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Semi-Fuzzy Graphs

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ABSTRACT: In this paper, we introduce the relatively new notions of semi-fuzzy graph and balanced semi-fuzzy graph. We study several operations on these graphs such as complement, union, join, composition, direct product, semi-strong product and strong product. In addition, we provide some classes of balanced semi-fuzzy graphs. Similar work is also done for Intuitionistic semi-fuzzy graphs.

Key Words: Semi-fuzzy graph, complete fuzzy graph, operations on semi-fuzzy graphs, balanced semi-fuzzy graph, intuitionistic semi-fuzzy graphs.

Contents

1. Introduction

Biggs [23] first introduced the notion of graphs in 1736. In the history of mathematics, the solution given by Euler of the well-known Konigsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graphs is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research , optimization theory and computer science.

In 1965, Zadeh [46] suggested a framework to describe the uncertain phoneme in real life. Graph theory has been witnessing an exponential growth in fuzzy graph theory. In 1975, Rosenfeld [40] considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs. During the same time Yeh and Bang [45] have also introduced various connectedness concepts in fuzzy graphs. The theory of fuzzy sets have been applied widely in areas like logic, information theory, pattern recognition, clustering, expert systems, database theory, control theory, robotics, networks and nanotechnology, see [19,26,27,28,32]. Fuzzy set theory has emerged as a potential area of interdisciplinary research and fuzzy graph theory is of recent interest.

There are many research papers on balanced fuzzy graphs that have been introduced by Al-Hawary [9] were he studied some operations of fuzzy graphs, see for example [1,2,3,4,5,6,7,8,10,11]. In this paper, we modify his definition of balanced Then we study some property of fuzzy graphs and its relation with balanced property.

Atanassov [17] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs (IFGs). Articles [9,17,38] motivated us to analyze balanced IFGs and their properties. We modify balanced IFGs notion to semi-IFGs and study their properties.

Next, we recall some definitions and results that will be needed throughout this paper.

Definition 1.1 *[26] A fuzzy subset of a set V is a mapping $\sigma : V \to [0.1]$. A fuzzy relation on σ is a mapping $\mu : V \times V \to [0.1]$. If μ and ω are fuzzy relations on σ, then $\mu \circ \omega(u, w) = Sup_{v \in V}\{\mu(u, v) \wedge \omega(v, w)\}$.*

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Definition 1.2 *[40] A* fuzzy graph *$G : (\sigma, \mu)$ where σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x, y) \in V \times V : \mu(x, y) > 0\}$.$H = (\sigma', \mu')$ is a* fuzzy subgraph *of G if there exists $X \subseteq V$ such that, $\sigma' : X \to [0, 1]$ is a fuzzy subset and $\mu' : X \times X \to [0, 1]$ is a fuzzy relation on σ' such that $\mu(x, y) \le \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.*

Definition 1.3 *[22] A fuzzy graph $G : (\sigma, \mu)$ is* strong *if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$ A fuzzy graph $G : (\sigma, \mu)$ is* complete *if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.*

It is clear that every complete fuzzy graph is strong.

Definition 1.4 *[22] An isomorphism between two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ is a bijective $h : V_1 \to V_2$ satisfying $\sigma_1(u) = \sigma_2(h(u))$ for all $u \in V_1$ and $\mu_1(u, v) = \mu_2(h(u), h(v))$ for all $u, v \in V_1$. We write $G_1 \simeq G_2$. If $G_1 = G_2$, h is called an automorphism.*

After the pioneering work of Rosenfeld [40] and Yeh Bang [45] in 1975, when some basic fuzzy graph theoretic and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts have been studied . This include fuzzy trees [25,42] , fuzzy line graphs [29] , operations on fuzzy graphs [30] , automorphism of fuzzy graphs [20,22] , fuzzy interval graphs [24], cycles and cocyles of fuzzy graphs [31] and metric aspects in fuzzy graphs [43].

Definition 1.5 *[44] A complement of a fuzzy graph $G : (\sigma, \mu)$ is a fuzzy graph $\bar{G} : (\bar{\sigma}, \bar{\mu})$ where $\bar{\sigma} = \sigma$ and $\bar{\mu}(u, v) = \sigma(u) \wedge \sigma(v) - \mu(u, v)$ for all $u, v \in V$. If $G \simeq \bar{G}$, we say G is self-complemetary.*

Theorem 1.1 *[44] a) Let $G : (\sigma, \mu)$ be a self-complemetary fuzzy graph. Then $\sum_{x,y\in V} \mu(x, y) = (1/2) \sum_{x,y\in V} (\sigma(x) \wedge \sigma(y))$*

b) Let $G : (\sigma, \mu)$ be a fuzzy graph satisfying $\mu(x, y) = (1/2)(\sigma(x) \wedge \sigma(y))$ for all $x, y \in V$.Then G is self-complemetary.

Definition 1.6 *[32] The* union *of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ where $(\sigma_1 \cup \sigma_2)(u, v) = \sigma_1(u)$ if $u \in V_1$ and $(\sigma_1 \cup \sigma_2)(u, v) = \sigma_2(v)$ if $v \in V_2$ and $(\mu_1 \cup \mu_2)(uv) = \mu_1(uv)$ if $uv \in E_1$ and $(\mu_1 \cup \mu_2)(uv) = \mu_2(uv)$ if $uv \in E_2$.*

The join $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ is $G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ where $(\sigma_1 + \sigma_2)(u, v) = (\sigma_1 \cup \sigma_2)(u, v)$ and $(\mu_1 + \mu_2)(uv) = (\mu_1 \cup \mu_2)(uv)$ if $uv \in E_1 \cup E_2$ and $(\mu_1 + \mu_2)(uv) = \sigma_1(u) \wedge \sigma_2(v)$ if $uv \in \acute{E}$ where $\acute{E}$ is the set of arcs joining the nodes of V_1 and V_2.

Theorem 1.2 *[32] Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs. Then*

(1) $\overline{G_1 + G_2} \simeq \overline{G_1} \cup \overline{G_2}$.

(2) $\overline{G_1 \cup G_2} \simeq \overline{G_1} + \overline{G_2}$.

Definition 1.7 *[32] The* Cartesian product *of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ where $(\sigma_1 \times \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$ for all $u \in V_1$, $v \in V_2$ and $(\mu_1 \times \mu_2)((u, u_2)(v, v_2)) = \sigma_1(u) \wedge \mu_2(u_2v_2)$ if $u \in V_1.u_2v_2 \in E_2$ and $(\mu_1 + \mu_2)((u_1, w)(v_1, w)) = \mu_1(u_1v_1) \wedge \sigma_2(w)$ if $u_1v_1 \in E_2$ and $w \in V_2$.*

The Composition of $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ is defined to be the fuzzy graph $G_1 \circ G_2 : (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ where $(\sigma_1 \circ \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$ for all $u \in V_1$, $v \in V_2$ and $(\mu_1 \circ \mu_2)((u, u_2)(v, v_2)) = \sigma_1(u) \wedge \mu_2(u_2v_2)$ if $u \in V_1.u_2v_2 \in E_2$, $(\mu_1 \circ \mu_2)((u_1, w)(v_1, w)) = \mu_1(u_1v_1) \wedge \sigma_2(w)$ if $u_1v_1 \in E_2$ and $w \in V_2$ and $(\mu_1 \circ \mu_2)((u_1, u_2)(v_1, v_2)) = \mu_1(u_1v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$ if $u_1v_1 \in E_2$ and $u_2v_2 \in E_2$.

Theorem 1.3 *[44] a) If G is a strong fuzzy graph, then $\bar{G}$ is strong.*

b) If G_1 and G_2 are two strong fuzzy graphs, then $G_1 \circ G_2$ is strong and $\overline{G_1 \circ G_2} \simeq \overline{G_1} \circ \overline{G_2}$.

Al-Hawary [9] introduced the concept of balanced fuzzy graphs and studied some operations of fuzzy graphs. In this chapter, we modify the definition of density and thus the definition of balanced fuzzy graphs . Some properties of self-complementary fuzzy graphs and the complement of fuzzy graphs and the direct product, semi-strong product , strong product and composition of fuzzy graphs, that were introduced in [33] , were studied. For more on the previous notions and the following ones , you can see [33,34,35,36,40,44].

There are several papers written on balanced extension of graphs [41] which has tremendous applications in artificial intelligence , signal processing , robotics, computer networks and decision making. First, we display some types of products of fuzzy graph, such like direct product, semi-strong product and strong product of two fuzzy graphs , then we provide examples on it.

Definition 1.8 *[9] The* direct product *of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \varnothing$, is defined to be the fuzzy graph $G_1 \sqcap G_2 : (\sigma_1 \sqcap \sigma_2, \mu_1 \sqcap \mu_2)$ with crisp graph $G : (V_1 \times V_2, E)$ where*

$$E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$(\sigma_1 \sqcap \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$ and $(\mu_1 \sqcap \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2)$.

The semi-strong product *of two fuzzy graphs G_1 and $G_2 : (\sigma_2, \mu_2)$ is defined to be the fuzzy graph $G_1 \cdot G_2 : (\sigma_1 \cdot \sigma_2, \mu_1.\mu_2)$ with crisp graph $G : (V_1 \times V_2, E)$ where*

$$E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\},$$

$(\sigma_1 \cdot \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2, (\mu_1.\mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$ and $(\mu_1.\mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2)$.

The strong product *of two fuzzy graphs G_1 and G_2 is defined to be the fuzzy graph $G_1 \otimes G_2 : (\sigma_1 \otimes \sigma_2, \mu \otimes \mu_2)$ with crisp graph $G : (V_1 \times V_2, E)$ where $E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \cup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\} \cup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}, (\sigma_1 \otimes \sigma_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v)$, for all $(u, v) \in V_1 \times V_2$, $(\mu_1 \otimes \mu_2)((u, v_1)(u, v_2)) = \sigma_1(u) \wedge \mu_2(v_1, v_2)$, $(\mu_1 \otimes \mu_2)((u_1, w)(u_2, w)) = \sigma_2(w) \wedge \mu_1(u_1, u_2)$ and $(\mu_1 \otimes \mu_2)((u_1, v_1)(u_2, v_2)) = \mu_1(u_1,u_2) \wedge \mu_2(v_1, v_2)$.*

Theorem 1.4 *[1]a) If G_1 and G_2 are strong fuzzy graphs, then $G_1 \sqcap G_2$ is strong.*
b) If G_1 and G_2 are complete fuzzy graphs, then $G_1 \sqcap G_2$ is strong.
c) If G_1 and G_2 are strong fuzzy graphs, then $G_1.G_2$ is strong.
d) If G_1 and G_2 are complete fuzzy graphs, then $G_1.G_2$ is strong.
e) If G_1 and G_2 are strong fuzzy graphs, then $G_1 \otimes G_2$ is strong.
f) If G_1 and G_2 are complete fuzzy graphs, then $G_1 \otimes G_2$ is complete.
g) If G_1 and G_2 are complete fuzzy graphs, then $\overline{G_1 \otimes G_2} \asymp \overline{G_1} \otimes \overline{G_2}$.

Definition 1.9 *[17] An* intuitionistic fuzzy graph *(simply, IFG) is of the form $G : (V, E)$ where*

(i) $V = \{\nu_0, \nu_1, ..., \nu_n\}$ such that $\mu_{1:}V \to [0, 1]$ and $\gamma_1 : V \to [0, 1]$, denotes the degree of membership and non-membership of the element $v_i \in V$, respectively and $0 \leq \mu_1(\nu_i) + \gamma_1(\nu_i) \leq 1, \forall\ \nu_i \in V, (i = 1, 2, ..., n)$,

(ii) $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0, 1]$ and $\gamma_2 : V \times V \to [0, 1]$ are such that and $0 \leq \mu_2(\nu_i, \nu_j) + \gamma_2(\nu_i, \nu_j) \leq 1$, for every $(\nu_i, \nu_j) \in E$, $(i, j = 1, 2, ..., n)$.

Definition 1.10 *[17] The* complement *of an IFG, $G : (V, E)$ is an IFG, $G^c : (V^c, E^c)$, where*

(i) $V^c = V$,

(ii) $\mu^c_{1i} = \mu_{1i}$ and $\gamma^c_{1i} = \gamma_{1i}$, $\forall i = 1, 2, ..., n$,

(iii) $\mu^c_{2ij} = min(\mu_{1i}, \mu_{1j}) - \mu_{2ij}$ and $\gamma^c_{2ij} = max(\gamma_{1i}, \gamma_{1j}) - \gamma_{2ij}$, $\forall i,\ j = 1, 2, ..., n$.

An IFG G is said to be complete IFG *if $\mu_{2ij} = min(\mu_{1i}, \mu_{1j})$ and $\gamma_{2ij} = max(\gamma_{1i}, \gamma_{1j})$, $\forall \nu_i,\ \nu_j \in V$.*

An IFG G is said to be strong *IFG if $\mu_{2ij} = min(\mu_{1i}, \mu_{1j})$ and $\gamma_{2ij} = max(\gamma_{1i}, \gamma_{1j})$, $\forall \nu_i,\ \nu_j \in E$.*

An IFG G is said to be regular IFG if all the vertices have the same closed neighborhood degree.

Let $G_1 : (V_1, E_1)$ and $G_2 : (V_2, E_2)$ be two IFG's. An isomorphism between G_1 and G_2 (denoted by $G_1 \simeq G_2$) is a bijective map $h : V_1 \to V_2$ which satisfies $\mu_1(\nu_i) = \acute{\mu}_1(h(\nu_i))$, $\nu_1(\nu_i) = \acute{\nu}_1(h(\nu_i))$, $\mu_2(\nu_i, \nu_j) = \acute{\mu}_2(h(\nu_i), h(\nu_j))$ and $\gamma_2(\nu_i, \nu_j) = \acute{\gamma}_2(h(\nu_i), h(\nu_j))$, $\forall \nu_i, \upsilon_j \in V$.

2. Balanced semi-fuzzy graphs

We begin this Section by defining semi-fuzzy graph and the density of a semi-fuzzy graph and balanced semi-fuzzy graphs .Then we show that any complete semi-fuzzy graph is balanced, but the converse needs not be true.

Definition 2.1 *A fuzzy graph $G:(\sigma,\mu)$ is a semi-fuzzy graph if $\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)<1$.*

Definition 2.2 *The Density of a semi-fuzzy graph $G:(\sigma,\mu)$ is $D'(G)=\frac{2\sum_{x,y\in V}\mu(xy)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$. A semi-fuzzy fuzzy graph G is balanced if $D\prime(H)\leq D\prime(G)$ for every fuzzy non-empty subgraph H of G.*

Theorem 2.1 *Every complete semi-fuzzy graph is balanced.*

Proof: *Let $G:(\sigma,\mu)$ be a complete semi-fuzzy graph. Then $\sum_{x,y\in V}\mu(xy)=\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)$. Hence $D\prime(G)=\frac{2}{\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)}$. If $H=(V_H,E_H)$ is non-empty fuzzy subgraph of G, then to make H complete, we might have to add some edges to it with weights as the minimum between the adjacent vertices. Let us call it $H'=(V'_H,E'_H)$. Since $H\prime$ is complete , then $D'(H')=\frac{2}{\sum_{x,y\in V_H}\sigma(x)\wedge\sigma(y)}$. Since $E_H\subseteq E_{H'}$, $\sum_{x,y\in V_H}\mu(xy)\leq\sum_{x,y\in V_{H'}}\mu(xy)$ and so $D'(H')\leq\frac{2\sum_{x,y\in V_{H'}}\mu(xy)}{(1-\sum_{x,y\in V_H}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V_H}\sigma(x)\wedge\sigma(y))}=D'(H')$ and since $V_H\subseteq V_{H'}$, $D'(H')\leq D\prime(G)$. Hence $D'(H)\leq D'(H')\leq D\prime(G)$ and therefore, G is balanced.* □

Theorem 2.2 *Any complete semi-fuzzy graph G has density $D\prime(G)>2$.*

Proof: *Let $G:(\sigma,\mu)$ be complete semi-fuzzy graph. Then $\mu(x,y)=\sigma(x)\wedge\sigma(y)$ for all $x,y\in V$. So $\sum_{x,y\in V}\mu(xy)\leq\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)$. Thus $D\prime(G)=\frac{2}{\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)}$ and since $\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)<1$, $1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)<1$ and thus $D\prime(G)>2$.* □

The covers of the above theorem need not be true.

Example 2.1 The semi-fuzzy graph $G:(\sigma,\mu)$ defined on two vertices a and b such that $\sigma(a)=0.5$, $\sigma(b)=0.8$, $\sigma(c)=0.2$ and $\mu(a,b)=0.5$ is a semi-fuzzy graph such that $D\prime(G)>2$, but it is not complete.

Theorem 2.3 *Every self-complementary semi-fuzzy graph has density more than or equal to 1.*

Proof: *Let G be self-complementary semi-fuzzy graph. Then $D\prime(G)=\frac{1}{1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)}$. Since $\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)<1$, $1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y)<1$ and thus $D\prime(G)>1$.* □

The converse of the above Theorem need not be true.

Example 2.2 The semi-fuzzy graph $G:(\sigma,\mu)$ defined on two vertices a,b and such that $\sigma(a)=0.1=\sigma(b),\sigma(c)=0.05$, $\mu(a,c)=\mu(a,b)=0.025$ and $\mu(b,c)=0.06$ is a semi-fuzzy graph that has density more than 1, but it is not self-complementary.

Theorem 2.4 *Let $G:(\sigma,\mu)$ be semi-fuzzy graph such that $\mu(u,v)=\frac{\sigma(x)\wedge\sigma(y)}{2}$ for all $u,v\in V$. Then $D\prime(G)>1$.*

Proof: *Let $G:(\sigma,\mu)$ be semi-fuzzy graph such that $\mu(u,v)=\frac{\sigma(x)\wedge\sigma(y)}{2}$ for all $u,v\in V$. Then G is self-complementary and thus by Theorem.2.3, $D\prime(G)>1$.* □

Lemma 2.1 *Let G_1 and G_2 be complete semi-fuzzy graphs. Then $D\prime(G_i)\leq D\prime(G_1\sqcap G_2)$ for $i-1,2$ and $\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2),(v_1,v_2))\leq\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)$ if and only if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\sqcap G_2)$.*

Proof: Since G_1 is complete, $\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)\leq\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)$. So $\frac{\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)}{\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}=1$ and by last Theorem , we have

$$\begin{aligned}
D\prime(G_i) &= \frac{2}{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}\\
&= \frac{2}{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}(\frac{\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)}{\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}\\
&\geq \frac{2}{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}(\frac{\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2)}{\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2)})\\
&= \frac{2}{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}(\frac{\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)\wedge\mu_2(v_1,v_2)}{\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)\wedge\sigma_2(v_1)\wedge\sigma_2(v_2)}).
\end{aligned}$$

Since G_2 is complete, $D\prime(G_i)\geq\frac{2}{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}(\frac{\sum_{u_1,u_2\in V_1}\mu_1\sqcap\mu_2((u_1,u_2)(v_1,v_2))}{\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))})$. Now we multiply both sides by $\frac{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}{1-\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))}$, we get

$$\frac{1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)}{1-\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))}D\prime(G_1)\geq D\prime(G_1\sqcap G_2)......(1)$$

Substitute $D\prime(G_1)$ in (1), we get $\frac{2\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)}{(1-\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2)))(\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2))}\geq D\prime(G_1\sqcap G_2)$. According to our condition, $\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2)$. Hence $\frac{2\sum_{u_1,u_2\in V_1}\mu_1(u_1,u_2)}{(1-\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2))(\sum_{u_1,u_2\in V_1}\sigma_1(u_1)\wedge\sigma_2(u_2))}\geq D\prime(G_1\sqcap G_2)$. Therefore $D\prime(G_1)\geq D\prime(G_1\sqcap G_2)$. □

Theorem 2.5 *Let G_1 and G_2 be balanced complete semi-fuzzy graphs and $\sum_{u_1,u_2\in V_1}\sigma_1\sqcap\sigma_2((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_i}\sigma_1(u_i)\wedge\sigma_2(u_i)$ for $i=1,2$. Then $G_1\sqcap G_2$ is balanced if and only if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\sqcap G_2)$.*

Proof: If $G_1\sqcap G_2$ is balanced, then $D\prime(G_1)\leq D\prime(G_1\sqcap G_2)$ for $i=1,2$, and by Lemma 2.1, $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\sqcap G_2)$.

Conversely, if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\sqcap G_2)$ and H is a fuzzy subgraph of $G_1\sqcap G_2$, then there are semi-fuzzy subgraphs H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D\prime(G_1)=D\prime(G_1)=\frac{n_1}{r_1}$, then $D'(H_1)\leq\frac{a_1}{b_1}\leq\frac{n_2}{r_1}$ and $D'(H_2)\leq\frac{a_2}{b_2}\leq\frac{n_1}{r_1}$. Thus $a_1r_1+a_2r_2\leq b_1n_1+b_2n_1$ and hence $D'(H)\leq\frac{a_1+a_2}{b_1+b_2}\leq\frac{n_1}{r_1}=D\prime(G_1\sqcap G_2)$. Therefore $D\prime(G_1\sqcap G_2)$ is balanced. □

By similar arguments to those in Lemma.2.1 and Theorem.2.5, we can prove the following result:

Theorem 2.6 *Let G_1 and G_2 be balanced complete semi-fuzzy graphs with $\sum_{u_1,u_2\in V_1}\sigma_1\otimes\sigma_2((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_i}\sigma_1(u_i)\wedge\sigma_2(u_i)$ and $\sum_{u_1,u_2\in V_1}\sigma_1.\sigma_2((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_i}\sigma_1(u_i)\wedge\sigma_2(u_i)$ for $i=1,2$. Then*

(1) $G_1\otimes G_2$ is balanced if and only if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\otimes G_2)$.

(2) $G_1.G_2$ is balanced if and only if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1.G_2)$.

Theorem 2.7 *Let G_1 and G_2 be isomorphic semi-fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

Proof: Let $h:V_1\to V_2$ be a bijection such that $\sigma_1(x)=\sigma_2(h(x))$ and $\mu_1(x,y)=\mu_2(h(x),h(y))$, for all $\sigma x,y\in V_1$. By Lemma 2.1, $\sum_{x\in V_i}\sigma_1(x)=\sum_{x\in V_2}\sigma_2(x)$ and $\sum_{x,y\in V_i}\mu_1(xy)=\sum_{x,y\in V_2}\mu_2(xy)$. If $H_1=(\sigma_1,\mu_1)$ is a fuzzy subgraph of G_1 with underlying set W, then $H_2=(\sigma_2',\mu\prime_2)$ is a fuzzy subgraph of G_2 with underlying set $h(W)$ where $\sigma_2'(h(x))=\sigma_1'(x)$ and $\mu_2\prime(h(x),h(y))=\mu_1\prime(x,y)$ for all $x,y\in W$.

Since G_2 is balanced , $D\prime(H_2) \leq D\prime(G_2)$ and so $\frac{2\sum_{x,y\in W}\mu_2\prime(h(x),h(y))}{(1-\sum_{x,y\in W}\sigma_2'(h(x))\wedge\sigma_2'(h(x)))(\sum_{x,y\in W}\sigma_2'(h(x))\wedge\sigma_2'(h(x)))} \leq \frac{2\sum_{x,y\in W}\mu_2(h(x),h(y))}{(1-\sum_{x,y\in W}\sigma_2(h(x))\wedge\sigma_2(h(x)))(\sum_{x,y\in W}\sigma_2(h(x))\wedge\sigma_2(h(x)))}$. Thus

$$\begin{aligned}\frac{2\sum_{x,y\in W}\mu_1(h(x),h(y))}{(1-\sum_{x,y\in W}\sigma_2'(h(x))\wedge\sigma_2'(h(x)))(\sum_{x,y\in W}\sigma_2'(h(x))\wedge\sigma_2'(h(x)))} &\leq \\ \frac{2\sum_{x,y\in W}\mu_2(h(x),h(y))}{(1-\sum_{x,y\in W}\sigma_2(h(x))\wedge\sigma_2(h(x)))(\sum_{x,y\in W}\sigma_2(h(x))\wedge\sigma_2(h(x)))} &\leq D'(G_1)\end{aligned}$$

and hence $D\prime(H_1) \leq D\prime(G_1)$.Therefore G_1 is balanced. □

3. Regular fuzzy graphs

In this section, we need to study the concept of balanced for several semi-fuzzy graphs types so we introduce regular fuzzy graphs concept and then we study balanced semi-fuzzy graphs and its relation to regular fuzzy graphs.

Definition 3.1 [36]*Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* = (V, E)$.if $d_G(v) = k$, for all $v \in V$. Then G is said to be a regular -fuzzy graph of degree k or k-regular fuzzy graph .The total degree of vertex $u \in V$ is defined by $td_G = \sum_{u,v\in V}\mu(uv) + \sigma(u) = d_G(u) + \sigma(u)$. If each vertex of G has the same total degree k, then G is said to by a totally regular fuzzy graph of total degree k or k-totally regular fuzzy graph. σ is called a constant function if $\sigma(u) = \sigma(v)$, for all $u \neq v \in V$. μ is called a constant function if $\mu(uv)$ are equivalent for all $u, v \in V$.*

Theorem 3.1 *If $G : (\sigma, \mu)$ is an r-regular semi-fuzzy graph, then G has a density $D'(G) = \frac{pr}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$, where $p = |V|$.*

Proof: Since G is an r-regular semi-fuzzy graph, $d_G(v) = r$ for all $v \in V$. Thus $\sum_{v\in V} d_G(v) = 2\sum_{u,v\in V}\mu(uv)$. Hence $\sum_{u,v\in V}\mu(uv) = \frac{\sum_{v\in V} r}{2} = \frac{pr}{2}$ and so $D'(G) = \frac{pr}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$. □

Theorem 3.2 *If $G : (\sigma, \mu)$ is an r-totally regular semi-fuzzy graph, then G has a density $D'(G) = \frac{pr-\sum_{x\in V}\sigma(x)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$, where $p = |V|$.*

Proof: Since G is an r-totally regular semi-fuzzy graph, $td_G(v) = r$ for all $v \in V$. So $\sum_{v\in V} r = \sum_{v\in V} d_G(v) + \sum_{v\in V}\sigma(v)$. Thus $pr = 2\sum_{u,v\in V}\mu(uv) + \sum_{v\in V}\sigma(v)$. Dividing both sides by $(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))$, we get $\sum_{u,v\in V}\mu(uv) = \frac{\sum_{v\in V} r}{2} = \frac{pr}{2}$ and so $\frac{pr}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))} = \frac{2\sum_{u,v\in V}\mu(uv)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))} + \frac{\sum_{v\in V}\sigma(v)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$. So
$D'(G) = \frac{pr}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))} - \frac{\sum_{v\in V}\sigma(v)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$
$= \frac{pr-\sum_{x\in V}\sigma(x)}{(1-\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))(\sum_{x,y\in V}\sigma(x)\wedge\sigma(y))}$ □

In general, some operations on semi-fuzzy graphs do not preserve totally regular as we see next.

Example 3.1 Consider the semi-fuzzy graphs G_1 and G_2 defined as follows: $\sigma_1(v_1) = 0.04, \sigma_1(v_2) = 0.08$, $\sigma_1(v_3) = 0.07$, $\mu_1(v_1, v_2) = 0.03, \mu_1(v_1, v_3) = 0.04, \mu_1(v_2, v_3) = 0$, $\sigma_2(u_1) = \sigma_2(u_2) = 0.01$ and $\mu_2(u_1, u_2) = 0.01$. Then both G_1 and G_2 are 0.11-totally regular, but $G_1 \sqcap G_2$, $G_1.G_2$ and $G_1 \otimes G_2$ are not.

Example 3.2 The fuzzy graph G on vertices a,b,c such that $\sigma(a) = 0.2, \sigma(b) = \sigma(c) = 0.1$ and $\mu(b, c) = 0.1$ is an 0.2-totally regular semi-fuzzy graph, but not balanced.

4. Balanced intuitionistic semi-fuzzy graphs

Next, we introduce and study properties of intuitionistic semi-fuzzy and emphasize the case when an intuitionistic semi-fuzzy graph is balanced.

Definition 4.1 *A fuzzy graph $G:(\sigma,\mu)$ is an intuitionistic semi-fuzzy graph if $\sum_{x,y\in V}\mu_1(x)\wedge\mu_1(y)<1$ and $\sum_{x,y\in V}v_1(x)\vee v_1(y)<1$. It is denoted by ISFG.*

Definition 4.2 *The density of a ISFG $G:(\sigma,\mu)$ is*
$D'(G)=(D'_\mu(G),D'_v(G))$, where $D'_\mu(G)=\frac{2\sum_{u,v\in V}\mu_2(uv)}{(1-\sum_{x,y\in V}\mu_1(x)\wedge\mu_1(y))(\sum_{x,y\in V}\mu_1(x)\wedge\mu_1(y))}$ and

$$D'_v(G)=\frac{2\sum_{u,v\in V}v_2(uv)}{(1-\sum_{x,y\in V}v_1(x)\wedge v_1(y))(\sum_{x,y\in V}v_1(x)\wedge v_1(y))}$$

.

Definition 4.3 *An ISFG $G:(\sigma,\mu)$ is balanced if $D'(H)\leq D'(G)$ for all subgraphs H. That is $D'_\mu(H)\leq D'_\mu(G)$ and $D'_v(H)\leq D'_v(G)$.*

Theorem 4.1 *Any complete ISFG has density $D'(G)=(D'_\mu(G),D'_v(G))>(2,2)$.*

Proof: Let $G:(\sigma,\mu)$ be a complete ISFG. Then $\mu_2(v_iv_j)=\mu_1(v_i)\wedge\mu_1(v_j)$ and $v_2(v_iv_j)=v_1(v_i)\vee v_1(v_j)$. So $\sum_{v_i,v_j\in V}\mu_2(v_iv_j)=\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)$ and $\sum_{v_i,v_j\in V}v_2(v_iv_j)=\sum_{v_i,v_j\in V}v_1(v_i)\vee v_1(v_j)$ and thus $D'_\mu(G)=\frac{2}{1-\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)}$ and $D'_v(G)=\frac{2}{1-\sum_{v_i,v_j\in V}v_1(v_i)\vee v_1(v_j)}$. Since both $1-\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)$ and $1-\sum_{v_i,v_j\in V}v_1(v_i)\vee v_1(v_j)$ is less than 1, $D'(G)=(D'_\mu(G),D'_v(G))>(2,2)$. □

Theorem 4.2 *Every complete ISFG is balanced.*

Proof: Let $G:(\sigma,\mu)$ be a complete ISFG. Then $D'_\mu(G)=\frac{2}{1-\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)}$ and $D'_v(G)=\frac{2}{1-\sum_{v_i,v_j\in V}v_1(v_i)\vee v_1(v_j)}$. If H is non-empty fuzzy subgraph of G, then to make H complete , we might have to add some edges to it with weights as the minimum and maximum between the adjacent vertices. Let's call $H'=(V_H,E_{H'})$. Since $H\prime$ is complete, now $D'_\mu(G)=\frac{2\sum_{v_i,v_j\in V_H}\mu_2(v_iv_j)}{(1-\sum_{v_i,v_j\in V_H}\mu_1(v_i)\wedge\mu_1(v_j))(\sum_{v_i,v_j\in V_H}\mu_1(v_i)\wedge\mu_1(v_j))}$ and $D'_v(G)=\frac{2\sum_{v_i,v_j\in V_H}\mu_{v2}(v_iv_j)}{(1-\sum_{v_i,v_j\in V_H}v_1(v_i)\vee v_1(v_j))(\sum_{v_i,v_j\in V_H}v_1(v_i)\vee v_1(v_j))}$ and since $E_H\subseteq E_{H'}$, $D'_\mu(H)\leq D'_\mu(H')$ and $D'_v(H)\leq D'_v(H')$. Since $V_H\subseteq V_{H'}$, $D'_\mu(H')\leq D'_\mu(G)$ and $D'_v(H')\leq D'_v(G)$. □

Corollary 4.1 *Let $G:(\sigma,\mu)$ be a self-complementary SIFG . Then*
$\sum_{v_i,v_j\in V}\mu_2(v_iv_j)=\frac{\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)}{2}$ and $\sum_{v_i,v_j\in V}v_2(v_iv_j)=\frac{\sum_{v_i,v_j\in V}v_1(v_i)\wedge v_1(v_j)}{2}$.

Theorem 4.3 *Every self-complementary ISFG has a density more than or equal to (1,1).*

Proof: Let G be self-complementary ISFG. Then $D'_\mu(G)=\frac{1}{1-\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)}$ and
$D'_v(G)=\frac{1}{1-\sum_{v_i,v_j\in V}v_1(v_i)\wedge v_1(v_j)}$. Since $1-\sum_{v_i,v_j\in V}\mu_1(v_i)\wedge\mu_1(v_j)<1$ and $1-\sum_{v_i,v_j\in V}v_1(v_i)\wedge v_1(v_j)<1$, $D'_\mu(G)>1$ and $D'_v(G)>1$ and so $D'(G)<(1,1)$. □

Lemma 4.1 *Let G_1 and G_2 be complete ISFGs such that $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)$ and $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}v_1\sqcap v'_1((u_1,u_2)(v_1,v_2))\leq\sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)$. Then $D\prime(G_i)\leq D\prime(G_1\sqcap G_2)$ if and only if $D\prime(G_1)=D\prime(G_2)=D\prime(G_1\sqcap G_2)$ for $i=1,2$.*

Proof: Since G_1 is complete, $D'_\mu(G_1) = \frac{1}{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}$ and $D'_v(G_1)=\frac{1}{1-\sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)}$. Thus $\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)=\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)$ and $\sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)=\sum_{u_1,u_2\in V_1}v_2(u_1u_2)$. So $\frac{\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)}{\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)} = \frac{\sum_{u_1,u_2\in V_1}v_2(u_1u_2)}{\sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)} = 1$ and hence

$$\begin{aligned} D'_\mu(G_1) &= \frac{2}{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}\frac{\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)}{\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)} \\ &\geq \frac{2}{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_2(u_1u_2)\wedge\mu'_1(v_1)\wedge\mu'_1(v_2)}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1(u_1)\wedge\mu_1(u_2)\wedge\mu'_1(v_1)\wedge\mu'_1(v_2)} \\ &= \frac{2}{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_2(u_1u_2)\wedge\mu'_2(v_1,v_2)}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1(u_1)\wedge\mu_1(u_2)\wedge\mu'_1(v_1)\wedge\mu'_1(v_2)}. \end{aligned}$$

Similarly, since G_2 is complete $D'_\mu(G_2) \geq \frac{2}{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_2\sqcap\mu'_2((u_1u_2)(v_1,v_2))}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1u_2)(v_1,v_2))}$. Now we multiply both sides by $\frac{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}{1-\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1u_2)(v_1,v_2))}$ to get

$$\frac{1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)}{1-\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1u_2)(v_1,v_2))}D'_\mu(G_1) \geq D\prime_\mu(G_1\sqcap G_2)$$

and as $D'_\mu(G_1) = \frac{2\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)}{(1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2))(\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2))}$, we have $\frac{2\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)}{(1-\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1u_2)(v_1,v_2)))(\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2))} \geq D\prime_\mu(G_1\sqcap G_2)$. According to our condition $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)$ and so $\frac{2\sum_{u_1,u_2\in V_1}\mu_2(u_1u_2)}{(1-\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2))(\sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2))} \geq D\prime_\mu(G_1\sqcap G_2)$. Thus $D\prime_\mu(G_1) \geq D\prime_\mu(G_1\sqcap G_2)$ and similarly $D\prime_\mu(G_2) \geq D\prime_\mu(G_1\sqcap G_2)$. Similar arguments give $D\prime_v(G_1) \geq D\prime_v(G_1\sqcap G_2)$ and $D\prime_v(G_2) \geq D\prime_v(G_1\sqcap G_2)$. Therefore, $D\prime(G_1) = D\prime(G_2) = D\prime(G_1\sqcap G_2)$. □

Theorem 4.4 *Let G_1 and G_2 be balanced complete ISFGs such that $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\sqcap\mu'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)$ and $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}v_1\sqcap v'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)$. Then $G_1\sqcap G_2$ is balanced if and only if $D\prime(G_1) = D\prime(G_2) = D\prime(G_1\sqcap G_2)$ for $i=1,2$.*

Proof: If $G_1\sqcap G_2$ is balanced, then $D\prime(G_i) \leq D\prime(G_1\sqcap G_2)$ for $i = 1,2$ and by the above Lemma $D\prime(G_1) = D\prime(G_2) = D\prime(G_1\sqcap G_2)$.

Conversely , if $D\prime(G_1) = D\prime(G_2) = D\prime(G_1\sqcap G_2)$ and H is an IFG subgraph of $G_1\sqcap G_2$, then there exist IFG subgraph H_1 of G_1 and H_2 of G_2. As G_1 and G_2 are balanced and $D'_\mu(G_1) = D'_\mu(G_2) = \frac{n_1}{r_1}$, then $D'_\mu(H_1) = \frac{a_1}{b_1} \leq \frac{n_2}{r_1}$ and $D'_\mu(H_2) = \frac{a_2}{b_2} \leq \frac{n_1}{r_1}$. Thus $a_1r_1 + a_2r_2 \leq b_1n_1 + b_2n_1$ and hence $D'_\mu(H) \leq \frac{a_1+a_2}{b_1+b_2} \leq \frac{n_1}{r_1} = D\prime_\mu(G_1\sqcap G_2)$. Similarly $D'_v(H) \leq D\prime_v(G_1\sqcap G_2)$ and thus $D'(H) \leq D\prime(G_1\sqcap G_2)$. Therefore $G_1\sqcap G_2$ is balanced. □

We can prove the following result by similar arguments to those in the above Theorem:

Theorem 4.5 *Let G_1 and G_2 be balanced complete ISFGs.*

(a) If $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1\otimes\mu'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)$ and $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}v_1\otimes v'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)$, then $G_1\otimes G_2$ is balanced if and only if $D\prime(G_1) = D\prime(G_2) = D\prime(G_1\otimes G_2)$ for $i=1,2$.

(b) If $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}\mu_1.\mu'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}\mu_1(u_1)\wedge\mu_1(u_2)$ and $\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2}v_1.v'_1((u_1,u_2)(v_1,v_2)) \leq \sum_{u_1,u_2\in V_1}v_1(u_1)\wedge v_1(u_2)$, then $G_1.G_2$ is balanced if and only if $D\prime(G_1) = D\prime(G_2) = D\prime(G_1.G_2)$ for $i=1,2$.

We end this section by stating that balanced notion of IFGs is preserved under isomorphism. The proof is easy and thus omitted.

Theorem 4.6 *Let G_1 and G_2 be isomorphic ISFGs. If one of them is balanced, the the other is balanced.*

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