \). □

It follows from Proposition 2.5 and by a mathematical computation that the mapping \( f(z_1, \ldots, z_n) = \prod_{j=1}^{n} (\alpha_j z_j^2 + \beta_j z_j^3) \) satisfies (2.4) and so this equation is said to be \textit{multi}plied \textit{quad}ratic-\textit{cubic} functional equation.

It is shown in [5, Proposition 2.1] that if a mapping \( f : V^n \rightarrow W \) is \( k \)-quadratic and \( n - k \)-cubic (multi)quad(cubic) mapping, then \( f \) satisfies equation

\[
\sum_{s \in \{-1,1\}^k} \sum_{t \in \{-1,1\}^{n-k}} f \left( x_k^t + sx_k^s, 2x_1^{n-k} + tx_2^{n-k} \right) = 2^k \sum_{m=0}^{n-k} 2^{n-k-m} 12^m \sum_{i \in \{1,2\}} f \left( x_i^t, M_{m}^{n-k} \right), \tag{2.8}
\]

for all \( x_k^t \in V^k \) and \( x_1^{n-k} = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k} \) where \( i \in \{1,2\} \) in which

\[
f \left( x_k^t, M_{m}^{n-k} \right) := \sum_{\mathcal{N}_n \in M_{m}^{n-k}} f \left( x_k^t, \mathcal{N}_n \right),
\]

whereas

\[
M_{m}^{n-k} = \{ \mathcal{N}_n = (N_{k+1}, \ldots, N_n) \, | \, N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\} \}
\]

and

\[
M_{m}^{n-k} = \{ \mathcal{N}_n = (N_{k+1}, \ldots, N_n) \in M_{m}^{n-k} \, | \, |\text{Card}\{N_j : N_j = x_{1j}\} = m \}.
\]

Note that in the the case \( k = n \) and \( k = 0 \), equation (2.8) converts to (1.2) and

\[
\sum_{t \in \{-1,1\}^n} f \left( 2x_1^t, tx_2^n \right) = \sum_{m=0}^{n} 2^{n-m} 12^m f \left( M_{m}^{n} \right), \tag{2.9}
\]
This finishes the proof of part (i).

**Proof.** (i) Let \( j \in \{1, \ldots, n\} \) be arbitrary and fixed. Set
\[
f_j^*(z) := f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n).
\]
Putting \( x_{1k} = 0 \) for all \( k \in \{1, \ldots, n\} \setminus \{j\} \), \( x_{1j} = z \) and \( x_2 = (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_n) \) in (2.4), using Lemma 2.4, we get
\[
(−3)^{n−1}[f(2z_1, \ldots, 2z_{j−1}, z + 2w, 2z_{j+1}, \ldots, 2z_n)
− f(2z_1, \ldots, 2z_{j−1}, z − 2w, 2z_{j+1}, \ldots, 2z_n) − 3f(2z_1, \ldots, 2z_{j−1}, 2w, 2z_{j+1}, \ldots, 2z_n)]
= (−12)^{n−1}[2f_j^*(z + w) − 2f_j^*(z − w) − 12f_j^*(w)].
\] (2.10)

Our assumption (2.10) converts to
\[
2^{2(n−1)}(−3)^{n−1}[f_j^*(z + 2w) − f_j^*(z − 2w) − 3f_j^*(2w)]
= (−12)^{n−1}[2f_j^*(z + w) − 2f_j^*(z − w) − 12f_j^*(w)],
\]
and so
\[
f_j^*(z + 2w) − f_j^*(z − 2w) − 3f_j^*(2w) = 2f_j^*(z + w) − 2f_j^*(z − w) − 12f_j^*(w).
\] (2.11)

This finishes the proof of part (i).

(ii) Similar to the proof of part (i), Putting \( x_{1k} = 0 \) for all \( k \in \{1, \ldots, n\} \setminus \{j\} \), \( x_{1j} = z \) and \( x_2 = (z_1, \ldots, z_{j−1}, w, z_{j+1}, \ldots, z_n) \) in (2.4), using our assumptions and Lemma 2.4, we obtain the left side of (2.4) as follows:
\[
\sum_{q_2=0}^{n−1} \left( \begin{array}{c} n−1 \\ q_2 \\ \end{array} \right) 2^{3(n−1−q_2)}(−3)^{q_2}2^{n−1−q_2}2^{q_2} (f_j^*(z + 2w) − f_j^*(z − 2w))
+ \sum_{q_2=1}^{n} \left( \begin{array}{c} n−1 \\ q_2−1 \\ \end{array} \right) 2^{3(n−1)}(−3)^{q_2}2^{n−q_2} f_j^*(w)
= 2^{3(n−1)}(−1)^{n−1}(f_j^*(z + 2w) − f_j^*(z − 2w) − 3) \times 2^{3(n−1)}(−1)^{n−1}f_j^*(2w). \] (2.12)

On the other hand, the right side of (2.4) will be
\[
\sum_{p_2=0}^{n−1} \left( \begin{array}{c} n−1 \\ p_2 \\ \end{array} \right) 2^{n−1−p_2}(−12)^{p_2}2^{n−p_2} (f_j^*(z + w) − f_j^*(z − w))
+ \sum_{p_2=1}^{n} \left( \begin{array}{c} n−1 \\ p_2−1 \\ \end{array} \right) 2^{n−p_2}(−12)^{p_2}2^{n−p_2} f_j^*(w)
= 2 \sum_{p_2=0}^{n−1} \left( \begin{array}{c} n−1 \\ p_2 \\ \end{array} \right) 4^{n−1−p_2}(−12)^{p_2} (f_j^*(z + w) − f_j^*(z − w))
− 12 \sum_{p_2=0}^{n} \left( \begin{array}{c} n−1 \\ p_2 \\ \end{array} \right) 4^{n−1−p_2}(−12)^{p_2} f_j^*(w)
= 2(−8)^{n−1}[f_j^*(z + w) − f_j^*(z − w)] − 12(−8)^{n−1}f_j^*(w). \] (2.13)

Comparing (2.12) and (2.13), we achieve (2.11). □
Corollary 2.7. Suppose that a mapping $f : V^n \rightarrow W$ satisfies equation (2.4).

(i) If $f$ is even in each variable and satisfies the quadratic condition in all variables, then it is multi-quadratic. Moreover, $f$ satisfies equation (1.2);

(ii) If $f$ is odd in each variable and satisfies the cubic condition in all variables, then it is multi-cubic. In addition, equation (2.9) is valid for $f$;

(iii) If $f$ is even in each of some $k$ variables with the quadratic condition and is odd in each of the other variables with the cubic condition, then it is multi-quadratic-cubic. In particular, $f$ satisfies equation (2.8).

Proof. (i) It is shown in Proposition 2.6 that for each $j$, $f^*_j$ satisfies (1.5). Putting $z = w = 0$ in (2.11), we have $f^*_j(0) = 0$. Letting $z = 0$ in (2.11), we get by the evenness of $f^*_j$ that $f^*_j(2w) = 4f^*_j(w)$ for all $w \in V$. The last equality converts (2.11) to

\[ f^*_j(z + 2w) - f^*_j(z - 2w) = 2[f^*_j(z + w) - f^*_j(z - w)], \]

for all $z, w \in V$. It is seen that (2.14) is the same relation (2.2) from [9]. Repeating the proof of Lemma 2.1 of [9], one can find (1.1) for $f^*_j$.

(ii) Putting $z = 0$ in (2.11) and using the oddness of $f^*_j$, we have $f^*_j(2w) = 8f^*_j(w)$ for all $w \in V$. Applying the last equality in (2.11), we arrive at

\[ f^*_j(z + 2w) - f^*_j(z - 2w) = 2[f^*_j(z + w) - f^*_j(z - w)] + 12f^*_j(w), \]

for all $z, w \in V$. Replacing $(z, w)$ by $(w, z)$ in (2.15), we obtain

\[ f^*_j(2z + w) + f^*_j(2z - w) = 2[f^*_j(z + w) + f^*_j(z - w)] + 12f^*_j(z), \]

for all $z, w \in V$. This completes the proof.

(iii) The result follows from the previous parts. \qed

3. Stability of the multimixed quadratic-cubic functional equations

We first recall some basic facts concerning quasi-$\beta$-normed space.

Definition 3.1. Let $\beta$ be a fixed real number with $0 < \beta < 1$, and $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $X$ be a linear space over $\mathbb{K}$. A quasi-$\beta$-norm is a real valued function on $X$ fulfilling the following conditions

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
2. $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and $t \in \mathbb{K}$;
3. There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

When $\beta = 1$, the norm above is a quasinorm. Recall that $K$ is the modulus of concavity of the norm $\| \cdot \|$. Moreover, if $\| \cdot \|$ is a quasi-$\beta$-norm on $X$, the pair $(X, \| \cdot \|)$ is said to be a quasi-$\beta$-normed space. A quasi-$\beta$-Banach space is a complete quasi-$\beta$-normed space. A quasi-$\beta$-norm $\| \cdot \|$ is called a $(\beta, p)$-norm ($0 < p \leq 1$) if $\|x + y\|^p \leq \|x\|^p + \|y\|^p$, for all $x, y \in X$. In this case, a quasi-$\beta$-Banach space is called a $(\beta, p)$-Banach space.

Given a $p$-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on $X$. By the Aoki-Rolewicz Theorem [25], each quasi-norm is equivalent to some $p$-norm; see also [2]. Since it is much easier to work with $p$-norms, here and subsequently, we restrict our attention mainly to $p$-norms. In this section, by using an idea of Găvruta [16], we prove the stability of (2.4) in quasi-$\beta$-normed spaces. Here, we need the following fundamental lemma which is a main tool to achieve our goal in this section taken from [31, Lemma 3.1].
Lemma 3.2. Let \( j \in \{-1, 1\} \) be fixed, \( a, s \in \mathbb{N} \) with \( a \geq 2 \). Suppose that \( X \) is a linear space, \( Y \) is a \((\beta, p)\)-Banach space with \((\beta, p)\)-norm \( \| \cdot \|_Y \). If \( \psi : X \rightarrow [0, \infty) \) is a function such that there exists an \( L < 1 \) with \( \psi(a^j x) \leq \sum_{k=1}^{n} \phi(z)^{\beta,p} \psi(x) \) for all \( x \in X \) and \( f : X \rightarrow Y \) is a mapping satisfying

\[
\|f(ax) - a^s f(x)\|_Y \leq \psi(x),
\]

for all \( x \in X \), then there exists a uniquely determined mapping \( F : X \rightarrow Y \) such that \( F(ax) = a^s F(x) \) and

\[
\|f(x) - F(x)\|_Y \leq \frac{1}{a^s - 1} \psi(x),
\]

for all \( x \in X \). Moreover, \( F(x) = \lim_{t \rightarrow \infty} \frac{f(a^j x)}{a^{st}} \) for all \( x \in X \).

From now on, for a mapping \( f : V^n \rightarrow W \), we consider the difference operator \( D_{qc} f : V^n \times V^n \rightarrow W \) by

\[
D_{qc} f(x_1, x_2) := \sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f \left( M_{q_1,q_2}^{n} \right) - \sum_{p_1=0}^{n-q_1} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f \left( N_{p_1,p_2}^{n} \right),
\]

where \( f \left( M_{q_1,q_2}^{n} \right) \) and \( f \left( N_{p_1,p_2}^{n} \right) \) are defined in (2.2) and (2.3), respectively. In the sequel, we assume that all mappings \( f : V^n \rightarrow W \) satisfy (have) zero condition.

**Theorem 3.3.** Let \( j \in \{-1, 1\} \) be fixed, \( V \) be a linear space and \( W \) be a \((\beta, p)\)-Banach space with \((\beta, p)\)-norm \( \| \cdot \|_W \) and \( \varphi : V^n \times V^n \rightarrow \mathbb{R}_+ \) be a function such that there exists an \( 0 < L < 1 \) with \( \varphi(2^j x_1, 2^j x_2) \leq 2^{(3n-k)j} L \varphi(x_1, x_2) \) for all \( x_1, x_2 \in V^n \). Suppose that \( f : V^n \rightarrow W \) is an even mapping in each of some \( k \) variables and is odd in each of the other variables and moreover fulfilling the inequality

\[
\|D_{qc} f(x_1, x_2)\|_W \leq \varphi(x_1, x_2),
\]

for all \( x_1, x_2 \in V^n \). Then, there exists a unique solution \( \mathcal{F} : V^n \rightarrow W \) of (2.4) such that

\[
\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{1 - L^j} \frac{1}{3^{k} \times 2^{(3n-k)j}} \varphi(0, x),
\]

for all \( x \in V^n \).

**Proof.** Without loss of generality, we assume that \( f \) is even in the \( k \) first of variables. Replacing \( (x_1, x_2) \) by \((0, x_1)\) in (3.1) and using the assumptions, we have

\[
\|(-3)^k T f(2x) - (-12)^k S f(x)\|_W \leq \varphi(0, x),
\]

for all \( x = x_1 \in V^n \), in which

\[
T = \sum_{q_2=0}^{n-k} \left( \begin{array}{c} n-k \end{array} \right) \left( -3 \right)^{q_2} 2^{n-k-q_2} = (-3 + 2)^{n-k} = (-1)^{n-k},
\]

and

\[
S = \sum_{p_2=0}^{n-k} \left( \begin{array}{c} n-k \end{array} \right) \left( -12 \right)^{p_2} 2^{n-k-p_2} = (4 - 12)^{n-k} = (-8)^{n-k}.
\]

A computational shows that inequality (3.3) is converted to

\[
\|(-3)^k(-1)^{n-k} f(2x) - (-12)^k(-8)^{n-k} f(x)\|_W \leq \varphi(0, x),
\]
for all \( x \in V^n \), and so
\[
\|f(2x) - 2^{3n-k}f(x)\|_W \leq \frac{1}{3k^j} \varphi(0, x),
\]
for all \( x \in V^n \). By Lemma 3.2, there exists a unique mapping \( \mathcal{F}: V^n \to W \) such that \( \mathcal{F}(2x) = 2^{3n-k}\mathcal{F}(x) \) and
\[
\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{|1-L^j|} \frac{1}{3k^j} \varphi(0, x)
\]
for all \( x \in V^n \). It remains to show that \( \mathcal{F} \) satisfies (2.4). Here, we note from Lemma 3.2 that for all \( x \in V^n \), \( \mathcal{F}(x) = \lim_{l \to \infty} \frac{f(2^{j\beta}x)}{2^{(3n-k)j\beta}} \). Now, by (3.1), we have
\[
\left\| \frac{D_q f(2^{j\beta}x_1, 2^{j\beta}x_2)}{2^{(3n-k)j\beta}} \right\|_W \leq 2^{-(3n-k)j\beta} \varphi(2^{j\beta}x_1, 2^{j\beta}x_2)
\leq 2^{-(3n-k)j\beta} (2^{(3n-k)j\beta}L)^j \varphi(x_1, x_2) = L^j \varphi(x_1, x_2),
\]
for all \( x_1, x_2 \in V^n \) and \( l \in \mathbb{N} \). Letting \( l \to \infty \) in the above inequality, we observe that \( D_q \mathcal{F}(x_1, x_2) = 0 \) for all \( x_1, x_2 \in V^n \). This means that \( \mathcal{F} \) satisfies (2.4).

We now have the next stability result for functional equation (2.4) in the special case of Theorem 3.3 when \( f \) is either an even or odd mapping in each of variable.

**Theorem 3.4.** Let \( j \in \{-1, 1\} \) be fixed, \( V \) be a linear space and \( W \) be a \((\beta, p)\)-Banach space with \((\beta, p)\)-norm \( \| \cdot \|_W \). Suppose that \( f: V^n \to W \) is a mapping such that
\[
\|D_q f(x_1, x_2)\|_W \leq \varphi(x_1, x_2),
\]
for all \( x_1, x_2 \in V^n \), where \( \varphi \) is as in Theorem 3.3.

(i) If \( f \) is even in each variable and there exists an \( 0 < L < 1 \) with \( \varphi(2^{j\beta}x_1, 2^{j\beta}x_2) \leq 4^{nj\beta}L \varphi(x_1, x_2) \) for all \( x_1, x_2 \in V^n \), then there exists a unique solution \( \Omega: V^n \to W \) of (2.4) such that
\[
\|f(x) - \Omega(x)\|_W \leq \frac{1}{|1-L^j|} \frac{1}{12^{nj\beta}} \varphi(0, x)
\]
for all \( x \in V^n \). In particular, if \( \Omega \) is even and has the quadratic condition in each variable, then it is multi-quadratic;

(ii) If \( f \) is odd in each variable and there exists an \( 0 < L < 1 \) with \( \varphi(2^{j\beta}x_1, 2^{j\beta}x_2) \leq 8^{nj\beta}L \varphi(x_1, x_2) \) for all \( x_1, x_2 \in V^n \), then there exists a unique solution \( \mathcal{C}: V^n \to W \) of (2.4) such that
\[
\|f(x) - \mathcal{C}(x)\|_W \leq \frac{1}{|1-L^j|} \frac{1}{8^{nj\beta}} \varphi(0, x)
\]
for all \( x \in V^n \). In particular, if \( \mathcal{C} \) is odd and has the cubic condition in each variable, then it is multi-cubic.

**Proof.** The results follow from Theorem 3.3 and Corollary 2.7. \( \square \)

The following corollary is a direct consequence of Theorem 3.4 concerning the stability of (2.4) when the norm of \( D_q f(x_1, x_2) \) is controlled by sum of variables norms of \( x_1 \) and \( x_2 \) with positive powers.

**Corollary 3.5.** Given the positive numbers \( \theta \) and \( \lambda \). Let \( V \) be a quasi-\( \alpha \)-normed space with quasi-\( \alpha \)-norm \( \| \cdot \|_V \), and \( W \) be a \((\beta, p)\)-Banach space with \((\beta, p)\)-norm \( \| \cdot \|_W \).
(i) If $\lambda \neq \frac{2n^2}{\alpha}$ and $f : V^n \rightarrow W$ is an even mapping in each variable fulfilling the inequality

$$\|D_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^{2n} \sum_{j=1}^{n} \|x_{kj}\|^\lambda_V,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\Omega : V^n \rightarrow W$ of (2.4) such that

$$\|f(x) - \Omega(x)\|_W \leq \begin{cases}
\frac{\theta}{8^{n^2}(4^n - 2^n)} \sum_{j=1}^{n} \|x_{1j}\|^\lambda_V & \lambda \in (0, \frac{2n^2}{\alpha}) , \\
\frac{2^n \lambda}{8^{n^2}(2^n \lambda - 4^n)} \sum_{j=1}^{n} \|x_{1j}\|^\lambda_V & \lambda \in (2n^2/\alpha, \infty),
\end{cases}$$

for all $x = x_1 \in V^n$. Moreover, if $\Omega$ is even and has the quadratic condition in each variable, then it is multi-quadratic;

(ii) If $\lambda \neq 3n^2/\alpha$ and $f : V^n \rightarrow W$ is an odd mapping in each variable fulfilling the inequality

$$\|D_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^{2n} \sum_{j=1}^{n} \|x_{kj}\|^\lambda_V,$$

for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \rightarrow W$ of (2.4) such that such that

$$\|f(x) - \mathcal{C}(x)\|_W \leq \begin{cases}
\frac{\theta}{8^{n^2}(2^n - 2^n)} \sum_{j=1}^{n} \|x_{1j}\|^\lambda_V & \lambda \in (0, \frac{3n^2}{\alpha}) , \\
\frac{2^n \lambda}{8^{n^2}(2^n \lambda - 8^n)} \sum_{j=1}^{n} \|x_{1j}\|^\lambda_V & \lambda \in (3n^2/\alpha, \infty),
\end{cases}$$

for all $x = x_1 \in V^n$. In particular, if $\mathcal{C}$ is odd and has the cubic condition in each variable, then it is multi-cubic.

Under some conditions the functional equation (2.4) can be hyperstable as follows.

**Corollary 3.6.** Given the positive number $\theta$ and $p_{ij} > 0$ for $i \in \{1, 2\}$, $j \in \{1, \ldots, n\}$. Let $V$ be a quasi-$\alpha$-normed space with quasi-$\alpha$-norm $\| \cdot \|_V$, and $W$ be a $(\beta, p)$-Banach space with $(\beta, p)$-norm $\| \cdot \|_W$.

(i) If $\sum_{i=1}^{2n} \sum_{j=1}^{n} p_{ij} \neq \frac{2n^2}{\alpha}$ and $f : V^n \rightarrow W$ is an even mapping and has the quadratic condition in each variable fulfilling the inequality

$$\|D_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^{2n} \prod_{j=1}^{n} \|x_{ij}\|^{p_{ij}},$$

for all $x_1, x_2 \in V^n$, then it is multi-quadratic;

(ii) If $\sum_{i=1}^{2n} \sum_{j=1}^{n} p_{ij} \neq \frac{3n^2}{\alpha}$ and $f : V^n \rightarrow W$ is an odd mapping and has the cubic condition in each variable fulfilling the inequality

$$\|D_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^{2n} \prod_{j=1}^{n} \|x_{ij}\|^{p_{ij}},$$

for all $x_1, x_2 \in V^n$, then it is multi-cubic.

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