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Approximation and Analysis Regarding the Structure of a Multiple Variable Mapping

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ABSTRACT: The article introduces a several variables mapping as the multimixed quadratic-cubic mapping in order to characterize such mappings. It reduces a system of equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. It is shown that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multiquadratic-cubic. Further, the generalized Hyers-Ulam stability and hyperstabilty for multimixed quadratic-cubic functional equations in quasi-β-normed spaces have been investigated.

Key Words: Hyers-Ulam stability, multi-quadratic mapping, multi-cubic mapping, multiquadratic-cubic mapping, multimixed quadratic-cubic mapping.

Contents

1. Introduction

The stability problem for functional equations, which Ulam [30] proposed for group homomorphisms, has been answered and explored for multiple variable mappings in recent decades. We recall that a functional equation Γ is said to be *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution. Moreover, Γ is called *hyperstable* if any function f satisfying the equation Γ approximately (in some senses) is actually a solution for it; for some stability results in one variable mappings and functional equations see for instance the papers and books [13], [19], [24], [26], [29] and references therein.

We now state some basic notions and developments about the structure and the stability of several variables mappings. Let V be a commutative group, W be a linear space, and $n \geq 2$ be an integer. A mapping $f : V^n \longrightarrow W$ is called

• *multi-additive* if it is additive (satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$) in each variable.

• *multi-quadratic* if it fulfills the quadratic functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.1}$$

in each variable [11]. A lot of information about the structure of multi-additive mappings and their Ulam stabilities are available in [10], [12] and [20, Sections 13.4 and 17.2]. C.-G. Park was the first author who studied the stability of multi-quadratic in the setting of Banach algebras in [22]. After that, Ciepliński [11] studied the generalized Hyers-Ulam stability of multi-quadratic mappings in Banach spaces. Zhao et al. [32] described the structure of multi-quadratic mappings and in fact showed that a mapping $f : V^n \longrightarrow W$ is multi-quadratic if and only if the equation

$$\sum_{s\in\{-1,1\}^n} f(x_1 + sx_2) = 2^n \sum_{j_1,j_2,\ldots,j_n\in\{1,2\}} f(x_{1j_1}, x_{2j_2}, \ldots, x_{nj_n}) \tag{1.2}$$

holds, where $x_j = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. Various versions of multi-quadratic mappings and their stability can be found in [7] and [28]. For the structure of multi-additive-quadratic, we refer to [1].

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Ghaemi et al. [15] introduced the multi-cubic mappings and then for a special case of such mappings have been studied in [8]. In fact, a mapping $f : V^n \longrightarrow W$ is called *multi-cubic* if it is cubic in each variable, i.e., satisfies the equation

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x) \tag{1.3}$$

in each variable [17]. In [8], the authors unified the system of functional equations defining a multi-cubic mapping to a single equation, namely, the multi-cubic functional equation. Furthermore, the general system of cubic functional equations which is defined in [15], characterized as a single equation in [14]. Other forms of cubic functional equations for instance are available in [3] and [23]. In [8], it is shown that every multi-cubic functional equation is stable and moreover such functional equations under some conditions can be hyperstable; for the miscellaneous versions of multi-cubic mappings and their stabilities in non-Archimedean normed and modular spaces, we refer to [14] and [21], respectively.

Chang and Jung [9] introduced the following mixed type quadratic and cubic functional equation

$$6f(x+y)-6f(x-y)+4f(3y)=3f(x+2y)-3f(x-2y)+9f(2y). \tag{1.4}$$

They established the general solution of the functional equation (1.4) and investigated the Hyers-Ulam stability of this equation; for a different form of mixed type quadratic-cubic functional equation, one can see [18].

The following mixed type quadratic-cubic functional was considered in [27] which is somewhat different from (1.4) as follows:

$$f(x+2y)-f(x-2y)=2[f(x+y)-f(x-y)]+3f(2y)-12f(y). \tag{1.5}$$

It is easily verified that the function $f(x)=ax^2+bx^3$ is a solution of equations (1.4) and (1.5). Recently, the first author and Mitrović [6] have studied the structure of multimixed quadratic-cubic mappings and established $\varepsilon-$stability (Hyers' stability) of such mappings in Banach spaces setting by applying an alternative fixed point method.

Motivated by equation (1.5), in this paper, we define multimixed quadratic-cubic mappings and present a characterization of such mappings. In other words, we reduce the system of n equations defining the multimixed quadratic-cubic mappings to obtain a single functional equation. We also show that under some mild conditions, every multimixed quadratic-cubic mapping can be multi-quadratic, multi-cubic and multiquadratic-cubic. We also prove the generalized Hyers-Ulam stability and hyperstabilty for multimixed quadratic-cubic functional equations in quasi-β-normed spaces.

2. Characterization of the multimixed quadratic-cubic mappings

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are the set of all positive integers, integers and rational numbers, respectively, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}, \mathbb{R}_+ := [0,\infty)$. For any $l \in \mathbb{N}_0, n \in \mathbb{N}$, $t=(t_1,\dots,t_n)\in\{-1,1\}^n$ and $x=(x_1,\dots,x_n)\in V^n$ we write $lx := (lx_1,\dots,lx_n)$ and $tx := (t_1x_1,\dots,t_nx_n)$, where lx stands, as usual, for the scaler product of l on x in the commutative group $(V,+)$.

Let V and W be linear spaces, $n\in\mathbb{N}$ and $k\in\{0,\dots,n\}$. Put

$$\mathbf{n} := \{1,\dots,n\}. \tag{2.1}$$

Each subset of $\mathbf{n}$ with m elements is denoted by $(m)_{\mathbf{n}}$. Recall from [5] that a mapping $f : V^n \longrightarrow W$ is called k-quadratic and $n-k$-cubic (briefly, multiquadratic-cubic) if f satisfies the following functional equations system.

$$\begin{cases} f(v_1,\dots,v_{i-1},v_i+v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},v_i-v_i',v_{i+1},\dots,v_n) \\ \quad =2f(v_1,\dots,v_n)+2f(v_1,\dots,v_i',\dots,v_n), \qquad i\in(k)_{\mathbf{n}}, \\ \\ f(v_1,\dots,v_{i-1},2v_i+v_i',v_{i+1},\dots,v_n)+f(v_1,\dots,v_{i-1},2v_i-v_i',v_{i+1},\dots,v_n) \\ \quad =2f(v_1,\dots,v_{i-1},v_i+v_i',v_{i+1},\dots,v_n)+2f(v_1,\dots,v_{i-1},v_i-v_i',v_{i+1},\dots,v_n) \\ \quad +12f(v_1,\dots,v_n), \qquad i\in(n-k)_{\mathbf{n}}. \end{cases}$$

Note that we can suppose for simplicity that f is quadratic in each of the first k variables, but one can obtain analogous results without this assumption. Let us note that for $k = n$ ($k = 0$), the above definition leads to the so-called multi-quadratic (multi-cubic) mappings; some basic facts on such mappings can be found for instance in [8] and [32].

Definition 2.1. *Let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$. A several variables mapping $f : V^n \longrightarrow W$ is called n-mixed quadratic-cubic or briefly multimixed quadratic-cubic if f fulfills (1.5) in each of its n arguments, that is*

$$\begin{aligned} &f(v_1, \ldots, v_{i-1}, v_i + 2v'_i, v_{i+1}, \ldots, v_n) - f(v_1, \ldots, v_{i-1}, v_i - 2v'_i, v_{i+1}, \ldots, v_n) \\ &\qquad - 3f(v_1, \ldots, v_{i-1}, 2v'_i, v_{i+1}, \ldots, v_n) \\ &= 2[f(v_1, \ldots, v_{i-1}, v_i + v'_i, v_{i+1}, \ldots, v_n) - f(v_1, \ldots, v_{i-1}, v_i - v'_i, v_{i+1}, \ldots, v_n)] \\ &\qquad - 12f(v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_n). \end{aligned}$$

Let $n \in \mathbb{N}$ with $n \geq 2$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1, 2\}$. We will write x_i^n simply x_i when no confusion can arise. For $x_1, x_2 \in V^n$, set

$$\mathcal{M}^n = \{\mathfrak{M}_n = (M_1, \ldots, M_n) |\ M_j \in \{x_{1j} \pm 2x_{2j}, 2x_{2j}\}\},$$

and

$$\mathcal{N}^n = \{\mathfrak{N}_n = (N_1, \ldots, N_n) |\ N_j \in \{x_{1j} \pm x_{2j}, x_{2j}\}\},$$

for all $j \in \{1, \ldots, n\}$. For $p_i, q_i \in \mathbb{N}_0$ with $0 \leq p_i, q_i \leq n$, consider the subsets $\mathcal{M}^n_{(q_1,q_2)}$ and $\mathcal{N}^n_{(p_1,p_2)}$ of $\mathcal{M}^n$ and $\mathcal{N}^n$, respectively, as follows:

$$\mathcal{M}^n_{(q_1,q_2)} := \{\mathfrak{M}_n \in \mathcal{M}^n |\ \mathrm{Card}\{M_j : M_j = x_{1j} - 2x_{2j}\} = q_1, \mathrm{Card}\{M_j : M_j = x_{2j}\} = q_2\},$$

$$\mathcal{N}^n_{(p_1,p_2)} := \{\mathfrak{N}_n \in \mathcal{N}^n |\ \mathrm{Card}\{N_j : N_j = x_{1j} - x_{2j}\} = p_1, \mathrm{Card}\{N_j : N_j = x_{2j}\} = p_2\}.$$

Hereafter, for a multimixed quadratic-cubic mappings f, we use the following notations:

$$f\left(\mathcal{M}^n_{(q_1,q_2)}\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}^n_{(q_1,q_2)}} f(\mathfrak{M}_n), \tag{2.2}$$

$$f\left(\mathcal{M}^n_{(q_1,q_2)}, z\right) := \sum_{\mathfrak{M}_n \in \mathcal{M}^n_{(q_1,q_2)}} f(\mathfrak{M}_n, z) \qquad (z \in V),$$

$$f\left(\mathcal{N}^n_{(p_1,p_2)}\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}^n_{(p_1,p_2)}} f(\mathfrak{N}_n), \tag{2.3}$$

and

$$f\left(\mathcal{N}^n_{(p_1,p_2)}, z\right) := \sum_{\mathfrak{N}_n \in \mathcal{N}^n_{(p_1,p_2)}} f(\mathfrak{N}_n, z) \qquad (z \in V).$$

For each $x_1, x_2 \in V^n$, we consider the equation

$$\begin{aligned} &\sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1} (-3)^{q_2} f\left(\mathcal{M}^n_{(q_1,q_2)}\right) \\ &\qquad = \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2} (-2)^{p_1} (-12)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}\right), \end{aligned} \tag{2.4}$$

where $f\left(\mathcal{M}^n_{(q_1,q_2)}\right)$ and $f\left(\mathcal{N}^n_{(p_1,p_2)}\right)$ are defined in (2.2) and (2.3), respectively.

We recall that the binomial coefficient for all $n, r \in \mathbb{N}_0$ with $n \geq r$ is defined and denoted by $\binom{n}{r} := \frac{n!}{r!(n-r)!}$.

Definition 2.2. *Let $r \in \mathbb{N}$. We say the mapping $f : V^n \longrightarrow W$*

(i) *satisfies (has) the r-power condition in the jth variable if*

$$f(z_1, \ldots, z_{j-1}, 2z_j, z_{j+1}, \ldots, z_n) = 2^r f(z_1, \cdots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n),$$

for all $z_1, \ldots, z_n \in V^n$. In particular, 2-power and 3-power conditions are called quadratic and cubic condition, respectively.

(ii) *has zero condition if $f(x) = 0$ for any $x \in V^n$ with at least one component which is equal to zero.*

(iii) *is odd in the jth variable if*

$$f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = -f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).$$

(iv) *is even in the jth variable if*

$$f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).$$

Here, we bring an elementary lemma from [4].

Lemma 2.3. *Let $n, k, p_l \in \mathbb{N}_0$, such that $k + \sum_{l=1}^m p_l \leq n$, where $l \in \{1, \ldots, m\}$. Then*

$$\begin{aligned}&\binom{n-k}{n-k-\sum_{l=1}^m p_l}\binom{\sum_{l=1}^m p_l}{\sum_{l=1}^{m-1} p_l}\cdots\binom{p_1+p_2}{p_1}\\ &\qquad = \binom{n-k}{p_1}\binom{n-k-p_1}{p_2}\cdots\binom{n-k-\sum_{l=1}^{m-1} p_l}{p_m}.\end{aligned}$$

Consider $\mathbf{n}$ as in (2.1). For a subset $T = \{j_1, \ldots, j_i\}$ of $\mathbf{n}$ with $1 \leq j_1 < \cdots < j_i \leq n$ and $x = (x_1, \ldots, x_n) \in V^n$,

$$_T x := (0, \ldots, 0, x_{j_1}, 0, \ldots, 0, x_{j_i}, 0, \ldots, 0) \in V^n$$

denotes the vector which coincides with x in exactly those components, which are indexed by the elements of T and whose other components are set equal zero. Note that $_0x = 0$, $_{\mathbf{n}}x = x$. We use these notations in the proof of upcoming lemma.

Next, we reduce the system of n equations defining the multimixed quadratic-cubic mapping in obtaining the single functional equation (2.4). For doing this, we need the next lemma.

Lemma 2.4. *If a mapping $f : V^n \longrightarrow W$ satisfies equation (2.4), then it has zero condition.*

Proof. We argue by induction on k that $f(_kx) = 0$, when $0 \leq k \leq n-1$. Putting $x_1 = x_2 = {}_0x$ in (2.4), we have

$$\begin{aligned}&\left[\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}\binom{n}{n-q_1-q_2}\binom{q_1+q_2}{q_2}(-1)^{q_1}(-3)^{q_2}\right] f(_0x)\\ &= \left[\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_2}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}\right] f(_0x).\end{aligned}\tag{2.5}$$

Here we compute the the left side of (2.5). Using Lemma 2.3 for $k = 0$, we have

$$\begin{aligned}&\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}\binom{n}{n-q_1-q_2}\binom{q_1+q_2}{q_2}(-1)^{q_1}(-3)^{q_2}\\ &\quad = \sum_{q_1=0}^{n}\binom{n}{q_1}(-1)^{q_1}\sum_{q_2=0}^{n-q_1}\binom{n-q_1}{q_2}1^{n-q_1-q_2}(-3)^{q_2}\\ &\quad = \sum_{q_1=0}^{n}\binom{n}{q_1}(-1)^{q_1}(-2)^{n-q_1} = (-1-2)^n = (-3)^n.\end{aligned}\tag{2.6}$$

Similarly, one can show from Lemma 2.3 that

$$\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\binom{n}{n-p_1-p_2}\binom{p_1+p_2}{p_2}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}=(-12)^n. \tag{2.7}$$

It follows from relations (2.5), (2.6) and (2.7) that $f({}_0x)=0$. Assume that $f({}_{k-1}x)=0$ for any $k\in\{1,\dots,n-1\}$. We show that $f({}_kx)=0$. Without loss of generality, we assume that the first k variables are non-zero. By our assumption, replacing (x_1,x_2) by $({}_kx_1,0)$ in equation (2.4), we have

$$\begin{aligned}&\left[\sum_{q_1=0}^{n-k}\sum_{q_2=0}^{n-k-q_1}\binom{n-k}{n-k-q_1-q_2}\binom{q_1+q_2}{q_2}(-1)^{q_1}(-3)^{q_2}\right]f({}_kx)\\ &=\left[\sum_{p_1=0}^{n-k}\sum_{p_2=0}^{n-k-p_1}\binom{n-k}{n-k-p_1-p_2}\binom{p_1+p_2}{p_2}2^{n-k-p_1-p_2}(-2)^{p_1}(-12)^{p_2}\right]f({}_kx).\end{aligned}$$

Similar the above and by using lemma 2.3, we can obtain $(-3)^{n-k}f({}_kx)=(-12)^{n-k}f({}_kx)$, and this implies that $f({}_kx)=0$. This finishes the proof. □

In the upcoming results which are our aim in this section, we unify the general system of quadratic-cubic functional equations defining a multimixed quadratic-cubic mapping to an equation and indeed this functional equation describe a multimixed quadratic-cubic mapping.

Proposition 2.5. *If a mapping $f:V^n\longrightarrow W$ is multimixed quadratic-cubic, then it satisfies equation* (2.4).

Proof. We proceed the proof by induction on n, and in fact we show that equation (2.4) is valid for f. Clearly, f satisfies equation (1.5) and this guarantees the assertion for $n=1$. If (2.4) holds for some positive integer $n>1$, then

$$\begin{aligned}
&\sum_{q_1=0}^{n+1}\sum_{q_2=0}^{n+1-q_1}(-1)^{q_1}(-3)^{q_2}f\left(\mathcal{M}^{n+1}_{(q_1,q_2)}\right)=\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}(-1)^{q_1}(-3)^{q_2}f\left(\mathcal{M}^{n}_{(q_1,q_2)},x_{1,n+1}+2x_{2,n+1}\right)\\
&\quad-\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}(-1)^{q_1}(-3)^{q_2}f\left(\mathcal{M}^{n}_{(q_1,q_2)},x_{1,n+1}-2x_{2,n+1}\right)-3\sum_{q_1=0}^{n}\sum_{q_2=0}^{n-q_1}(-1)^{q_1}(-3)^{q_2}f\left(\mathcal{M}^{n}_{(q_1,q_2)},2x_{2,n+1}\right)\\
&=\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{1,n+1}+2x_{2,n+1}\right)\\
&\quad-\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{1,n+1}-2x_{2,n+1}\right)\\
&\quad-3\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},2x_{2,n+1}\right)\\
&=2\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{1,n+1}+x_{2,n+1}\right)\\
&\quad-2\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{1,n+1}-x_{2,n+1}\right)\\
&\quad-12\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n}_{(p_1,p_2)},x_{2,n+1}\right)\\
&=\sum_{p_1=0}^{n+1}\sum_{p_2=0}^{n+1-p_1}2^{n+1-p_1-p_2}(-2)^{p_1}(-12)^{p_2}f\left(\mathcal{N}^{n+1}_{(p_1,p_2)}\right).
\end{aligned}$$

This means that (2.4) holds for $n+1$. □

It follows from Proposition 2.5 and by a mathematical computation that the mapping $f(z_1,\dots,z_n)=\prod_{j=1}^n(\alpha_jz_j^2+\beta_jz_j^3)$ satisfies (2.4) and so this equation is said to be *multimixed quadratic-cubic* functional equation.

It is shown in [5, Proposition 2.1] that if a mapping $f:V^n\longrightarrow W$ is k-quadratic and $n-k$-cubic (multiquadratic-cubic) mapping, then f satisfies equation

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-1,1\}^{n-k}}f\left(x_1^k+sx_2^k,2x_1^{n-k}+tx_2^{n-k}\right)=2^k\sum_{m=0}^{n-k}2^{n-k-m}12^m\sum_{i\in\{1,2\}}f\left(x_i^k,\mathbb{M}_m^{n-k}\right),\tag{2.8}$$

for all $x_i^k=(x_{i1},\dots,x_{ik})\in V^k$ and $x_i^{n-k}=(x_{i,k+1}\dots,x_{in})\in V^{n-k}$ where $i\in\{1,2\}$ in which

$$f\left(x_i^k,\mathbb{M}_m^{n-k}\right):=\sum_{\mathfrak{N}_n\in\mathcal{M}_m^{n-k}}f\left(x_i^k,\mathfrak{N}_n\right),$$

whereas

$$\mathbb{M}^{n-k}=\{\mathfrak{N}_n=(N_{k+1},\dots,N_n)|\ N_j\in\{x_{1j}\pm x_{2j},x_{1j}\}\}$$

and

$$\mathbb{M}_m^{n-k}:=\left\{\mathfrak{N}_n=(N_{k+1},\dots,N_n)\in\mathbb{M}^{n-k}|\ \mathrm{Card}\{N_j:\ N_j=x_{1j}\}=m\right\}.$$

Note that in the the case $k=n$ and $k=0$, equation (2.8) converts to (1.2) and

$$\sum_{t\in\{-1,1\}^n}f\left(2x_1^n+tx_2^n\right)=\sum_{m=0}^{n}2^{n-m}12^mf\left(\mathbb{M}_m^n\right),\tag{2.9}$$

respectively. In addition, it is proved in [32, Theorem 2] (resp., [8, Proposition 2.2]) that if the mapping $f : V^n \longrightarrow W$ is multi-quadratic (resp. multi-cubic), then it satisfies the equation (1.2) (resp., (2.9)).

Proposition 2.6. *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation (2.4). Under one of the following conditions, it is multimixed quadratic-cubic.*

(i) *f is even in each variable and satisfies the quadratic condition for all variables;*

(ii) *f is odd in each variable and satisfies the cubic condition for all variables.*

Proof. (i) Let $j \in \{1, \dots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f(z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n).$$

Putting $x_{1k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$, $x_{1j} = z$ and $x_2 = (z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_n)$ in (2.4), using Lemma 2.4, we get

$$\begin{aligned}(-3)^{n-1}[&f(2z_1, \dots, 2z_{j-1}, z + 2w, 2z_{j+1}, \dots, 2z_n)\\ &- f(2z_1, \dots, 2z_{j-1}, z - 2w, 2z_{j+1}, \dots, 2z_n) - 3f(2z_1, \dots, 2z_{j-1}, 2w, 2z_{j+1}, \dots, 2z_n)]\\ &= (-12)^{n-1}[2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w)]. \end{aligned} \tag{2.10}$$

Our assumption (2.10) converts to

$$\begin{aligned}2^{2(n-1)}(-3)^{n-1}&[f_j^*(z + 2w) - f_j^*(z - 2w) - 3f_j^*(2w)]\\ &= (-12)^{n-1}[2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w)],\end{aligned}$$

and so

$$f_j^*(z + 2w) - f_j^*(z - 2w) - 3f_j^*(2w) = 2f_j^*(z + w) - 2f_j^*(z - w) - 12f_j^*(w). \tag{2.11}$$

This finishes the proof of part (i).

(ii) Similar to the proof of part (i), Putting $x_{1k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$, $x_{1j} = z$ and $x_2 = (z_1, \dots, z_{j-1}, w, z_{j+1}, \dots, z_n)$ in (2.4), using our assumptions and Lemma 2.4, we obtain the left side of (2.4) as follows:

$$\begin{aligned}&\left[\sum_{q_2=0}^{n-1} \binom{n-1}{q_2} 2^{3(n-1-q_2)}(-3)^{q_2} 2^{n-1-q_2} 2^{3q_2}\right] (f_j^*(z + 2w) - f_j^*(z - 2w))\\ &+ \left[\sum_{q_2=1}^{n} \binom{n-1}{q_2-1} 2^{3(n-1)}(-3)^{q_2} 2^{n-q_2}\right] f_j^*(w)\\ &= 2^{3(n-1)}(-1)^{n-1}(f_j^*(z + 2w) - f_j^*(z - 2w)) - 3 \times 2^{3(n-1)}(-1)^{n-1} f_j^*(2w). \end{aligned} \tag{2.12}$$

On the other hand, the right side of (2.4) will be

$$\begin{aligned}&\left[\sum_{p_2=0}^{n-1} \binom{n-1}{p_2} 2^{n-1-p_2}(-12)^{p_2} 2^{n-p_2}\right] (f_j^*(z + w) - f_j^*(z - w))\\ &+ \left[\sum_{p_2=1}^{n} \binom{n-1}{p_2-1} 2^{n-p_2}(-12)^{p_2} 2^{n-p_2}\right] f_j^*(w)\\ &= 2\left[\sum_{p_2=0}^{n-1} \binom{n-1}{p_2} 4^{n-1-p_2}(-12)^{p_2}\right] (f_j^*(z + w) - f_j^*(z - w))\\ &- 12\left[\sum_{p_2=0}^{n} \binom{n-1}{p_2} 4^{n-1-p_2}(-12)^{p_2}\right] f_j^*(w)\\ &= 2(-8)^{n-1}[f_j^*(z + w) - f_j^*(z - w)] - 12(-8)^{n-1} f_j^*(w). \end{aligned} \tag{2.13}$$

Comparing (2.12) and (2.13), we achieve (2.11). □

Corollary 2.7. *Suppose that a mapping $f: V^n \longrightarrow W$ satisfies equation (2.4).*

(i) *If f is even in each variable and satisfies the quadratic condition in all variables, then it is multi-quadratic. Moreover, f satisfies equation (1.2);*

(ii) *If f is odd in each variable and satisfies the cubic condition in all variables, then it is multi-cubic. In addition, equation (2.9) is valid for f;*

(iii) *If f is even in each of some k variables with the quadratic condition and is odd in each of the other variables with the cubic condition, then it is multiquadratic-cubic. In particular, f satisfies equation (2.8).*

Proof. (i) It is shown in Proposition 2.6 that for each j, f_j^* satisfies (1.5). Putting $z=w=0$ in (2.11), we have $f_j^*(0)=0$. Letting $z=0$ in (2.11), we get by the evenness of f_j^* that $f_j^*(2w)=4f_j^*(w)$ for all $w\in V$. The last equality converts (2.11) to

$$f_j^*(z+2w)-f_j^*(z-2w)=2[f_j^*(z+w)-f_j^*(z-w)], \tag{2.14}$$

for all $z,w\in V$. It is seen that (2.14) is the same relation (2.2) from [9]. Repeating the proof of Lemma 2.1 of [9], one can find (1.1) for f_j^*.

(ii) Putting $z=0$ in (2.11) and using the oddness of f_j^*, we have $f_j^*(2w)=8f_j^*(w)$ for all $w\in V$. Applying the last equality in (2.11), we arrive at

$$f_j^*(z+2w)-f_j^*(z-2w)=2[f_j^*(z+w)-f_j^*(z-w)]+12f_j^*(w), \tag{2.15}$$

for all $z,w\in V$. Replacing (z,w) by (w,z) in (2.15), we obtain

$$f_j^*(2z+w)+f_j^*(2z-w)=2[f_j^*(z+w)+f_j^*(z-w)]+12f_j^*(z),$$

for all $z,w\in V$. This completes the proof.

(iii) The result follows from the previous parts. □

3. Stability of the multimixed quadratic-cubic functional equations

We first recall some basic facts concerning quasi-β-normed space.

Definition 3.1. *Let β be a fixed real number with $0<\beta<1$, and $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm is a real valued function on X fulfilling the following conditions*

(1) $\|x\|\geq 0$ *for all* $x\in X$ *and* $\|x\|=0$ *if and only if* $x=0$;

(2) $\|tx\|=|t|^\beta\|x\|$ *for all* $x\in X$ *and* $t\in\mathbb{K}$;

(3) *There is a constant* $K\geq 1$ *such that* $\|x+y\|\leq K(\|x\|+\|y\|)$ *for all* $x,y\in X$.

When $\beta=1$, the norm above is a quasinorm. Recall that K is the *modulus of concavity* of the norm $\|\cdot\|$. Moreover, if $\|\cdot\|$ is a quasi-β-norm on X, the pair $(X,\|\cdot\|)$ is said to be a quasi-β-normed space. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $\|\cdot\|$ is called a (β,p)-norm $(0<p\leq 1)$ if $\|x+y\|^p\leq\|x\|^p+\|y\|^p$, for all $x,y\in X$. In this case, a quasi-β-Banach space is called a (β,p)-Banach space.

Given a p-norm, the formula $d(x,y):=\|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [25], each quasi-norm is equivalent to some p-norm; see also [2]. Since it is much easier to work with p-norms, here and subsequently, we restrict our attention mainly to p-norms. In this section, by using an idea of Găvruţa [16], we prove the stability of (2.4) in quasi-β-normed spaces. Here, we need the following fundamental lemma which is a main tool to achieve our goal in this section taken from [31, Lemma 3.1].

Lemma 3.2. *Let $j \in \{-1, 1\}$ be fixed, $a, s \in \mathbb{N}$ with $a \geq 2$. Suppose that X is a linear space, Y is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. If $\psi : X \longrightarrow [0, \infty)$ is a function such that there exists an $L < 1$ with $\psi(a^j x) < La^{js\beta}\psi(x)$ for all $x \in X$ and $f : X \longrightarrow Y$ is a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \psi(x),$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \longrightarrow Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}|1 - L^j|}\psi(x),$$

for all $x \in X$. Moreover, $F(x) = \lim_{l\to\infty} \frac{f(a^{jl}x)}{a^{jls}}$ for all $x \in X$.

From now on, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathbf{D}_{qc}f : V^n \times V^n \longrightarrow W$ by

$$\begin{aligned}\mathbf{D}_{qc}f(x_1, x_2) := &\sum_{q_1=0}^{n} \sum_{q_2=0}^{n-q_1} (-1)^{q_1}(-3)^{q_2} f\left(\mathcal{M}^n_{(q_1,q_2)}\right) \\ &- \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} 2^{n-p_1-p_2}(-2)^{p_1}(-12)^{p_2} f\left(\mathcal{N}^n_{(p_1,p_2)}\right),\end{aligned}$$

where $f\left(\mathcal{M}^n_{(q_1,q_2)}\right)$ and $f\left(\mathcal{N}^n_{(p_1,p_2)}\right)$ are defined in (2.2) and (2.3), respectively. In the sequel, we assume that all mappings $f : V^n \longrightarrow W$ satisfy (have) zero condition.

Theorem 3.3. *Let $j \in \{-1, 1\}$ be fixed, V be a linear space and W be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_W$ and $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ be a function such that there exists an $0 < L < 1$ with $\varphi(2^j x_1, 2^j x_2) \leq 2^{(3n-k)j\beta} L\varphi(x_1, x_2)$ for all $x_1, x_2 \in V^n$. Suppose that $f : V^n \longrightarrow W$ is an even mapping in each of some k variables and is odd in each of the other variables and moreover fulfilling the inequality*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \varphi(x_1, x_2), \tag{3.1}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $\mathcal{F} : V^n \longrightarrow W$ of (2.4) such that

$$\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{|1 - L^j|} \frac{1}{3^{k\beta} \times 2^{(3n-k)\beta}} \varphi(0, x), \tag{3.2}$$

for all $x \in V^n$.

Proof. Without loss of generality, we assume that f is even in the k first of variables. Replacing (x_1, x_2) by $(0, x_1)$ in (3.1) and using the assumptions, we have

$$\left\|(-3)^k\, Tf(2x) - (-12)^k\, Sf(x)\right\|_W \leq \varphi(0, x), \tag{3.3}$$

for all $x = x_1 \in V^n$, in which

$$T = \sum_{q_2=0}^{n-k} \binom{n-k}{n-k-q_2} (-3)^{q_2} 2^{n-k-q_2} = (-3 + 2)^{n-k} = (-1)^{n-k}$$

and

$$S = \sum_{p_2=0}^{n-k} \binom{n-k}{n-k-p_2} 2^{n-k-p_2} 2^{n-k-p_2} (-12)^{p_2} = (4 - 12)^{n-k} = (-8)^{n-k}.$$

A computational shows that inequality (3.3) is converted to

$$\left\|(-3)^k (-1)^{n-k} f(2x) - (-12)^k\, (-8)^{n-k} f(x)\right\|_W \leq \varphi(0, x),$$

for all $x \in V^n$, and so

$$\left\|f(2x) - 2^{3n-k} f(x)\right\|_W \leq \frac{1}{3^{k\beta}}\varphi(0,x),$$

for all $x \in V^n$. By Lemma 3.2, there exists a unique mapping $\mathcal{F} : V^n \longrightarrow W$ such that $\mathcal{F}(2x) = 2^{3n-k}\mathcal{F}(x)$ and

$$\|f(x) - \mathcal{F}(x)\|_W \leq \frac{1}{|1-L^j|}\frac{1}{3^{k\beta} \times 2^{(3n-k)\beta}}\varphi(0,x),$$

for all $x \in V^n$. It remains to show that $\mathcal{F}$ satisfies (2.4). Here, we note from Lemma 3.2 that for all $x \in V^n$, $\mathcal{F}(x) = \lim_{l\to\infty} \frac{f(2^{jl}x)}{2^{(3n-k)jl}}$. Now, by (3.1), we have

$$\begin{aligned}\left\|\frac{\mathbf{D}_{qc}f(2^{jl}x_1, 2^{jl}x_2)}{2^{(3n-k)jl}}\right\|_W &\leq 2^{-(3n-k)jl\beta}\varphi(2^{jl}x_1, 2^{jl}x_2)\\ &\leq 2^{-(3n-k)jl\beta}(2^{(3n-k)j\beta}L)^l\varphi(x_1,x_2) = L^l\varphi(x_1,x_2),\end{aligned}$$

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}$. Letting $l \to \infty$ in the above inequality, we observe that $\mathbf{D}_{qc}\mathcal{F}(x_1,x_2) = 0$ for all $x_1, x_2 \in V^n$. This means that $\mathcal{F}$ satisfies (2.4). □

We now have the next stability result for functional equation (2.4) in the special case of Theorem 3.3 when f is either an even or odd mapping in each of variable.

Theorem 3.4. *Let $j \in \{-1,1\}$ be fixed, V be a linear space and W be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_W$. Suppose that $f : V^n \longrightarrow W$ is a mapping such that*

$$\|\mathbf{D}_{qc}f(x_1,x_2)\|_W \leq \varphi(x_1,x_2),$$

for all $x_1, x_2 \in V^n$, where φ is as in Theorem 3.3.

(i) *If f is even in each variable and there exists an $0 < L < 1$ with $\varphi(2^jx_1, 2^jx_2) \leq 4^{nj\beta}L\varphi(x_1,x_2)$ for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{Q} : V^n \longrightarrow W$ of (2.4) such that*

$$\|f(x) - \mathcal{Q}(x)\|_W \leq \frac{1}{|1-L^j|}\frac{1}{12^{n\beta}}\varphi(0,x)$$

for all $x \in V^n$. In particular, if $\mathcal{Q}$ is even and has the quadratic condition in each variable, then it is multi-quadratic;

(ii) *If f is odd in each variable and there exists an $0 < L < 1$ with $\varphi(2^jx_1, 2^jx_2) \leq 8^{nj\beta}L\varphi(x_1,x_2)$ for all $x_1, x_2 \in V^n$, then there exists a unique solution $\mathcal{C} : V^n \longrightarrow W$ of (2.4) such that*

$$\|f(x) - \mathcal{C}(x)\| \leq \frac{1}{|1-L^j|}\frac{1}{8^{n\beta}}\varphi(0,x)$$

for all $x \in V^n$. In particular, if $\mathcal{C}$ is odd and has the cubic condition in each variable, then it is multi-cubic.

Proof. The results follow from Theorem 3.3 and Corollary 2.7. □

The following corollary is a direct consequence of Theorem 3.4 concerning the stability of (2.4) when the norm of $\mathbf{D}_{qc}f(x_1,x_2)$ is controlled by sum of variables norms of x_1 and x_2 with positive powers.

Corollary 3.5. *Given the positive numbers θ and λ. Let V be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_V$, and W be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_W$.*

(i) *If* $\lambda \neq 2n\frac{\beta}{\alpha}$ *and* $f : V^n \longrightarrow W$ *is an even mapping in each variable fulfilling the inequality*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|_V^{\lambda},$$

for all $x_1, x_2 \in V^n$*, then there exists a unique solution* $\mathcal{Q} : V^n \longrightarrow W$ *of* (2.4) *such that*

$$\|f(x) - \mathcal{Q}(x)\|_W \leq \begin{cases} \frac{\theta}{3^{n\beta}(4^{n\beta}-2^{\alpha\lambda})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda} & \lambda \in \left(0, 2n\frac{\beta}{\alpha}\right), \\ \frac{2^{\alpha\lambda}\theta}{12^{n\beta}(2^{\alpha\lambda}-4^{n\beta})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda} & \lambda \in \left(2n\frac{\beta}{\alpha}, \infty\right), \end{cases}$$

for all $x = x_1 \in V^n$*. Moreover, if* $\mathcal{Q}$ *is even and has the quadratic condition in each variable, then it is multi-quadratic;*

(ii) *If* $\lambda \neq 3n\frac{\beta}{\alpha}$ *and* $f : V^n \longrightarrow W$ *is an odd mapping in each variable fulfilling the inequality*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \sum_{k=1}^{2}\sum_{j=1}^{n} \|x_{kj}\|_V^{\lambda},$$

for all $x_1, x_2 \in V^n$*, then there exists a unique solution* $\mathcal{C} : V^n \longrightarrow W$ *of* (2.4) *such that such that*

$$\|f(x) - \mathcal{C}(x)\|_W \leq \begin{cases} \frac{\theta}{(8^{n\beta}-2^{\alpha\lambda})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda} & \lambda \in \left(0, 3n\frac{\beta}{\alpha}\right), \\ \frac{2^{\alpha\lambda}\theta}{8^{n\beta}(2^{\alpha\lambda}-8^{n\beta})}\sum_{j=1}^{n}\|x_{1j}\|_V^{\lambda} & \lambda \in \left(3n\frac{\beta}{\alpha}, \infty\right), \end{cases}$$

for all $x = x_1 \in V^n$*. In particular, if* $\mathcal{C}$ *is odd and has the cubic condition in each variable, then it is multi-cubic.*

Under some conditions the functional equation (2.4) can be hyperstable as follows.

Corollary 3.6. *Given the positive number* θ *and* $p_{ij} > 0$ *for* $i \in \{1, 2\}$, $j \in \{1, \dots, n\}$*. Let* V *be a quasi-*α*-normed space with quasi-*α*-norm* $\|\cdot\|_V$*, and* W *be a* (β, p)*-Banach space with* (β, p)*-norm* $\|\cdot\|_W$*.*

(i) *If* $\sum_{i=1}^{2}\sum_{j=1}^{n} p_{ij} \neq 2n\frac{\beta}{\alpha}$ *and* $f : V^n \longrightarrow W$ *is an even mapping and has the quadratic condition in each variable fulfilling the inequality*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^{2}\prod_{j=1}^{n} \|x_{ij}\|_V^{p_{ij}},$$

for all $x_1, x_2 \in V^n$*, then it is multi-quadratic;*

(ii) *If* $\sum_{i=1}^{2}\sum_{j=1}^{n} p_{ij} \neq 3n\frac{\beta}{\alpha}$ *and* $f : V^n \longrightarrow W$ *is an odd mapping and has the cubic condition in each variable fulfilling the inequality*

$$\|\mathbf{D}_{qc}f(x_1, x_2)\|_W \leq \theta \prod_{i=1}^{2}\prod_{j=1}^{n} \|x_{ij}\|_V^{p_{ij}},$$

for all $x_1, x_2 \in V^n$*, then it is multi-cubic.*

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