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An Embedding Theorem and Spectral Equality for Semigroups Involving Demicompactness Classes

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ABSTRACT: Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ denote the strongly continuous semigroups of operators in a Banach space X. In this paper, we give a sufficient condition guaranteeing that $(S(t))_{t\geq 0}$ can be embedded in a C_0-group on X. Moreover, we characterize the demicompactness of $I-(S(t)-T(t))$ for $t>0$. Our theoretical results will be illustrated by investigating the spectral equality for uniformly continuous semigroups for an upper semi-Fredholm spectrum.

Key Words: demicompact linear operator, strongly continuous semigroups, Fredholm and semi-Fredholm operators, spectral equality, upper semi-Fredholm spectrum.

Contents

1. Introduction

Throughout this work we will denote by X a Banach space and by $\mathcal{C}(X)$ the set of all closed densely defined linear operators. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. The subset of all compact operators of $\mathcal{L}(X)$ is designed by $\mathcal{K}(X)$. For $A\in\mathcal{C}(X)$, we use $\sigma(A)$, $R(A)$ and $N(A)$ to denote the spectrum, the range and the null space of A, respectively. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in X. If λ belongs to the resolvent set of A, denoted by $\rho(A)$, then $R(\lambda, A)$ denotes the resolvent operator $(\lambda I-A)^{-1}$. The classes of upper semi-Fredholm and lower semi-Fredholm operators are defined respectively by

$$\Phi_+(X) := \{A\in\mathcal{C}(X) \text{ such that } \alpha(A)<\infty \text{ and } R(A) \text{ is closed in } X\},$$

and

$$\Phi_-(X) := \{A\in\mathcal{C}(X) \text{ such that } \beta(A)<\infty \text{ and } R(A) \text{ is closed in } X\}.$$

By $\Phi(X) := \Phi_+(X)\cap\Phi_-(X)$ we denote the set of Fredholm operators while $\Phi_\pm(X) := \Phi_+(X)\cup\Phi_-(X)$ is the set of semi-Fredholm operators. The index of an operator $A\in\Phi_\pm(X)$ is $i(A) := \alpha(A)-\beta(A)$. The upper semi-Fredholm spectrum of A is defined by

$$\sigma_{uf}(A) := \{\lambda\in\mathbb{C} \text{ such that } \lambda-A\notin\Phi_+(X)\} \text{ (see e.g. [15]).}$$

Many equations of mathematical physics can be cast in the abstract form:

$$u'(t)=Au(t),\quad t\geq 0,\quad u(0)=u_0\quad (CP1)$$

on a Banach space X. Here A is a given linear operator with domain $\mathcal{D}(A)$ and the initial value u_0. The semigroups can be used to solve a large class of problems commonly known as the Cauchy problem of first order. The solution of $(CP1)$ will be given by $u(t)=T(t)u$ for an operator semigroup $(T(t))_{t\geq 0}$ on X. In order to understand the behavior of the solutions in terms of the data concerning A, one

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seeks information about the spectrum of $T(t)$ in terms of the spectrum of A. The study of the relation between the spectrum $\sigma(A)$ of the generator A and the spectrum $\sigma(T(t))$ of the semigroup represented by investigating the below equality that we call spectral equality:

$$\sigma(T(t))\backslash\{0\} = \{e^{\lambda t},\ \lambda \in \sigma(A)\} \ \ (1).$$

This work push to ask the following question: Does this spectral equality hold for the upper semi-fredholm spectrum σ_{uf}?
It is worth noting that there is a relation between the equality (1) and when the difference $e^{\lambda t} - T(t)$ of two C_0-semigroups $(e^{\lambda t})_{t\geq 0}$ and $(T(t))_{t\geq 0}$ is upper semi-Fredholm.

This gives rise to the general study when the difference $S(t) - T(t)$ of two given C_0-semigroups has the property of being upper semi-Fredholm.

By definition, a family of bounded linear operators $(S(t))_{t\geq 0}$ in a Banach space X is called strongly continuous semigroup if
(1) $S(0) = I$.
(2) $S(t + s) = S(t)S(s)$ for $t, s \geq 0$.
(3) $S(t)$ is strongly continuous in t for $t \geq 0$.
It then follows that there exist constants $M \geq 1$ and $w \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. The linear operator A defined by

$$\mathcal{D}(A) = \Big\{x \in X \text{ such that } \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists}\Big\}$$

and

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t} = \frac{d}{dt} S(t)x|_{t=0} \text{ for } x \in \mathcal{D}(A)$$

is the infinitesimal generator of the semigroup $S(t)$ and $\mathcal{D}(A)$ is the domain of A. The resolvent set of the generator A contains the ray $]\omega, \infty[$ (see [13]). Recall that $(S(t))_{t\geq 0}$ is a uniformly continuous semigroup if it is a strongly continuous semigroup such that $\lim_{t\to 0^+} \|S(t) - I\| = 0$ holds. In [13, Theorem 1.2, p. 2], it is proved that a linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.
For the necessary concepts in the theories of semigroups of linear operators we refer the reader to the monographs [1,5,13].

Let us recall from [14] that an operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$, where $\mathcal{D}(T)$ is a subspace of X, is said to be demicompact if, for every bounded sequence $\{x_n\}$ in the domain $\mathcal{D}(T)$ such that $\{x_n - Tx_n\}$ converges to $x \in X$, there is a convergent subsequence of $\{x_n\}$. Note that each compact operator is demicompact, but the opposite is not always true. In fact, let $Id_X : X \longrightarrow X$ be the identity operator of a Banach space X of infinite dimension. It is clear that $-Id_X$ is demicompact but it is not compact. It is clear that the sum of demicompact and compact operators is demicompact. The concept of demicompactness was introduced by Petryshyn [14] in order to discuss fixed points. Jeribi [8] used the class of demicompact operators to obtain some results on Fredholm and spectral theories. In 2018, the authors [9] proved that an upper semi-Fredholm operator can be characterized by means of demicompactness concept as follows:

Theorem 1.1 [9, Theorem 2.1] *Let X be a Banach space and $A \in \mathcal{C}(X)$. Then, A is demicompact if and only if $I - A \in \Phi_+(X)$.*

In [2], Benkhaled et al established some specific demicompactness properties of semigroups of operators. More precisely, they considered a class of strongly continuous semigroups $(S(t))_{t\geq 0}$ having the property for some value of t, $S(t)$ is demicompact. They established that if $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X with infinitesimal generator A and if $\Theta = \{t > 0$ such that $S(t)$ is demicompact$\}$, under some conditions, then the following statements are equivalent:
(a) $\Theta =]0, +\infty[$,
(b) $I - A$ is demicompact,

(c) $\lambda R(\lambda, A)$ is demicompact for some (and then for all) $\lambda > \omega$.
Benkhaled et al's result was extended to strongly continuous cosine families of operators (see [3]).

The purpose of this work is to pursue the analysis started in [2] and to continue in this direction by studying the demicompactness of the operators $(I - (S(t) - T(t)))$ when $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are C_0-semigroups

An outline of this article is as follows. In Section 2, we recall some definitions and results needed in the sequel of the paper. In Section 3, we give a sufficient condition guaranteeing that $(S(t))_{t\geq 0}$ can be embedded in a C_0-group on X (see Theorem 3.1). Based on this result, we present some properties of two strongly continuous semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ when $I - (S(t) - T(t))$ is demicompact for $t > 0$ (see Theorem 3.4). The obtained results are applied to investigate the spectral equality for a uniformly continuous semigroup for upper semi-Fredholm spectrum (see Theorem 3.3 and Corollary 3.1).

2. Preliminary results

The aim of this section is to collect the most important definitions and elementary results which are used throughout this paper.

Definition 2.1 [10] Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be $\inf\{d > 0$ such that D can be covered by a finite number of sets of diameter less than or equal to d$\}$.

Definition 2.2 [11] Let $A : \mathcal{D}(A) \subseteq X \longrightarrow X$ be a continuous operator and let $\gamma(.)$ be the Kuratowski measure of noncompactness in X. Let $k \geq 0$. A is said to be a k-set-contraction, if for any bounded subset B of $\mathcal{D}(A)$, $A(B)$ is a bounded subset of X and $\gamma(A(B)) \leq k\gamma(B)$.

Theorem 2.1 [8] *Let $A \in \mathcal{C}(X)$. If A is a demicompact 1-set-contraction, then $I - A$ is a Fredholm operator of index zero.*

Theorem 2.2 [5] *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Definition 2.3 [13] A strongly continuous group on X is a family of operators $(S(t))_{t\geq 0}$ satisfying the conditions of definition of a strongly continuous semigroup but with $\mathbb{R}_+$ replaced by $\mathbb{R}$.

Remark 2.1 [13] Let $(S(t))_{t\in\mathbb{R}}$ be a strongly continuous group with generator A on the Banach space X. It is clear that for $t \geq 0$, $S(-t)$ is a strongly continuous semigroup with the infinitesimal generator $-A$.

Proposition 2.1 [5] *Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup with generator $(A, \mathcal{D}(A))$ on the Banach space X. Let $(B, \mathcal{D}(B))$ be the generator of a second strongly continuous semigroup $(T(t))_{t\geq 0}$ commuting with $(S(t))_{t\geq 0}$, i.e., $S(t)T(t) = T(t)S(t)$ for all $t \geq 0$. Then the operators*

$$U(t) = S(t)T(t) \text{ for } t \geq 0$$

form a strongly continuous semigroup $(U(t))_{t\geq 0}$, called the product semigroup of $(S(t)_{t\geq 0}$ and $(T(t))_{t\geq 0}$ and its generator is $Cy = Ay + By$ with domain $\mathcal{D}(C) = \mathcal{D}(A) \cap \mathcal{D}(B)$.

Theorem 2.3 [4, Theorem 1] *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of bounded operators. If for some $t_0 > 0$, $T(t_0) - I$ is compact, then $T(t)$ is invertible for every $t > 0$ and $(T(t))_{t\geq 0}$ can be embedded in a C_0-group.*

Theorem 2.4 [12, Theorem 2.1] *A strongly continuous semigroup $(S(t))_{t\geq 0}$ can be embedded in a strongly continuous group on X if and only if there exists $t_0 > 0$ such that $S(t_0) \in \Phi(X)$.*

Proposition 2.2 [6, Lemma 1] *Let $t_0 > 0$ and let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X.*
(i) If $S(t_0) \in \Phi_+(X)$, then $S(t) \in \Phi_+(X)$ and $\alpha(S(t)) = 0$ for all $t \geq 0$.
(ii) If $S(t_0) \in \Phi_-(X)$, then $S(t) \in \Phi_-(X)$ and $\beta(S(t)) = 0$ for all $t \geq 0$.
(iii) If $S(t_0) \in \Phi(X)$, then $S(t) \in \Phi(X)$ and $i(S(t)) = 0$ for all $t \geq 0$.

Theorem 2.5 [2, Theorem 3.1] *Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup with generator A on the Banach space X. If $S(t)$ is demicompact for $t > 0$, then $I - A$ is demicompact.*

3. Main results

Let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ denote the strongly continuous semigroups generated by linear operators A and B respectively on a Banach space X. In all the sequel of this section, we need to define the following set by

$$\mathcal{D}(t)=\{t>0 \text{ such that } I-S(t) \text{ is demicompact and a 1-set-contraction}\}.$$

We start by the following theorem which gives some conditions on semigroups guaranteeing the embedded in groups.

Theorem 3.1 *Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If $\mathcal{D}(t)\neq\emptyset$, then $(S(t))_{t\geq 0}$ can be embedded in a strongly continuous group on X and $S(t)$ is invertible for every $t\geq 0$.*

Proof: By hypothesis there exists $t_0>0$ such that $I-S(t_0)$ is demicompact and a 1-set-contraction. From Theorem 2.1, we have $S(t_0)$ is a Fredholm operator of index zero. On the one hand, thanks to Theorem 2.4, we obtain $(S(t))_{t\geq 0}$ can be embedded in a group $(S(t))_{t\in\mathbb{R}}$ on X. On the other hand, following Proposition 2.2, we infer that $S(t)\in\Phi(X)$ and $\alpha(S(t))=\beta(S(t))=0$ for all $t\geq 0$. Therefore, $S(t)$ is invertible for all $t\geq 0$. □

Remark 3.1 Theorem 3.1 is a generalisation of Theorem 1 of Cuthbert [4] and Theorem 6.6 in page 24 of Pazy [13].

The following examples illustrate Theorem 3.1.

Example 3.1 *Let Ω is a domain in $\mathbb{R}$ and $X=C_0(\Omega)$ the space of continuous functions vanishing at infinity.*

1. *Let $(S(t))_{t\geq 0}$ be the strongly continuous semigroup given by $S(t)=I$ for all $t\geq 0$. Clearly, $\mathcal{D}(t)=]0,\infty[$.*
2. *Consider $q:\Omega\longrightarrow\mathbb{R}$ be a continuous function such that $0\neq\sup_{s\in\Omega}q(s)<\infty$. The operators:*

$$S(t)f=e^{tq}f,\quad t\geq 0 \text{ and } f\in X$$

define a strongly continuous semigroup on the space X. $(S(t))_{t\geq 0}$ is called the multiplication semigroup (see [5, Definition I.4.3]). In this case, we have $t_0=\dfrac{1}{2\sup_{s\in\Omega}q(s)}\in\mathcal{D}(t)$.

Remark 3.2 Note that, in Examples 3.1 (2), the condition assumed in Cuthbert's result ($I-S(t_0)$ is compact) is not satisfied.

In the next, we devoted to discus the relationship between the demicompactness of $I-(A-B)$, $I-(R(\lambda,A)-R(\lambda,B))$ and $I-(S(t)-T(t))$.

Theorem 3.2 *Let A and B generate uniformly continuous semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$, respectively, on a Banach space X, and $\|S(t)\|$, $\|T(t)\|\leq Me^{wt}$ for some constants $M\geq 1$, $w\in\mathbb{R}$. Then, the following statements are equivalent:*

(1) *$I-(A-B)$ is demicompact.*

(2) *$I-(R(\lambda,A)-R(\lambda,B))$ is demicompact for all $\lambda>\omega$.*

Proof: Let $\lambda>\omega$ and $\{x_n\}$ be a bounded sequence of X such that $(R(\lambda,A)-R(\lambda,B))x_n$ converges to y. Since we can write

$$(R(\lambda,A)-R(\lambda,B))x_n=R(\lambda,A)(A-B)R(\lambda,B)x_n,$$

then $R(\lambda,A)(A-B)R(\lambda,B)x_n$ converges to y. On the other hand, since $(S(t))_{t\geq 0}$ is uniformly continuous, $A\in\mathcal{L}(X)$. Hence $(A-B)R(\lambda,B)x_n$ converges to $(\lambda-A)y$. Thus, the demicompactness of $I-(A-B)$

implies that $\{R(\lambda, B)x_n\}$ has a strongly convergent subsequence $\{R(\lambda, B)x_{n_k}\}$ which converges to z. Since $(T(t))_{t\geq 0}$ is uniformly continuous, $B \in \mathcal{L}(X)$. Hence, we infer that $\{x_n\}$ has a strongly convergent subsequence $\{x_{n_k}\}$ which converges to $(\lambda - B)z$ and so $I - (R(\lambda, A) - R(\lambda, B))$ is demicompact.
Conversely, let $\{x_n\}$ be a bounded sequence of X such that $(A - B)x_n$ converges to y. Hence, from the relation,

$$(A - B)x_n = (\lambda - A)(R(\lambda, A) - R(\lambda, B))(\lambda - B)x_n \ , \ \lambda > \omega,$$

it follows that $(\lambda-A)(R(\lambda, A)-R(\lambda, B))(\lambda-B)x_n$ converges to y. This means that $(R(\lambda, A)-R(\lambda, B))(\lambda-B)x_n$ converges to $R(\lambda, A)y$. In view of the demicompactness of $I - (R(\lambda, A) - R(\lambda, B))$, we obtain that $\{(\lambda - B)x_n\}$ has a strongly convergent subsequence $\{(\lambda - B)x_{n_k}\}$ which converges to z. Then, $\{x_n\}$ contains a strongly convergent subsequence. Therefore, $I - (A - B)$ is demicompact. □

Proposition 3.1 *Let A and B generate strongly continuous semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$, respectively, on a Banach space X, and $\|S(t)\|$, $\|T(t)\| \leq Me^{wt}$ for some constants $M \geq 1$ and $w \in \mathbb{R}$. If $I - (R(\lambda, A) - R(\lambda, B))$ is demicompact for some $\lambda > \omega$, then $I - (S(t) - T(t))$ is demicompact for $t > 0$.*

Proof: Let $t > 0$ and $\{x_n\}$ be a bounded sequence of X such that $(S(t) - T(t))x_n$ is convergent. First, setting $\varphi(t) = \lim_{n\to\infty} e^{-\lambda t}(S(t) - T(t))x_n$ for some $\lambda > \omega$ and $w \in \mathbb{R}$. Furthermore, there exists $\gamma \in \mathbb{R}$ such that for every $n \geq \gamma$ we have $t \longmapsto e^{-\lambda t}(S(t) - T(t))x_n$ is integrable on $[0, \infty[$. Also, there exists constant $M_1 \geq 1$ such that

$$\|e^{-\lambda t}(S(t) - T(t))x_n\| \leq M_1 e^{-(\lambda-\omega)t}$$

for $t > 0$ and $\lambda > \omega$. Thus, we have

$$t \longmapsto M_1 e^{-(\lambda-\omega)t}$$

is integrable on $[0, \infty[$. Therefore, thanks to Lebesgues dominated convergence Theorem 3.7.9 in [7], we have

$$\lim_{n\to\infty} \int_0^\infty e^{-\lambda t}(S(t) - T(t))x_n dt = \int_0^\infty \varphi(t)dt.$$

On the other hand, from the integral representation

$$(R(\lambda, A) - R(\lambda, B))x_n = \int_0^\infty e^{-\lambda t}(S(t) - T(t))x_n dt$$

for $\lambda > \omega$, we conclude that

$$(R(\lambda, A) - R(\lambda, B))x_n \text{ converges to } \int_0^\infty \varphi(t)dt.$$

Therefore, the demicompactness of $I - (R(\lambda, A) - R(\lambda, B))$ ensures that $\{x_n\}$ contains a strongly convergent subsequence and hence we deduce that $I - (S(t) - T(t))$ is demicompact for $t > 0$. □

Now, we ask: what effect does the demicompactness of $I - (S(t) - T(t))$ have on $I - (A - B)$? For this, we need to prove the following lemma.

Lemma 3.1 *Let $A \in \mathcal{C}(X)$. Then, $I - A$ is demicompact if and only if $I + A$ is demicompact.*

Proof: Let $\{x_n\}$ be a bounded sequence of X such that $-Ax_n$ converges to y. This implies that Ax_n converges to $-y$. Using the demicompactness of A, we infer that $\{x_n\}$ has a convergent subsequence and we conclude that $I + A$ is demicompact.
Conversely, we establish the same reasoning by replacing $-A$ with A. □

Thanks to Theorem 3.1 and Lemma 3.1, we have:

Proposition 3.2 *Suppose that the strongly continuous semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ commute and $\mathcal{D}(B) \subseteq \mathcal{D}(A)$. Assume that $\mathcal{D}(t) \neq \emptyset$. If $I - (S(t) - T(t))$ is demicompact for $t > 0$, then $I - (A - B)$ is demicompact.*

Proof: Our proof is divided into two steps.
Step 1: After the use of Theorem 3.1 to obtain that the strongly continuous semigroup $(S(t))_{t\geq 0}$ can be embedded in a group $(S(t))_{t\in\mathbb{R}}$ on X and by Remark 2.1, $(S(-t))_{t\geq 0}$ is a strongly continuous semigroup with the infinitesimal generator $-A$. We claim that, for all $t>0$, $S(-t)T(t)$ is demicompact.
Let $t>0$ and $\{x_n\}$ be a bounded sequence of X such that $(I-S(-t)T(t))x_n$ converges to y. Since $S(t)\in\mathcal{L}(X)$, we have $S(t)(I-S(-t)T(t))x_n$ converges to $S(t)y$. From the following identity

$$(S(t)-T(t))x_n = S(t)(I-S(-t)T(t))x_n,$$

we obtain $(S(t)-T(t))x_n$ converges to $S(t)y$. As $I-(S(t)-T(t))$ is demicompact, $\{x_n\}$ contains a strongly convergent subsequence and so that $S(-t)T(t)$ is demicompact for every $t>0$.
Step 2: We prove that $I-(A-B)$ is demicompact.
On the one hand, $(S(-t)T(t))_{t\geq 0}$ is a strongly continuous semigroup on X with generator $B-A$ (see Proposition 2.1). On the other hand, using Theorem 2.5, we infer that the operator $I-(B-A)$ is demicompact. In view of Lemma 3.1, we obtain $I-(A-B)=I+(B-A)$ is demicompact. □

Remark 3.3 Note that, if we consider the strongly continuous semigroup $(S(t))_{t\geq 0}$ given by $S(t)=I$ for all $t\geq 0$. In this case, we have its infinitesimal generator $A=0$ and so we recover the result obtained in [2, Theorem 3.1].

The aim of the following result is to apply Proposition 3.2 to investigate the spectral inclusion for strongly continuous semigroups for an upper semi-Fredholm spectrum.

Theorem 3.3 *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup and let A be its infinitesimal generator. Then, we have the following spectral inclusion:*

$$e^{t\sigma_{uf}(A)} \subseteq \sigma_{uf}(T(t))\backslash\{0\}.$$

Proof: Step 1: We claim that if $I-(e^{\lambda t}-T(t))$ is demicompact then $I-(\lambda-A)$ is demicompact.
Our idea is to used Proposition 3.2 which we consider the strongly continuous semigroup $(S(t)_{t\geq 0}$ given by $S(t)=e^{\lambda t}$ for $t\geq 0$ and $\lambda\in\mathbb{C}$. The infinitesimal generator of $(S(t))_{t\geq 0}$ is given by $B.f=\lambda.f$ with domain $D(B)=\{f\in X$ such that $\lambda.f\in X\}$. It is not hardy to see that $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ commute. Furthermore, there exists $t_0>0$ such that $I-S(t_0)$ is demicompact and a 1-set-contraction. Indeed, Let $\{f_n\}$ be a bounded sequence of X such that $f_n-(I-S(t_0))f_n\to g\in X$. We have to show that $\{f_n\}$ has a convergent subsequence. For each n, we have, $\|f_n-(I-S(t_0))f_n-g\|=\|S(t_0)f_n-g\|=\|e^{t_0\lambda}f_n-g\|=e^{t_0\Re\lambda}\|f_n-e^{-t_0\lambda}g\|$. Since $\|f_n-(I-S(t_0))f_n-g\|\to 0$, then $\|f_n-e^{-t_0\lambda}g\|\to 0$. So, $I-S(t_0)$ is demicompact.
Let D be a bounded subset of X. It suffices to show that $\gamma((I-S(t_0))D)\leq\gamma(D)$. We have

$$\gamma((I-S(t_0))D)=\gamma((1-e^{t_0\lambda})D)=|1-e^{t_0\lambda}|\gamma(D).$$

Since $\lim\limits_{t_0\to 0} e^{\lambda t_0}=1$, then for $\varepsilon=1$, there exists $\eta>0$ such that $\forall\ t_0\in]0,\eta[$, we have $|1-e^{\lambda t_0}|\leq 1$. Therefore, $\gamma((I-S(t_0))D)\leq\gamma(D)$. Then, $I-S(t_0)$ is a 1-set-contraction.
Step 2: We prove that if $e^{\lambda t}-T(t)$ is upper semi-Fredholm then $\lambda-A$ is upper semi-Fredholm.
Suppose that $e^{\lambda t}-T(t)$ is upper semi-Fredholm. Using Theorem 1.1, we infer that $I-(e^{\lambda t}-T(t))$ is demicompact. By Step 1, we obtain that $I-(\lambda-A)$ is demicompact. In view of the same Theorem 1.1, we conclude that $\lambda-A$ is upper semi-Fredholm. □

Example 3.2 *Let $X=C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic continuous functions on $\mathbb{R}$. Consider the translation group on the space X (see* [5, Paragraph I. 4.15]*). Its infinitesimal generator is denoted by A. By virtue of* [5, Exmp 2.6.iv]*, we obtain $\sigma(A)=i\mathbb{Z}$. The use of Theorem 3.3 asserts that*

$$\{e^{\lambda t},\ \lambda\in\sigma_{uf}(A)\}\subseteq\sigma_{uf}(T(t))\backslash\{0\}.$$

The following conclusion summarizes all the previous results by which we can characterize the demicompactness of $I-(S(t)-T(t))$ by the demicompactness of $I-(A-B)$ and $I-(R(\lambda,A)-R(\lambda,B))$.

Theorem 3.4 *Let A and B generate uniformly continuous semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$, respectively, on a Banach space X, and $\|S(t)\|$, $\|T(t)\| \leq Me^{wt}$ for some constants $M \geq 1$, $w \in \mathbb{R}$. Assume that $\mathcal{D}(t) \neq \emptyset$. Then, the following statements are equivalent:*

(1) *$I - (S(t) - T(t))$ is demicompact for every $t > 0$.*

(2) *$I - (A - B)$ is demicompact.*

(3) *$I - (R(\lambda, A) - R(\lambda, B))$ is demicompact for all $\lambda > \omega$.*

Proof: (1) $\Longrightarrow$ (2) Since $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are uniformly continuous semigroup with generators A and B respectively. Then, $S(t) = e^{tA}$ and $T(t) = e^{tB}$ for all $t \geq 0$ (see [5, Theorem I.3.7]). Clearly that $e^{tA}e^{tB} = e^{tB}e^{tA}$ for all $t \geq 0$. Thus, the semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ commute. So, we can use Proposition 3.2 to obtain the result.
(2) $\Longleftrightarrow$ (3) See Theorem 3.2.
(3) $\Longrightarrow$ (1) It follows from Proposition 3.1. □

To clarify our result, we derive the following example:

Example 3.3 *Let $X = C_0(\Omega)$ with Ω is a domain in $\mathbb{R}$. We consider the uniformly continuous semigroup $(S(t))_{t\geq 0}$ defined in Examples 3.1 (2) by*

$$S(t)f = e^{tq}f, \quad t \geq 0 \text{ and } f \in X.$$

Its generator is given by the multiplication operator $Af = q.f$ with domain $\mathcal{D}(A) = \{f \in X$ such that $q.f \in X\}$. It is worth noting that $\mathcal{D}(t) \neq \emptyset$. Suppose now that $T(t) = e^{tq_1}$ such that $q_1 : \Omega \longrightarrow \mathbb{R}$ be a bounded function such that $q_1 \neq q$. Then, the family operators $(T(t))_{t\geq 0}$ define a uniformly continuous semigroup on the space X and its generator is denoted by $Bf = q_1.f$. For $t > 0$, $I - (S(t) - T(t))$ is demicompact. The use of Theorem 3.4 asserts that $I - (A - B)$ is demicompact.

As a consequence, we can apply the results obtained in Theorems 1.1 and 3.4 to investigate the spectral equality for uniformly continuous semigroups for an upper semi-Fredholm spectrum.

Corollary 3.1 *Let $(T(t))_{t\geq 0}$ be a uniformly continuous semigroup with generator A on the Banach space X. Then,*

$$e^{t\sigma_{uf}(A)} = \sigma_{uf}(T(t))\backslash\{0\}.$$

Proof: We establish the same reasoning as the proof of Theorem 3.3. □

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