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A New Generalized Beta Function Associated with Statistical Distribution and Fractional Kinetic Equation

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ABSTRACT: Several authors have extensively investigated beta function, hypergeometric function, and confluent hypergeometric function, their extensions, and generalizations due to their several applications in many areas of engineering, probability theory, and science. The main purpose of this paper is to present a new generalization of the extended beta function, hypergeometric function, and confluent hypergeometric function with the help of the m-parameter Mittag-Leffler function, as well as examine some important properties like integral representations, differential formulas, and summation formulas. We also examine the generalized Caputo fractional derivative operator with the help of the m-parameter Mittag-Leffler function and associated properties using the generalized beta function. We define a new beta distribution involving the new generalized beta function. The mean, variance, coefficient of variance, moment generating function, characteristic function, and cumulative distribution are derived. Further, we derive the solution of a fractional kinetic equation involving generalized hypergeometric functions.

Key Words: Extended beta function, generalized m-parameter Mittag-Leffler function, statistical distribution, Caputo fractional derivative operator, fractional Kinetic equation, Laplace transform.

Contents

1. Introduction

Special functions are renowned for their significance and applications in several domains, especially in engineering sciences, statistics, and mathematical physics like string theory, astronomy, etc. Several generalizations and extensions of well-known special functions like the Bessel-Maitland function, the Mittag-Leffler function, etc., have been investigated by numerous researchers ([1], [6], [7], [8], [23]). One of the most well-known and valuable special function with various applications, classified as beta function, was taken into consideration by many authors ([2], [4], [5], [15], [22]).
Fractional calculus has substantial applications in various branches of science, including control theory, fluid mechanics, signal processing, and bioengineering. The fractional calculus, integral and differential operators involving a variety of special functions have been utilized and shown to play a significant role in various domains of applied mathematical analysis ([21], [25]). The Caputo fractional derivative operator

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also plays an essential role in fractional calculus due to its numerous scientific applications. We introduce the novel generalization of the beta function and investigate its many properties and applications in diverse fields, all of which are motivated by the prior literature. We start by recalling the classical definitions. The classical beta function is defined as [20]

$$B(\chi_1,\chi_2)=\int_0^1 t^{\chi_1-1}(1-t)^{\chi_2-1}\,dt=\frac{\Gamma(\chi_1)\Gamma(\chi_2)}{\Gamma(\chi_1+\chi_2)}, \tag{1.1}$$

where $\Re(\chi_1),\Re(\chi_2)>0$. The Gauss hypergeometric and confluent hypergeometric functions [20] are defined as

$$_2F_1(\chi_1,\chi_2;\chi_3;z)=\sum_{n=0}^{\infty}\frac{(\chi_1)_n(\chi_2)_n}{(\chi_3)_n}\frac{z^n}{n!}, \tag{1.2}$$

where $|z|<1$, $\chi_1,\chi_2,\chi_3\in\mathbb{C}$; $\chi_3\neq 0,-1,-2,\ldots$ and

$$_1\phi_1(\chi_2;\chi_3;z)=\sum_{n=0}^{\infty}\frac{(\chi_2)_n}{(\chi_3)_n}\frac{z^n}{n!}, \tag{1.3}$$

respectively, where $\chi_2,\chi_3\in\mathbb{C}$ and $\chi_3\neq 0,-1,-2,\ldots$ and $(\chi)_n$ is the pochhammer symbol defined as [20], for $\chi\neq 0,-1,-2,\ldots$.

$$(\chi)_n=\prod_{k=1}^{n}(\chi+k-1);n\in\mathbb{N}\;and\;(\chi)_n=\frac{\Gamma(\chi+n)}{\Gamma(\chi)}. \tag{1.4}$$

Rahman et al. [19] extended the beta function as follows:

$$B_{\alpha}^{p,\lambda}(\chi_1,\chi_2)=\int_0^1 t^{\chi_1-1}(1-t)^{\chi_2-1}E_{\alpha}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right)dt, \tag{1.5}$$

where $\Re(\chi_1)>0,\Re(\chi_2)>0,\Re(p)\geq 0,\Re(\alpha)>0,\Re(\lambda)>0$ and E_α is the function defined by Mittag-Leffler [17] as

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\alpha\in\mathbb{C},\Re(\alpha)>0. \tag{1.6}$$

Orbay et al. [18] generalized the beta function by using the generalized 2-parameter Mittag-Leffler function as follows:

$$B_{\alpha,\tau}^{p,\lambda}(\chi_1,\chi_2)=\int_0^1 t^{\chi_1-1}(1-t)^{\chi_2-1}E_{\alpha,\tau}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right)dt, \tag{1.7}$$

where $\Re(\chi_1)>0,\Re(\chi_2)>0,\Re(p)\geq 0,\Re(\tau)>0,\Re(\lambda)>0,\Re(\alpha)>0\,and$ $E_{\alpha,\tau}$ is the generalized Mittag-Leffler function defined by (see [26])

$$E_{\alpha,\tau}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\tau)},\;\alpha,\tau\in\mathbb{C},\Re(\alpha)>0,\Re(\tau)>0. \tag{1.8}$$

Clearly, for $p=0\,and\,\lambda=\alpha=\tau=1$, then equation (1.7) reduces to classical beta function (1.1).
Further, Oraby et al. [18] generalized the hypergeometric and confluent hypergeometric functions using the generalized beta function as follows:

$$F_{\alpha,\tau}^{p,\lambda}(\chi_1,\chi_2;\chi_3;z)=\sum_{n=0}^{\infty}(\chi_1)_n\frac{B_{\alpha,\tau}^{p,\lambda}(\chi_2+n,\chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}, \tag{1.9}$$

where $\Re(\lambda)>0,\Re(\alpha)>0,\Re(\tau)>0,\Re(\chi_1)>0,\Re(\chi_3)>\Re(\chi_2)>0,\Re(p)\geq 0$ and $|z|<1$ and

$$\phi_{\alpha,\tau}^{p,\lambda}(\chi_2;\chi_3;z)=\sum_{n=0}^{\infty}\frac{B_{\alpha,\tau}^{p,\lambda}(\chi_2+n,\chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\frac{z^n}{n!}, \tag{1.10}$$

where $\Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\tau) > 0, \Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0$ and $|z| < 1$, with the following integral representations:

$$F_{\alpha,\tau}^{p,\lambda}(\chi_1, \chi_2; \chi_3; z) = \frac{1}{B(\chi_2, \chi_3 - \chi_2)} \int_0^1 t^{\chi_2 - 1}(1-t)^{\chi_3 - \chi_2 - 1}(1 - zt)^{-\chi_1} E_{\alpha,\tau}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt, \quad (1.11)$$

where $\Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\tau) > 0, \Re(\chi_1) > 0, \Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0$ and $|z| < 1$ and

$$\phi_{\alpha,\tau}^{p,\lambda}(\chi_2; \chi_3; z) = \frac{1}{B(\chi_2, \chi_3 - \chi_2)} \int_0^1 t^{\chi_2 - 1}(1-t)^{\chi_3 - \chi_2 - 1} e^{zt} E_{\alpha,\tau}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt, \quad (1.12)$$

where $\Re(\lambda) > 0, \Re(\alpha) > 0, \Re(\tau) > 0, \Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0$ and $|z| < 1$.
Agarwal et al. [1] defined the generalized m-parameter Mittag-Leffler function as follows:

$$E_{\alpha,\tau;(\beta,k)_s}^{(\mu,\nu)_r}(t) = \sum_{n=0}^{\infty} \frac{(\mu_1)_{\nu_1 n}(\mu_2)_{\nu_2 n} \dots (\mu_r)_{\nu_r n}}{\Gamma(\alpha n + \tau)(\beta_1)_{\kappa_1 n} \dots (\beta_s)_{\kappa_s n}} t^n, \quad (1.13)$$

where $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re\{\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j\} > 0$ for $i = 1 \dots, r$ and $j = 1 \dots s$ with r+s= m-2, m be any positive integer and t is a complex variable.
The classical Caputo fractional derivative operator (see [13]) is defined as follows:

$$D_z^y[f(z)] = \frac{1}{\Gamma(n-y)} \int_0^z (z-t)^{n-y-1} \frac{d^n}{dt^n} f(t)\, dt, \quad (1.14)$$

where $n - 1 < \Re(y) < n, n \in \mathbb{N}$.
Goyal et al. [10] introduced the extended Caputo fractional operator as follows:

$$D_{z,\alpha,\tau}^{y,p}[f(z)] = \frac{1}{\Gamma(n-y)} \int_0^z (z-t)^{n-y-1} E_{\alpha,\tau}\left(\frac{-pz^2}{t(z-t)}\right) \frac{d^n}{dt^n} f(t)\, dt, \quad (1.15)$$

where min $\{\Re(\alpha), \Re(\tau)\} > 0, \Re(p) > 0,\ n - 1 < \Re(y) < n, n \in \mathbb{N}$ and $E_{\alpha,\tau}$ is the 2-parameter Mittag-Leffler function.

2. Generalization of Beta function

In this section, a new generalization of beta function using generalized m-parameter Mittag-Leffler function has been introduced.

Definition 2.1. *The generalized form of the beta function is defined as*

$$B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_1, \chi_2) = \int_0^1 t^{\chi_1 - 1}(1-t)^{\chi_2 - 1} E_{\alpha,\tau;(\beta,\kappa)_s}^{(\mu,\nu)_r}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt, \quad (2.1)$$

where, $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \dots r; j = 1, \dots, s$, r+s= m-2, where m be any positive integer and $\Re(\chi_1) > 0, \Re(\chi_2) > 0, \Re(p) \geq 0, \Re(\lambda) > 0$, and $E_{\alpha,\tau;(\beta,\kappa)_s}^{(\mu,\nu)_r}(t)$ is the generalized m-parameter Mittag-Leffler function defined by Agarwal et al. [1].

Remark 2.2. *For r=s=0, $\tau = \alpha = 1 = \lambda$ and p=0, then equation (2.1) reduces to classical beta function given by equation (1.1).*

Remark 2.3. *For r=s=0 and $\tau = 1$, equation (2.1) reduces to the extension of beta function given by equation (1.5).*

Remark 2.4. *For r=s=0, equation (2.1) reduces to the further extension given by Oraby et al. [18] defined by equation (1.7).*

3. Some properties of the generalized beta function

In this section, different integral representations and summation formulas of generalized beta function are obtained.

Theorem 3.1 (Integral representation)**.** *Each of the following integral representation holds true:*

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = 2\int_0^{\frac{\pi}{2}} cos^{2\chi_1-1}\theta\ sin^{2\chi_2-1}\theta\, E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{cos^{2\lambda}\theta sin^{2\lambda}\theta}\right) d\theta, \tag{3.1}$$

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \int_0^{\infty} \frac{u^{\chi_1-1}}{(1+u)^{\chi_1+\chi_2}} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(1+u)^{2\lambda}}{u^{\lambda}}\right) du, \tag{3.2}$$

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = (b-a)^{1-\chi_1-\chi_2}\int_a^{b} (u-a)^{\chi_1-1}\ (b-u)^{\chi_2-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(b-a)^{2\lambda}}{(u-a)^{\lambda}(b-u)^{\lambda}}\right) du, \tag{3.3}$$

Proof. Substituting $t = cos^2\theta$, $t = \frac{u}{1+u}$ and $t = \frac{u-a}{b-a}$ in equation (2.1), we get equations (3.1) - (3.3) respectively. □

Theorem 3.2. *The following result holds true for* $B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)$*:*

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \sum_{n=0}^{l}\binom{l}{n} B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,\chi_2+l-n), \tag{3.4}$$

$l \in \mathbb{N}_0$.

Proof. Using equation (2.1),we have

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \int_0^1 t^{\chi_1}(1-t)^{\chi_2-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right) dt +$$
$$\int_0^1 t^{\chi_1-1}(1-t)^{\chi_2} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right) dt,$$

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2) + B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2+1),$$

again applying the same argument to each of two terms, we get

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+2,\chi_2) + 2B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2+1) + B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2+2),$$

proceeding in the same manner, using induction on n, we get the desired result. □

Theorem 3.3. *The following result holds true for* $B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)$*:*

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,1-\chi_2) = \sum_{n=0}^{\infty}\frac{(\chi_2)_n}{n!} B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,1). \tag{3.5}$$

Proof.

$$LHS = B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,1-\chi_2) = \int_0^1 t^{\chi_1-1}(1-t)^{-\chi_2} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right) dt.$$

Using $(1-t)^{-\chi_2} = \sum_{n=0}^{\infty}\frac{(\chi_2)_n}{n!}t^n,\ |t|<1$,

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,1-\chi_2) = \int_0^1 t^{\chi_1-1}\sum_{n=0}^{\infty}\frac{t^n}{n!}(\chi_2)_n E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^{\lambda}(1-t)^{\lambda}}\right) dt.$$

Now, interchanging the order of integration and summation, we get

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1, 1-\chi_2) = \sum_{n=0}^{\infty} \frac{(\chi_2)_n}{n!} \int_0^1 t^{\chi_1+n-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt.$$

Using equation (2.1), we get the desired result. □

Theorem 3.4. *The following result holds true for* $B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)$*:*

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \sum_{n=0}^{\infty} B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n, \chi_2+1). \tag{3.6}$$

Proof.

$$LHS = B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \int_0^1 t^{\chi_1-1}(1-t)^{\chi_2-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt.$$

$Using\ (1-t)^{\chi_2-1} = (1-t)^{\chi_2}\sum_{n=0}^{\infty} t^n, |t|<1,$

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \int_0^1 t^{\chi_1-1}(1-t)^{\chi_2} \sum_{n=0}^{\infty} t^n E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt.$$

Now, interchanging the order of integration and summation

$$B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2) = \sum_{n=0}^{\infty} \int_0^1 t^{\chi_1+n-1}(1-t)^{\chi_2} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) dt.$$

Using equation (2.1), we get the desired result. □

4. Generalized hypergeometric and confluent hypergeometric functions

In this section, we introduce a new generalization of hypergeometric and confluent hypergeometric function using our newly defined beta function.

Definition 4.1. *The generalization of hypergeometric function with the help of newly defined beta function involving m-parameter Mittag-Leffler function is*

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \sum_{n=0}^{\infty}(\chi_1)_n \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2+n, \chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)} \frac{w^n}{n!}, \tag{4.1}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j \in \mathbb{C}$ *with min* $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j) > 0$ *for* $i = 1,\dots r; j = 1,\dots,s$ *r+s= m-2, where m be any positive integer and* $\Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0, \Re(\lambda) > 0$ *and* $|w| < 1$.

Definition 4.2. *The generalization of confluent hypergeometric function using our newly defined beta function involving m-parameter Mittag-Leffler function is*

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w) = \sum_{n=0}^{\infty} \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2+n, \chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)} \frac{w^n}{n!}, \tag{4.2}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j \in \mathbb{C}$ *with min* $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j) > 0$ *for* $i = 1,\dots r; j = 1,\dots,s$ *r+s= m-2, where m be any positive integer and* $\Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0, \Re(\lambda) > 0$ *and* $|w| < 1$.

Remark 4.3. *For r=s=0, equations (4.1) and (4.2) reduces to Gauss hypergeometric function and confluent hypergeometric function defined by Oraby et al. [18] given by equations (1.9) and (1.10) respectively.*

4.1. Integral representation

Theorem 4.4. *The following integral representation for the generalized hypergeometric function holds true:*

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 t^{\chi_2-1}(1-t)^{\chi_3-\chi_2-1}(1-tw)^{-\chi_1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right)dt, \tag{4.3}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j \in \mathbb{C}$ *with min* $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j) > 0$ *for* $i = 1,\ldots r; j = 1,\ldots,s$ *and* $\Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0, \Re(\lambda) > 0$ *and* $|w| < 1$.

Proof. From equation (4.1) and equation (2.1), we have

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \sum_{n=0}^{\infty}\frac{(\chi_1)_n}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 t^{\chi_2+n-1}(1-t)^{\chi_3-\chi_2-1} \times E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right)\frac{(w)^n}{n!}dt,$$

using $\sum_{n=0}^{\infty}\frac{(\chi_1)_n(tw)^n}{n!} = (1-tw)^{-\chi_1}$, we get the desired result. □

Theorem 4.5. *The following integral representation for the generalized confluent hypergeometric function holds true:*

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w) = \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^1 t^{\chi_2-1}(1-t)^{\chi_3-\chi_2-1}e^{wt}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right)dt, \tag{4.4}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j \in \mathbb{C}$ *with min* $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j) > 0$ *for* $i = 1,\ldots r; j = 1,\ldots,s$ *and* $\Re(\chi_3) > \Re(\chi_2) > 0, \Re(p) \geq 0, \Re(\lambda) > 0$ *and* $|w| < 1$.

Proof. From equation (4.2) and equation (2.1), we have

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \frac{1}{B(\chi_2,\chi_3-\chi_2)}\sum_{n=0}^{\infty}\int_0^1 t^{\chi_2+n-1}(1-t)^{\chi_3-\chi_2-1} \times E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right)\frac{w^n}{n!}dt.$$

using $\sum_{n=0}^{\infty}\frac{(tw)^n}{n!} = e^{wt}$, we get the desired result. □

Theorem 4.6. *The following integral representations for the generalized hypergeometric function hold true:*

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^\infty a^{\chi_2-1}(1+a)^{\chi_1-\chi_3} \times(1+a(1-w))^{-\chi_1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(1+a)^{2\lambda}}{a^\lambda}\right)da, \tag{4.5}$$

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \frac{2}{B(\chi_2,\chi_3-\chi_2)}\int_0^{\frac{\pi}{2}} cos^{2\chi_2-1}\theta\ sin^{2\chi_3-2\chi_2-1}\theta \times(1-wcos^2\theta)^{-\chi_1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{cos^{2\lambda}\theta sin^{2\lambda}\theta}\right)d\theta, \tag{4.6}$$

$$F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w) = \frac{(c-u)^{1+\chi_1-\chi_3}}{B(\chi_2,\chi_3-\chi_2)}\int_u^c (a-u)^{\chi_2-1}(c-a)^{\chi_3-\chi_2-1} \times((c-u)-(a-u)w)^{-\chi_1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(c-u)^{2\lambda}}{(a-u)^\lambda(c-a)^\lambda}\right)da, \tag{4.7}$$

Proof. Substituting t= $\frac{a}{1+a}$, t=$cos^2\theta$ and $t = \frac{a-u}{c-u}$ in equation (4.3), we get equations (4.5) - (4.7) respectively. ∎

Theorem 4.7. *The following integral representations for the generalized confluent hypergeometric function hold true:*

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w) = \frac{1}{B(\chi_2,\chi_3-\chi_2)}\int_0^\infty a^{\chi_2-1}(1+a)^{-\chi_3}e^{\frac{wa}{(1+a)}} \times E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(1+a)^{2\lambda}}{a^\lambda}\right)da, \tag{4.8}$$

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w) = \frac{2}{B(\chi_2,\chi_3-\chi_2)}\int_0^{\pi/2} cos^{2\chi_2-1}\theta\ sin^{2\chi_3-2\chi_2-1}\theta\ e^{wcos^2\theta} \times E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{cos^{2\lambda}\theta sin^{2\lambda}\theta}\right)d\theta, \tag{4.9}$$

$$\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w) = \frac{(c-u)^{1-\chi_3}}{B(\chi_2,\chi_3-\chi_2)}\int_u^c (a-u)^{\chi_2-1}(c-a)^{\chi_3-\chi_2-1}e^{\left(\frac{w(a-u)}{(c-u)}\right)} \times E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p(c-u)^{2\lambda}}{(a-u)^\lambda(c-a)^\lambda}\right)da. \tag{4.10}$$

Proof. Substituting t= $\frac{a}{1+a}$, t=$cos^2\theta$ and $t = \frac{a-u}{c-u}$ in equation (4.4), we get equations (4.8) - (4.10) respectively. ∎

4.2. Differentiation formula

Theorem 4.8. *The following differentiation formula for the generalized hypergeometric and confluent hypergeometric functions hold true:*

$$\frac{d^n}{dw^n}\{F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w)\} = \frac{(\chi_1)_n(\chi_2)_n}{(\chi_3)_n}F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,\chi_2+n;\chi_3+n;w), \tag{4.11}$$

and

$$\frac{d^n}{dw^n}\{\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2;\chi_3;w)\} = \frac{(\chi_2)_n}{(\chi_3)_n}\phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2+n;\chi_3+n;w). \tag{4.12}$$

Proof. Differentiating equation (4.1) with respect to w, we have

$$\frac{d}{dw}\{F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w)\} = \sum_{n=1}^\infty(\chi_1)_n\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2+n,\chi_3-\chi_2)}{B(\chi_2,\chi_3-\chi_2)}\frac{w^{n-1}}{(n-1)!},$$

replacing n by n+1, and using $B(\chi_2,\chi_3-\chi_2) = \frac{\chi_3}{\chi_2}B(\chi_2+1,\chi_3-\chi_2)$, we have

$$\frac{d}{dw}\{F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w)\} = \frac{(\chi_1)(\chi_2)}{(\chi_3)}\sum_{n=0}^\infty\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2+n+1,\chi_3-\chi_2)}{B(\chi_2+1,\chi_3-\chi_2)}(\chi_1+1)_n\frac{w^n}{(n)!}.$$

$$\frac{d}{dw}\{F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w)\} = \frac{(\chi_1)(\chi_2)}{(\chi_3)}F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2+1;\chi_3+1;w), \tag{4.13}$$

again differentiating equation (4.13) with respect to w, we get

$$\frac{d^2}{dw^2}\{F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2;\chi_3;w)\} = \frac{(\chi_1)(\chi_1+1)(\chi_2)(\chi_2+1)}{(\chi_3)(\chi_3+1)}F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+2,\chi_2+2;\chi_3+2;w),$$

continue this n times, we get the required result (4.11).
We can prove our result (4.12) in a similar way. ∎

5. Generalization of Caputo Fractional Derivative 0perator

In this section, we define the generalization of Caputo fractional derivative operator with the help of m-parameter Mittag-Leffler function and investigate its properties using our new generalized beta function.

Definition 5.1. *The generalized Caputo fractional derivative is defined as*

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[f(z)] = \frac{1}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)\frac{d^n}{dt^n}f(t)\,dt, \tag{5.1}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j \in \mathbb{C}$ with min $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j) > 0$ for $i = 1,\dots r; j = 1,\dots,s$ with r+s= m-2, m be any positive integer and $n-1 < \Re(y) < n, n \in \mathbb{N}$ and $E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(t)$ is the generalized m-parameter Mittag-Leffler function.

Remark 5.2. *For $\lambda = 1$, r=s=0, we get the extended Caputo fractional derivative operator given by equation (1.15).*

Theorem 5.3. *Consider $n-1 < \Re(y) < n, n \in \mathbb{N}$and $\Re(y) < \Re(a)$. Then*

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^a] = \frac{\Gamma(a+1)}{\Gamma(a-y+1)}\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau,(\beta,\kappa)_s}(a-n+1,n-y)}{B(a-n+1,n-y)}z^{a-y}. \tag{5.2}$$

Proof. From the definition of generalized Caputo fractional Derivative Operator (5.1), we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^a] = \frac{1}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)\frac{d^n}{dt^n}t^a\,dt.$$

$$= \frac{[a(a-1)\dots(a-n+1)]}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)t^{a-n}\,dt.$$

$$= \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)t^{a-n}\,dt.$$

On substituting t=xz , we get

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^a] = \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n-y)}z^{a-y}$$
$$\int_0^1 x^{a-n}(1-x)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{x^{\lambda}(1-x)^{\lambda}}\right)dx,$$

using the definition of generalized beta function given by equation (2.1), we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^a] = \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n-y)}z^{a-y}B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau,(\beta,\kappa)_s}(a-n+1,n-y), \tag{5.3}$$

using $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we get our desired result. □

Remark 5.4. *For $a = 0,1,\dots n-1$, then $D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^a] = 0$.*

Theorem 5.5. *Assume that f(z) is an analytic function in the disc $|z| < \delta$ with the Taylor series expansion $f(z) = \sum_{l=0}^{\infty} b_l z^l$. Then*

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[f(z)] = \sum_{l=0}^{\infty} b_l D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^l], \tag{5.4}$$

where $n-1 < \Re(y) < n, n \in \mathbb{N}$.

Proof. We have the definition of generalized Caputo fractional derivative operator (5.1), we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[f(z)] = \frac{1}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)\frac{d^n}{dt^n} f(t)\,dt.$$

Applying the Taylor series expansion of f, we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[f(z)] = \frac{1}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)\frac{d^n}{dt^n}\left(\sum_{l=0}^{\infty} b_l t^l\right) dt,$$

since this power series converges uniformly and the integral converges absolutely, changing the order of integration and summation

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[f(z)] = \sum_{l=0}^{\infty} b_l \frac{1}{\Gamma(n-y)}\int_0^z (z-t)^{n-y-1} E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-pz^{2\lambda}}{t^{\lambda}(z-t)^{\lambda}}\right)\frac{d^n}{dt^n}(t^l)dt,$$

using the definition of generalized Caputo fractional derivative Operator (5.1), we get the desired result. □

Theorem 5.6. *Assume that f(z) is an analytic function in disc $|z| < \delta$ with the Taylor series $f(z) = \sum_{l=0}^{\infty} b_l z^l$. Then*

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^{\gamma-1}f(z)] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-y)} z^{\gamma-y-1}\sum_{l=0}^{\infty}(b_l)\frac{(\gamma)_l}{(\gamma-n)_l}\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau,(\beta,\kappa)_s}(\gamma+l-n,n-y)}{B(\gamma-n,n-y)} z^l, \tag{5.5}$$

where $n-1 < \Re(y) < n < \Re(\gamma)$.

Proof. Using Theorem (5.5), we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^{\gamma-1}f(z)] = \sum_{l=0}^{\infty} b_l D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^{\gamma+l-1}].$$

Further, using Theorem (5.3), we have

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^{\gamma-1}f(z)] = \sum_{l=0}^{\infty} b_l \frac{\Gamma(\gamma+l) z^{\gamma+l-1-y}}{\Gamma(\gamma+l-y)}\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau,(\beta,\kappa)_s}(\gamma+l-n,n-y)}{B(\gamma+l-n,n-y)},$$

using $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we get

$$D^{y,p,\lambda;(\mu,\nu)_r}_{z,\alpha,\tau;(\beta,\kappa)_s}[z^{\gamma-1}f(z)] = \frac{\Gamma(\gamma)}{\Gamma(\gamma-y)} z^{\gamma-y-1}\sum_{l=0}^{\infty} b_l \frac{(\gamma)_l}{(\gamma-y)_l}\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau,(\beta,\kappa)_s}(\gamma+l-n,n-y)}{B(\gamma+l-n,n-y)} z^l,$$

again using the identities $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we get the desired result. □

6. Applications

6.1. Statistical distribution

In an extensive analysis of generalised pdf and their statistical characteristics, special functions have been crucial(see for details [9],[12],[14],[16]). In this section, we are describing new generalized beta distribution as an application of our newly defined beta function.

Definition 6.1. *We define the following new beta distribution involving generalized Mittag-Leffler function as:*

$$f(t)=\begin{cases}\frac{1}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}t^{\chi_1-1}(1-t)^{\chi_2-1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{t^\lambda(1-t)^\lambda}\right) & \text{if } 0<t<1,\\ 0 & \text{otherwise}\end{cases} \tag{6.1}$$

where $\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j\in\mathbb{C}$ *with min* $\Re(\alpha,\tau,\mu_i,\nu_i,\beta_j,\kappa_j)>0$ *for* $i=1,\dots r; j=1,\dots,s$ *with* $r+s=m-2$, *m be any positive integer and* $\Re(\chi_1)>0,\Re(\chi_2)>0,\Re(p)\geq 0,\Re(\lambda)>0$.

If n is any real number, then the n^{th} moment of the p.d.f f(t) for a random variable T is defined as

$$E(T^n)=\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,\chi_2)}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}. \tag{6.2}$$

If n=1, then the mean of the distribution

$$E(T)=\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2)}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}.$$

The variance of the beta distribution is given by

$$Var(T)=E(T^2)-[E(T)]^2$$

$$Var(T)=\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+2,\chi_2)B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)-[B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2)]^2}{[B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)]^2}. \tag{6.3}$$

The Coefficient of variation of distribution is

$$C.V=\sqrt{\frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+2,\chi_2)B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}{[B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+1,\chi_2)]^2}-1}. \tag{6.4}$$

The moment generating function of our beta distribution is

$$M_T(y)=\sum_{n=0}^{\infty}\frac{y^n}{n!}E(T^n)$$

$$M_T(y)=\frac{1}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}\sum_{n=0}^{\infty}B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,\chi_2)\frac{y^n}{n!}. \tag{6.5}$$

The characteristic function of beta distribution

$$E(e^{ity})=\sum_{n=0}^{\infty}\frac{i^ny^n}{n!}E(T^n).$$

$$E(e^{ity})=\frac{1}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}\sum_{n=0}^{\infty}B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1+n,\chi_2)\frac{i^ny^n}{n!}. \tag{6.6}$$

The cumulative distribution is given by

$$F(t)=\frac{B^{p,\lambda;(\mu,\nu)_r}_{t,\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)}. \tag{6.7}$$

with

$$B^{p,\lambda;(\mu,\nu)_r}_{t,\alpha,\tau;(\beta,\kappa)_s}(\chi_1,\chi_2)=\int_0^t y^{\chi_1-1}(1-y)^{\chi_2-1}E^{(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}\left(\frac{-p}{y^\lambda(1-y)^\lambda}\right)dy. \tag{6.8}$$

6.2. Fractional Kinetic Equation

In 2000, Haubold and Mathai [11] form a fractional differential equation defined as follows:

$$\frac{dM}{dt} = -d(M_t) + q(M_t), \tag{6.9}$$

where, M=M(t) is the rate of reaction, d=d(M) is the rate of destruction, q=q(M) the rate of production and M_t is the function defined by $M_t(t^*) = M(t - t^*), t^* > 0$.
Further, they considered a special case of (6.9), for spatial fluctuations and inhomogeneities in M(t) quantities are neglected, is equation

$$\frac{dM}{dt} = -c_i M_i(t), \tag{6.10}$$

with the condition that $M_i(t = 0) = M_0$ is the number density of species i at time t=0 and $c_i > 0$. If we drop the index i in (6.10) and integrate, we get the solution

$$M(t) - M_0 = -c \, {}_0D_t^{-1} M(t), \tag{6.11}$$

here ${}_0D_t^{-1}$ denotes the particular case of the Riemann-Liouville integral operator ${}_0D_t^{-w}$, which is defined as

$${}_0D_t^{-w} f(t) = \frac{1}{\Gamma(w)} \int_0^t (t-s)^{w-1} f(s) \, ds. \tag{6.12}$$

$t > 0, \Re(w) > 0$. After that, the generalization into arbitrary order of the standard Kinetic equation (6.11) is given by Haubold and Mathai [11] as follows:

$$M(t) - M_0 = -c^w \, {}_0D_t^{-w} M(t), \tag{6.13}$$

and solution of this equation is given as

$$M(t) = M_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(wk+1)} (ct)^{wk}. \tag{6.14}$$

In 2008, Saxena and Kalla [21] defined the fractional Kinetic equation as follows:

$$M(t) - M_0 f(t) = -c^w \, {}_0D_t^{-w} M(t), (\Re(w) > 0) \tag{6.15}$$

where M(t) represent the number density of a given species at time t, M_0 is the number density of that species at time t=0, c is a constant.
Applying laplace transform to equation (6.15), we get

$$L\{M(t); q\} = M_0 \frac{F(q)}{1 + c^w q^{-w}} = M_0 \left(\sum_{n=0}^{\infty} (-c^w)^n q^{-wn} \right) F(q), \tag{6.16}$$

$(n \in M_0, |\frac{c}{q}| < 1)$ and laplace transform [24] is given by

$$F(q) = L\{M(t); q\} = \int_0^{\infty} e^{-qt} f(t) \, dt, (\Re(q) > 0). \tag{6.17}$$

6.2.1. Solutions of the Generalized Fractional Kinetic Equations. In this section, we investigated the solution of the generalized fractional Kinetic equations involving the new generalized hypergeometric function and confluent hypergeometric function.

Theorem 6.2. *If* $a > 0, d > 0, w > 0$; $\Re(\chi_3) > \Re(\chi_2) > 0$ *and* $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ *with min* $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ *for* $i = 1, \ldots r; j = 1, \ldots, s$ *with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1, \chi_2; \chi_3; d^w t^w) = -a^w \, {}_0D_t^{-w} M(t) \tag{6.18}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) E_{w,wn+1}(-a^w t^w). \tag{6.19}$$

Proof. Laplace transform of Riemann-Liouville fractional Integral operator is given by following (see for details [3], [24])

$$L\{{}_0D_t^{-w} f(t); q\} = q^{-w} F(q), \tag{6.20}$$

where F(q) is defined above in equation (6.17) and applying the laplace transform on equation (6.18)

$$L\{M(t); q\} = M_0 L\{F_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_1, \chi_2; \chi_3; d^w t^w; q) - a^w L\{{}_0D_t^{-w} M(t); q\}, \tag{6.21}$$

$$M(q) = M_0 \int_0^{\infty} e^{-qt} \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \, dt - a^w q^{-w} M(q). \tag{6.22}$$

Interchanging the order of integration and summation

$$M(q) + a^w q^{-w} M(q) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w)^n}{n!} \int_0^{\infty} e^{-qt} t^{wn} \, dt, \tag{6.23}$$

$$M(q) + a^w q^{-w} M(q) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w)^n}{n!} \frac{\Gamma(wn+1)}{q^{wn+1}},$$

$$M(q) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w)^n}{n!} \Gamma(wn+1) \left\{ q^{-(wn+1)} \sum_{l=0}^{\infty} \left[-\left(\frac{q}{a}\right)^{-w} \right]^l \right\}, \tag{6.24}$$

taking laplace inverse of equation (6.24) and using $L^{-1}\{q^{-w}; t\} = \frac{t^{w-1}}{\Gamma(w)}$ $(\Re(w) > 0)$,

$$\begin{aligned} L^{-1}\{M(q)\} = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w)^n}{n!} \Gamma(wn+1) \\ L^{-1}\left\{ \sum_{l=0}^{\infty} (-1)^l a^{wl} q^{-[w(n+l)+1]} \right\}, \end{aligned} \tag{6.25}$$

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w)^n}{n!} \Gamma(wn+1) \left\{ \sum_{l=0}^{\infty} (-1)^l a^{wl} \frac{t^{w(n+l)}}{\Gamma(w(n+l)+1)} \right\},$$

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) \left\{ \sum_{l=0}^{\infty} (-1)^l \frac{(a^w t^w)^l}{\Gamma(w(n+l)+1)} \right\},$$

which can be written as

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B_{\alpha,\tau;(\beta,\kappa)_s}^{p,\lambda;(\mu,\nu)_r}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) E_{w,wn+1}(-a^w t^w). \tag{6.26}$$

□

Theorem 6.3. *If $a > 0, d > 0, w > 0, \chi_2, \chi_3 \in \mathbb{C}$; $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \ldots r; j = 1, \ldots, s$, with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 \phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2; \chi_3; d^w t^w) = -a^w \, {}_0D_t^{-w} M(t) \tag{6.27}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) E_{w,wn+1}(-a^w t^w). \tag{6.28}$$

Theorem 6.4. *If $d > 0, w > 0, \chi_1, \chi_2, \chi_3 \in \mathbb{C}$; $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \ldots r; j = 1, \ldots, s$ with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1, \chi_2; \chi_3; d^w t^w) = -d^w \, {}_0D_t^{-w} M(t) \tag{6.29}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) E_{w,wn+1}(-d^w t^w). \tag{6.30}$$

Theorem 6.5. *If $d > 0, w > 0, \chi_2, \chi_3 \in \mathbb{C}$; $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \ldots r; j = 1, \ldots, s$ with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 \phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2; \chi_3; d^w t^w) = -d^w \, {}_0D_t^{-w} M(t) \tag{6.31}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} \frac{(d^w t^w)^n}{n!} \Gamma(wn+1) E_{w,wn+1}(-d^w t^w). \tag{6.32}$$

Theorem 6.6. *If $d > 0, w > 0, \chi_1, \chi_2, \chi_3 \in \mathbb{C}$; $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \ldots r; j = 1, \ldots, s$ with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 F^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_1, \chi_2; \chi_3; t) = -d^w \, {}_0D_t^{-w} M(t) \tag{6.33}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} (\chi_1)_n \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} t^n E_{w,n+1}(-d^w t^w). \tag{6.34}$$

Theorem 6.7. *If $d > 0, w > 0, \chi_2, \chi_3 \in \mathbb{C}$; $\Re(\chi_3) > \Re(\chi_2) > 0$ and $\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j \in \mathbb{C}$ with min $\Re(\alpha, \tau, \mu_i, \nu_i, \beta_j, \kappa_j) > 0$ for $i = 1, \ldots r; j = 1, \ldots, s$ with r+s= m-2, m be any positive integer, then the solution of the equation*

$$M(t) - M_0 \phi^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2; \chi_3; t) = -d^w \, {}_0D_t^{-w} M(t) \tag{6.35}$$

is given by the following formula

$$M(t) = M_0 \sum_{n=0}^{\infty} \frac{B^{p,\lambda;(\mu,\nu)_r}_{\alpha,\tau;(\beta,\kappa)_s}(\chi_2 + n, \chi_3 - \chi_2)}{B(\chi_2, \chi_3 - \chi_2)} t^n E_{w,n+1}(-d^w t^w). \tag{6.36}$$

Similarly, we can prove all the theorems.

7. Conclusion

In this paper, we introduce a generalization of the beta function with the help of a generalized m-parameter Mittag-Leffler function. Firstly, we have derived some properties of our proposed beta function. Further, we have generalized the hypergeometric function and confluent hypergeometric function using our newly defined beta function and investigated some properties like integral representations and the differential formula. We presented further generalization of the Caputo fractional derivative operator using the m-parameter Mittag-Leffler function, and some of its properties have been discussed using the new generalized beta function. As an application of our newly defined beta function, we have introduced a new type of statistical distribution as well as examined several functions associated with it like mean, variance, coefficient of variance, moment generating function, characteristic function, and cumulative distribution. We investigated a further fractional generalization of the kinetic equation that involves a hypergeometric function. The newly defined generalized beta function and other generalizations defined in this paper will be applicable in various fields of science.

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