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Soft prime groups, strong conjugation and applications of Sylow Theorems to soft groups

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ABSTRACT: In classical algebra, p-groups, conjugate groups and Sylow Theorems are of great importance to understand the arbitrary finite group structures. Our interest, in this paper, is to transfer these important structures to soft group theory. First, we define soft prime group, conjugate group and soft conjugate group. Then, we examine their properties under group mappings, group homomorphisms and soft homomorphisms. Also, as a strong case of conjugate group, strong conjugate group is defined and the relationship between the conjugation and strong conjugation is derived and it is showed that strong conjugation is an equivalence relation on the set of all soft groups over G with the parameter set A. Additionally, we convey Cauchy's Theorem to soft groups. Moreover, in order to understand the structure of an arbitrary finite soft group, we define soft Sylow p-subgroup and obtain the corresponding Sylow Theorems in soft group theory with this concept. By this way, we bring a new aspect to soft group theory by expanding the theory with the fundamental concepts.

Key Words: Soft set, soft group, soft prime group, conjugate group, soft conjugate group, strong conjugate group, soft Sylow p-group

Contents

1. Introduction

Group theory is a major area in abstract algebra and is a very important tool in mathematics. In classical algebra, finite abelian groups are fully classified up to isomorphism, but non-abelian finite groups are much more complex. According to the fundamental theorem of arithmetic, also called the unique factorization theorem, every positive integer greater than 1 can be represented in exactly one way apart from rearrangement as a product of one or more primes, then it is an eligible to start to understand the structure of an arbitrary finite group by making use of prime numbers dividing the order of the group. At this point, in group theory, the important place of p-groups and conjugate groups arise. In particular, the Sylow subgroups of any finite group are p-groups. As p-groups have some special properties, it is easier to understand and classify them rather than arbitrary groups, but they are useful since they are the building blocks for arbitrary groups via the Sylow theorems. Also, it is known that the inverse of Lagrange's theorem does not hold in general. Both p-groups and conjugate groups are used to investigate the problem of finding out in which cases the inverse of Lagrange's theorem is satisfied. Also, in group theory, Cauchy's theorem states that if G is a finite group and p is a prime number dividing the order of G, then G contains an element of order p. This theorem is related to Lagrange's theorem and it is generalized by Sylow's first theorem.

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In 1999, Molodtsov [27] proposed a completely new approach called Soft Set Theory for modeling vagueness and uncertainty. Since this approach is free from the problem of setting the membership function, this theory can be easily applied to many different fields including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Some of these applications have already been indicated in [27].
For more than twenty years, works on soft set theory have been progressing rapidly in both theoretical and practical studies. In [2,4] Aktaş and Çağman compared soft sets to fuzzy sets and rough sets, gave some basic concepts of soft set theory and defined the concept of soft group. In the papers [5,7,25,28,33], the authors introduced several operations and gave some of algebraic properties of soft sets. Since the paper [2], many authors have studied the soft algebraic structures and soft operations [1,11,17,18,19,22,23,24,32,34,35,36,37].
Soft matrix theory is introduced by Çağman and Enginoğlu [9]. They defined soft matrices and some of their operations, then constructed a soft max-min decision making problems that contain uncertainties. The theory of soft sets and soft matrices have also been applied to decision making problems succesfully in [6,7,9,26,29].
With the start of soft group theory by the paper [2], many concepts in group theory are tried to be transferred to soft group theory. At first, in [31], normalistic soft group and normalistic soft group homomorphism are introduced, their several related properties are derived. In [3], order of the soft group and cyclic soft group by using definition of the soft group was defined and the relationships between cyclic soft groups and classical groups are obtained. In [21], the concept of cyclicizer soft set, centralizer soft set, normalizer soft set, cosetial soft set, orbital soft set and stabilizer soft set using some group concepts such as cyclic, centralizer and normalizer of an element, coset and group action are given and studied with the basic soft set theoretic properties. Also, some necessary and sufficient conditions for two groups to be isomorphic are derived by using the similarities of soft sets.

Since p groups and conjugate groups are of great importance to examine the problem when the inverse of Lagrange's theorem is satisfied, in this study we investigate how these important structures are transferred to soft group theory. For this aim, we first define two special cases of soft groups, that is, soft prime group and soft conjugate group. We then examine their properties under group mappings, group homomorphisms and soft homomorphisms and obtain the transformations where these structures are preserved. We show that being a soft prime group is preserved under soft conjugation. Additionally, as a strong case of conjugate group, strong conjugate group is defined and the relationship between the conjugation and strong conjugation is obtained and it is showed that strong conjugation is an equivalence relation on the set of all soft groups over G with the parameter set A. Moreover, since Cauchy's Theorem is related to Lagrange's theorem, we convey this theorem to soft groups over a finite group. And from the fact that Sylow theorems are used to understand the structure of an arbitrary finite group, we define soft Sylow p-subgroup for the soft set theory in order to investigate the structures of soft groups over a finite group and obtain the corresponding Sylow Theorems in soft group theory with the concept of soft Sylow p-subgroups.

2. Preliminaries

From now on, let p be a prime number, G be a group with identity e_G. G is called a **p-group** if every element of G has order a power of the prime p. If H is a subgroup of G (denoted by $H \leq G$) and H itself is a p-group, then H is called a **p-subgroup** of G. $\{e_G\}$ is a p-subgroup of G for every prime p. In G, the order of an element u will be denoted by $o(u)$. A subgroup P of a group G is said to be a Sylow p-subgroup for a prime number p if P is a maximal p-subgroup of G. We refer to [13,15] for all undefined concepts and notations regarding group theory.

Definition 2.1 *([10]) Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$. A soft set (F, A) or simply F_A on the universe U is defined by the ordered pairs*

$$(F, A) = \{(x, F(x)) | x \in E, F(x) \in P(U)\},$$

where $F : E \to P(U)$ such that $F(x) = \emptyset$ if $x \notin A$.

In other words, a soft set over U is a parameterized family of subsets of the universe U.

Definition 2.2 *([28]) Let (F, A) and (G, B) be soft sets over U.*

a) If $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$, then (F, A) is a soft subset of (G, B), denoted by $(F, A)\tilde{\subseteq}(G, B)$.

b) If $(F, A)\tilde{\subseteq}(G, B)$ and $(G, B)\tilde{\subseteq}(F, A)$, then (F, A) and (G, B) is said to be soft equal and denoted by $(F, A) = (G, B)$.

Definition 2.3 *([11]) Let (F, A) be soft set over U. Then,*

$$supp(F, A) = \{x \in A | F(x) \neq \emptyset\}$$

is called the support of the soft set (F, A). The null soft set is a soft set with empty support and denoted by $\emptyset_E$. Hence, a soft set (F, A) is called non-null if $supp(F, A) \neq \emptyset$.

Definition 2.4 *([2]) Let (F, A) be a soft set over G. Then, (F, A) is called a soft group over G if and only if $F(x)$ is a subgroup of G for all $x \in A$.*

Definition 2.5 *([2]) Let (F, A) and (H, K) be two soft groups over G. Then, (H, K) is a soft subgroup of (F, A), written $(H, K)\tilde{<}(F, A)$, if*

a) $K \subset A$,

b) $H(x) < F(x)$ for all $x \in K$.

Definition 2.6 *([2]) Let (F, A) be a soft group over G and $(H, B)\tilde{<}(F, A)$. (H, B) is said to be a normal soft subgroup of (F, A), denoted by $(H, B)\tilde{\lhd}(F, A)$, if $H(x)$ is a normal subgroup of $F(x)$ for all $x \in B$.*

3. Soft Prime Groups

In this section, two special soft groups are introduced and their properties under group mappings, group homomorphisms and soft homomorphisms are examined.

From now on, for a soft group (F, A) over G, $supp(F, A) = A$ unless otherwise stated. Thus, the parameters whose images are empty set will be eliminated from the very beginning.

Definition 3.1 *For a soft set (F, A) over G, if for each $x \in A$, there exist prime numbers p_x such that $F(x)$ is a p_x-group, then the soft set (F, A) is called a **soft prime group** over G.*

As a special case of the concept of soft prime groups, we have the following;

Definition 3.2 *For a soft set (F, A) over G, if there exists a constant prime number p such that $F(x)$ are p-groups for all $x \in A$, then the soft set (F, A) is called a **soft p-group** over G.*

Clearly, every soft p-group is also a soft prime group; but the converse does not hold in general.

Example 3.1 Let the symmetric group $G = S_{20}$, the parameter set $A = \{e_1, e_2, e_3\} \subset E$ and $F : A \to P(G)$ be a soft set given by

$$F(e_1) = \{(1), (12)\},\ F(e_2) = A_3 \text{ and } F(e_3) = D_4.$$

Since $F(e_1)$, $F(e_3)$ are 2-groups and $F(e_2)$ is a 3-group, then (F, A) is a soft prime group; but it is not a soft p-group over G.
Obviously, if $B = \{e_1, e_3\} \subset E$ and $F(e_1)$ and $F(e_3)$ are defined as above, then the soft set (F, B) is a soft 2-group over G.

Proposition 3.1 *Let (F, A) be a soft prime group over G and $(H, B)\tilde{<}(F, A)$. Then, (H, B) is a soft prime group over G.*

Proof: Since (F,A) is a soft prime group over G, then there exist prime numbers p_x such that $F(x)$ is a p_x-group. Since $(H,B)\tilde{<}(F,A)$, then $B \subset A$ and $H(x)$ is a subgroup of $F(x)$ for all $x \in B$. Thus, for each $x \in B$, $H(x)$ is a p_x-group. Therefore, (H,B) is also a soft prime group over G. □

Proposition 3.2 *Let (F,A) be a soft p-group over G and $(H,B)\tilde{<}(F,A)$. Then, (H,B) is a soft p-group over G.*

Proof: The proof is similar to the proof of Proposition 3.1, hence omitted. □

The inverse of Proposition 3.1 and Proposition 3.2 are not true in general.

Example 3.2 Let (F,A) be the soft group over G in Example 3.1 and let the soft subgroup of (F,A) be (F,B) in Example 3.1. It is easily seen that (F,B) is a soft 2-group over G; but (F,A) is not a soft 2-group over G.

Now, let the soft group (T,C) over G such that $C = \{e_1, e_2, e_3, e_4\} \subset E$ and $T : C \to P(G)$ is a soft set given by

$$T(e_1) = \{(1),(12)\},\ T(e_2) = A_3,\ T(e_3) = D_4 \text{ and } T(e_4) =< (123456) >,$$

where $< (123456) >$ is a cyclic group generated by the element (123456). It is easily seen that $(F,A)\tilde{<}(T,C)$. Moreover, (F,A) is a soft prime group but (T,C) is not a soft prime group over G, since $o(123456) = 6$.

In the remaining part of this section, it is investigated under which transformations the structure of soft prime group (soft p-group) is preserved and it is given an application of Cauchy's Theorem for soft prime groups (soft p-groups).

Definition 3.3 *(cf. [2,31]) Let (F,A) and (H,B) be two soft sets over G_1 and G_2, respectively, and let $f : G_1 \to G_2$ and $g : A \to B$ be two functions. Then, $(f,g) : (F,A) \to (H,B)$ is a soft homomorphism if the following conditions are satisfied:*

a) f is a group homomorphism,

b) g is a mapping and

c) $f(F(x)) = H(g(x))$ for all $x \in A$.

Theorem 3.1 *Let (F,A) be a soft prime group over a group G_1 and (H,B) be a soft set over a group G_2. If there exists a group monomorphism f such that $f : G_1 \to G_2$, a surjective function g such that $g : A \to B$, and a soft homomorphism (f,g) such that $(f,g) : (F,A) \to (H,B)$, then (H,B) is a soft prime group over G_2.*

Proof: Assume that (F,A) is a soft prime group over a group G_1, (H,B) is a soft set over a group G_2, $f : G_1 \to G_2$ is a group homomorphism, $g : A \to B$ is a surjective function and $(f,g) : (F,A) \to (H,B)$ is a soft homomorphism. Then, for each $x \in A$ there exist prime numbers p_x such that $F(x)$ is a p_x-group and thus $f(F(x))$ is a subgroup of G_2 for each $x \in A$. Let $y \in B$. Since g is surjective, there exists $x \in A$ such that $g(x) = y$. Thus, $H(y) = H(g(x)) = f(F(x))$ is a subgroup of G_2 for all $y \in B$, i.e. (H,B) is a soft group over G_2. Now, it is enough to show that for each $y \in B$, $H(y)$ is a p_y-group for prime numbers p_y. Assume that $u \in H(y)$ and $o(u) = k$. Since $u \in H(y) = H(g(x)) = f(F(x))$, then there exists an $a \in F(x)$ such that $u = f(a)$. Since f is a group monomorphism,

$$u^k = e_{G_2} = [f(a)]^k = f[(a)^k] = f(e_{G_1})$$

which implies that $o(a) \mid k$. Similarly,

$$u^{o(a)} = [f(a)]^{o(a)} = f[(a)^{o(a)}] = f(e_{G_1}) = e_{G_2}$$

which implies $k \mid o(a)$. Hence, $k = o(a) = o(u)$. Since (F,A) is a soft prime group over G_1, $a \in F(x)$ and $o(a) = k$, then $k = (p_x)^s$ where p_x is a prime number and s is a non-negative integer. Thus, $o(u) = (p_x)^s$. Hence, (H,B) is a soft prime group over G_2. □

Corollary 3.1 *Theorem 3.1 is also valid when (F, A) is a soft p-group over G_1.*

In the paper [31], the method obtaining two soft sets using the image and inverse image of a mapping between groups is given as follows:

Let G_1 and G_2 be two groups, (F, A) and (H, B) be soft sets over G_1 and G_2, respectively and $f : G_1 \to G_2$ be a mapping of groups. Then $(f(F), supp(F, A))$ is a soft set over G_2, where $f(F) : supp(F, A) \to P(G_2)$ is given by $f(F)(x) = f(F(x))$ for all $x \in supp(F, A)$. On the other hand, if $f : G_1 \to G_2$ is a bijective mapping, then $(f^{-1}(H), supp(H, B))$ is a soft set over G_1, where $f^{-1}(H) : supp(H, B) \to P(G_1)$ is given by $f^{-1}(H)(y) = f^{-1}(H(y))$ for all $y \in supp(H, B)$. Here, it is easily seen that $supp(F, A) = supp(f(F), supp(F, A))$ and $supp(H, B) = supp(f^{-1}(H), supp(H, B))$. Now, we have the following:

Theorem 3.2 *Let G_1 and G_2 be two groups, (F, A) and (H, B) be soft sets over G_1 and G_2, respectively and $f : G_1 \to G_2$ be a group isomorphism.*

a) If (F, A) is a soft prime group over G_1, then $(f(F), A)$ is a soft prime group over G_2.

b) If (H, B) is a soft prime group over G_2, then $(f^{-1}(H), B)$ is a soft prime group over G_1.

Proof: a) Since (F, A) is a soft prime group over G_1, for each $x \in A$ there exist prime numbers p_x such that $F(x)$ is a p_x-group. Then, $supp(F, A) = A$ and $f(F) : A \to P(G_2)$ is given by $f(F)(x) = f(F(x))$ for all $x \in A$. Let $u \in f(F(x))$ and $o(u) = k$. Then, there exists an $a \in F(x)$ such that $u = f(a)$. Similar to proof of Theorem 3.1, $k = o(a)$. Hence $(f(F), A)$ is a soft prime group over G_2.
b) Since (H, B) is a soft prime group over G_2, there exist prime numbers p_y such that $H(y)$ is a p_y-group for each $y \in B$. Then, $supp(H, B) = B$ and $f^{-1}(H) : B \to P(G_1)$ is given by $f^{-1}(H)(y) = f^{-1}(H(y))$ for all $y \in B$. Since $H(y)$ is a subgroup of G_2 and f is a group isomorphism, then $f^{-1}(H(y))$ is a subgroup of G_1 for all $y \in B$. Let $v \in H(y)$. Since $H(y)$ is a p_y-group, then there exists a non-negative integer s such that $o(v) = (p_y)^s$. Since f is an isomorphism, there exists a $u \in f^{-1}(H(y))$ such that $f(u) = v$ and $o(u) = o(f(u)) = o(v) = (p_y)^s$. Hence $f^{-1}(H(y))$ is a p_y-group for each $y \in B$. Therefore, $(f^{-1}(H), B)$ is a soft prime group over G_1. □

Corollary 3.2 *Theorem 3.2 is also valid when (F, A) is a soft p-group over G_1 and (H, B) is a soft p-group over G_2.*

Lemma 3.1 *([13])**(Cauchy's Theorem)** Let p be a prime number and G a finite group such that p divides $|G|$, where $|G|$ is the order of G. Then, G has an element of order p and, consequently, a subgroup of order p. Furthermore, G is a p-group if and only if $|G|$ is a power of p.*

The Cauchy's Theorem can be transferred to soft groups over a finite group G as the following;

Theorem 3.3 *Let p be a prime number and G be a finite group such that p divides $|G|$. Then,*

a) There exists a soft p-group (soft prime group) over G.

b) G is a p-group if and only if every soft group over G is a p-group.

Proof: a) Under assumptions, there exists a $g \in G$ such that $o(g) = p$. Let the soft set (F, A) over G be given by $F(x) = \langle g \rangle$ for all $x \in A$, then obviously, (F, A) is a soft p-group over G by Definition 3.2. Since every soft p-group is also a soft prime group, (F, A) is also a soft prime group over G.
b) Assume that G is a p-group and (F, A) is a soft group over G. Since $F(x)$ is a subgroup of G for all $x \in A$ and G is a p-group, then $F(x)$ is a p-group for all $x \in A$. Therefore, every soft group over G is a p-group.
Now, assume that all soft groups over G is a p-group. If we take the soft set (H, B) over G, such that $H(x) = G$ for all $x \in B$, then (H, B) is a soft group over G. Under assumption, since (H, B) is a soft p-group, then G is a p-group. □

4. Soft Conjugate Groups

In this section, conjugate group, the well-known concept in classical algebra, is transferred to soft subgroups with the following definition:

Definition 4.1 *Let G be a group and (F,A), (H,B) be soft groups over G.*

*a) If there exist $g_x \in G$ and $y \in B$ such that $g_x F(x) g_x^{-1} = H(y)$ for each $x \in A$, then (F,A) is called a **conjugate group** to (H,B) and denoted by $(F,A)\widetilde{\rightarrow}(H,B)$.*

*b) If $(F,A)\widetilde{\rightarrow}(H,B)$ and $(H,B)\widetilde{\rightarrow}(F,A)$, then (F,A) and (H,B) are called **soft conjugate groups** of G and denoted by $(F,A)\widetilde{\leftrightarrow}(H,B)$.*

Theorem 4.1 *Let (F,A), (H,B) and (T,C) be soft groups over G.*

a) $(F,A)\widetilde{\leftrightarrow}(F,A)$. The relation of being a soft conjugate group is reflective.

b) If $(F,A)\widetilde{\rightarrow}(H,B)\widetilde{\rightarrow}(T,C)$, then $(F,A)\widetilde{\rightarrow}(T,C)$. The relation of being a conjugate group is transitive.

c) If $(F,A)\widetilde{\leftrightarrow}(H,B)$ and $(F,A)\widetilde{\rightarrow}(T,C)$, then $(H,B)\widetilde{\rightarrow}(T,C)$.

Proof: a) If for each $x \in A$, we take $g_x = e_G$, the proof is seen by Definition 4.1(a).
b) Since $(F,A)\widetilde{\rightarrow}(H,B)$, there exist $g_x \in G$ and $y \in B$ such that $g_x F(x) g_x^{-1} = H(y)$ for each $x \in A$ and $(H,B)\widetilde{\rightarrow}(T,C)$ implies that there exist $g \in G$ and $z \in C$ such that $gH(y)g^{-1} = T(z)$ for each $y \in B$. Then, there exist $g_1 = gg_x \in G$, $z \in C$ such that $T(z) = gH(y)g^{-1} = gg_x F(x) g_x^{-1} g^{-1} = g_1 F(x) g_1^{-1}$ for each $x \in A$. Therefore, $(F,A)\widetilde{\rightarrow}(T,C)$.
c) It is seen by (b) and Definition 4.1(b). □

The relation of being a conjugate group is not symmetric so it is not a equivalence relations (On the contrary to the conjugation, strong conjugation is an equivalence relation on $S_A(G)$.) Here, there is an also important reminding that we should immediately point out: The concepts of conjugate group and soft subgroup are different from each other. We have the following example:

Example 4.1 Let the symmetric group $G = S_{20}$, the parameter sets $A = \{e_1, e_2, e_3\}$, $B = \{e_1, e_2, e_3, e_4\}$ and $F : A \to P(G)$, $H : B \to P(G)$ soft sets given by

$$F(e_1) = \{(1),(12)\},\ F(e_2) = \{(1),(13)\} \text{ and } F(e_3) = A_3.$$

$$H(e_1) = \{(1),(23)\},\ H(e_2) = A_3,\ H(e_3) = S_3 \text{ and } H(e_4) = D_4.$$

Since $F(e_1)$ is not a subgroup of $H(e_1)$, then (F,A) is not a soft subgroup of (H,B). But, for $e_1 \in A$, there exists $g_{e_1} = (13) \in G$ such that

$$(13)F(e_1)(13)^{-1} = H(e_1),$$

for $e_2 \in A$, there exists $g_{e_2} = (12) \in G$ such that

$$(12)F(e_2)(12)^{-1} = H(e_1),$$

for $e_3 \in A$, there exists $g_{e_3} = (1) \in G$ such that

$$(1)F(e_3)(1)^{-1} = H(e_2),$$

then $(F,A)\widetilde{\rightarrow}(H,B)$, i.e. (F,A) is a conjugate group to (H,B). As is seen, being a conjugate group does not require to be a soft subgroup.

In order to show that the opposite case is not provided, let the soft group (T,C) over G be such that $C = \{e_1, e_2, e_3\}$ and $T : C \to P(G)$ is a soft set given by

$$T(e_1) = \{(1), (12)\} \text{ and } T(e_2) = T(e_3) = S_3.$$

Since $F(x)$ is a subgroup of $T(x)$ for all $x \in A$, then $(F, A)\tilde{<}(T, C)$. However, there are not any $g_{e_3} \in G$ and $y \in C$ such that $g_{e_3} F(e_3) g_{e_3}^{-1} = T(y)$. Hence (F, A) is not a conjugate group to (T, C). As is seen, being a soft subgroup does not require to be a conjugate group.

With the following theorem, it is proved that the conjugate group to a soft prime group is also a soft prime group.

Theorem 4.2 *Let (F, A) and (H, B) be soft groups over G. If $(F, A)\widetilde{\rightrightarrows}(H, B)$ and (H, B) is a soft prime group over G, then so is (F, A).*

Proof: Since (H, B) is a soft prime group over G, there exist prime numbers p_y such that $H(y)$ is a p_y-group for each $y \in B$. Since $(F, A)\widetilde{\rightrightarrows}(H, B)$, there exist $g_x \in G$ and $y \in B$ such that $g_x F(x) g_x^{-1} = H(y)$ for each $x \in A$. Let an arbitrary $u \in F(x)$ such that $o(u) = k$. Since $g_x F(x) g_x^{-1} = H(y)$, there exists $v \in H(y)$ such that $g_x u g_x^{-1} = v$. Now, $v^k = (g_x u g_x^{-1})^k = g_x^k u^k g_x^{-k} = g_x^k g_x^{-k} = e_G$, then $o(v) \mid k$. Similarly, $v^{o(v)} = e_G = (g_x u g_x^{-1})^{o(v)} = g_x^{o(v)} u^{o(v)} g_x^{-o(v)}$, then $g_x^{o(v)} u^{o(v)} g_x^{-o(v)} = e_G$ implies that $g_x^{o(v)} u^{o(v)} = g_x^{o(v)}$ and that $u^{o(v)} = e_G$. Since $o(u) = k$, then $k \mid o(v)$. Hence $o(u) = o(v)$. Therefore (F, A) is a soft prime group over G. □

Corollary 4.1 *Let (F, A) and (H, B) be soft groups over G. If $(F, A)\widetilde{\leftrightarrows}(H, B)$, then (F, A) is a soft prime group over G if and only if (H, B) is a soft prime group over G.*

Corollary 4.2 *Theorem 4.2 and Corollary 4.1 are valid when the soft prime groups are soft p-groups, as well.*

With the following theorem, it is proved that the structure of the conjugate group is preserved under a soft homomorphism.

Theorem 4.3 *Let G_1 and G_2 be two groups, (F_1, A_1), (H_1, B_1) and (F_2, A_2), (H_2, B_2) be soft groups over G_1 and G_2, respectively. Assume that there exist a group homomorphism f such that $f : G_1 \to G_2$, a bijective mapping g_1 such that $g_1 : A_1 \to A_2$, a mapping g_2 such that $g_2 : B_1 \to B_2$, soft homomorphisms $(f, g_1)$and (f, g_2) such that $(f, g_1) : (F_1, A_1) \to (F_2, A_2)$, $(f, g_2) : (H_1, B_1) \to (H_2, B_2)$. If $(F_1, A_1)\widetilde{\rightrightarrows}(H_1, B_1)$, then $(F_2, A_2)\widetilde{\rightrightarrows}(H_2, B_2)$.*

Proof: Since $(f, g_1) : (F_1, A_1) \to (F_2, A_2)$ and $(f, g_2) : (H_1, B_1) \to (H_2, B_2)$ are soft homomorphisms, then $f(F_1(x)) = F_2(g_1(x))$ and $f(H_1(y)) = H_2(g_2(y))$ for all $x \in A_1$, $y \in B_1$. Let an arbitrary $x_2 \in A_2$. Since $g_1 : A_1 \to A_2$ is a bijective mapping, there exists an $x_1 \in A_1$ such that $g_1(x_1) = x_2$. $(F_1, A_1)\widetilde{\rightrightarrows}(H_1, B_1)$ implying that for $x_1 \in A_1$, there exist $g_{x_1} \in G_1$ and $y_1 \in B_1$ such that

$$g_{x_1} F_1(x_1) g_{x_1}^{-1} = H_1(y_1).$$

If we take the images of the both sides of this last equation under the homomorphism f, then there exist $g_{x_2} \in G_2$ and $y_2 \in B_2$ such that

$$\begin{aligned} f(g_{x_1} F_1(x_1) g_{x_1}^{-1}) = f(H_1(y_1)) &\implies f(g_{x_1}) f(F_1(x_1)) f(g_{x_1}^{-1}) = f(H_1(y_1)) \\ &\implies f(g_{x_1}) F_2(g_1(x_1)) f(g_{x_1}^{-1}) = H_2(g_2(y_1)) \\ &\implies g_{x_2} F_2(x_2) g_{x_2}^{-1} = H_2(y_2) \end{aligned}$$

Thus, $(F_2, A_2)\widetilde{\rightrightarrows}(H_2, B_2)$. □

Corollary 4.3 *Let G_1 and G_2 be two groups, (F_1, A_1), (H_1, B_1) and (F_2, A_2), (H_2, B_2) be soft groups over G_1 and G_2, respectively. Assume that there exist a group homomorphism f such that $f : G_1 \to G_2$, bijective mappings g_1 and g_2 such that $g_1 : A_1 \to A_2$, $g_2 : B_1 \to B_2$, soft homomorphisms (f, g_1) and (f, g_2) such that $(f, g_1) : (F_1, A_1) \to (F_2, A_2)$ and $(f, g_2) : (H_1, B_1) \to (H_2, B_2)$. If $(F_1, A_1)\widetilde{\leftrightarrows}(H_1, B_1)$, then $(F_2, A_2)\widetilde{\leftrightarrows}(H_2, B_2)$.*

Proof: Considering that the mapping g_2 is also bijective, the proof is obtained from Theorem 4.3 and Definition 4.1. □

5. Strong Conjugation on Soft Groups

In soft set theory, since both parameter sets and images are used, constructing algebraic structures on soft sets causes the emergence of different concepts compared to classical algebraic structures. If the conjugation of images of the same parameters is used instead of the conjugation relation defined by the conjugation of images of different parameters, then the new concept of strong conjugation given below is obtained.
Let H be a subgroup of G. From now on, N_H denotes the normalizer of H in G and $(G : H)$ denotes the index of H in G. And depending on the parameter set A, the set of all soft groups over G will be represented by $S_A(G)$.

Definition 5.1 *Let G be a group and (F, A), (H, A) be soft groups over G. If there exists $g_x \in G$ such that $g_xF(x)g_x^{-1} = H(x)$ for each $x \in A$, then (F, A) is called a* ***strong conjugate group*** *to (H, A) and denoted by $(F, A)\widetilde{\rightarrow}_s(H, A)$.*

On the contrary to conjugation, strong conjugation is an equivalence relation on $S_A(G)$. We have the following;

Theorem 5.1 *The strong conjugation is an equivalence relation on $S_A(G)$.*

Proof: Let $(F, A), (H, A), (T, A) \in S_A(G)$. The reflexivity is seen by taking $g_x = e_G \in G$ for all $x \in A$. To show the symmetry, assume $(F, A)\widetilde{\rightarrow}_s(H, A)$. Then, there exists $g_x \in G$ such that $g_xF(x)g_x^{-1} = H(x)$ for each $x \in A$, which implies that there exists $g_x^{-1} \in G$ such that $g_x^{-1}H(x)g_x = F(x)$ for each $x \in A$ and hence $(H, A)\widetilde{\rightarrow}_s(F, A)$. Finally, assume that $(F, A)\widetilde{\rightarrow}_s(H, A)\widetilde{\rightarrow}_s(T, A)$. Since $(F, A)\widetilde{\rightarrow}_s(H, A)$, there exists $g_x \in G$ such that $g_xF(x)g_x^{-1} = H(x)$ for each $x \in A$ and since $(H, A)\widetilde{\rightarrow}_s(T, A)$, there exists $u_x \in G$ such that $u_xH(x)u_x^{-1} = T(x)$ for each $x \in A$. Then, there exists $k_x = u_xg_x$ such that $T(x) = u_xH(x)u_x^{-1} = u_xg_xF(x)g_x^{-1}u_x^{-1} = k_xF(x)k_x^{-1}$ for each $x \in A$, i.e. transitivity is valid. □

In the following proposition, we give the relationship between the conjugation and strong conjugation:

Proposition 5.1 *Let G be a group and (F, A), (H, A) be soft groups over G. If $(F, A)\widetilde{\rightarrow}_s(H, A)$, then $(F, A)\widetilde{\leftrightarrows}(H, A)$.*

Proof: The result is obvious by Definitions 4.1, 5.1 and Theorem 5.1. □

Corollary 5.1 *Let $(F, A), (H, A) \in S_A(G)$ such that $(F, A)\widetilde{\rightarrow}_s(H, A)$. Then,*

a) (F, A) is a soft prime group over G if and only if (H, A) is a soft prime group over G.

b) (F, A) is a soft p-group over G if and only if (H, A) is a soft p-group over G.

Proof: The proofs are seen by Proposition 5.1 and Corollaries 4.1,4.2. □

Let $(F, A), (H, A) \in S_A(G)$. Since $F(x)$ is a subgroup of G for all $x \in A$, then $N_{F(x)} = \{g \in G|gF(x)g^{-1} = F(x)\}$ is the normalizer of $F(x)$, for all $x \in A$. Since the normalizer is also a subgroup, then $N_{F(x)}$ is also a subgroup of G, for all $x \in A$. Now assume that $(F, A)\widetilde{\rightarrow}_s(H, A)$ and let the subset $N_{FH(x)} = \{g \in G|gF(x)g^{-1} = H(x)\}$ for $x \in A$. In this case, the following question may come to mind: "If we extend the concept of normalizer by using the strong conjugate relation, can we still obtain a subgroup? The answer is no with the following example:

Example 5.1 Let the symmetric group $G = S_4$, the parameter set $A = \{e_1, e_2\} \subset E$. $F : A \to P(G)$ and $H : A \to P(G)$ are soft sets given by $F(e_1) = V_4$, $F(e_2) = \{(1), (12)\}$ and $H(e_1) = V_4$, $H(e_2) = \{(1), (14)\}$. Then $(F, A)\widetilde{\rightarrow}_s(H, A)$. Since $e_G \notin N_{FH(e_2)}$, then $N_{FH(e_2)}$ is not a subgroup of G.

Lemma 5.1 *([13]) Let H be a p-subgroup of a finite group G. Then,*

$$(N_H : H) \equiv (G : H)(mod\ p).$$

Proposition 5.2 *Let G be a finite group, E be a set of parameters and $A \subseteq E$.*

a) If the number of different subgroups of G is n and $m = |A|$ is the number of parameters in A, then $|S_A(G)| = n^m$.

b) If (F, A) is a soft p-group over G, then for all $x \in A$,

$$(N_{F(x)} : F(x)) \equiv (G : F(x))(mod\ p).$$

c) If (F, A) is a soft prime group over G, then for each $x \in A$, there exists a prime number p_x such that

$$(N_{F(x)} : F(x)) \equiv (G : F(x))(mod\ p_x).$$

Proof: a) Let $(F, A) \in S_A(G)$. Since $F : A \to P(G)$ is a function and $F(x)$ is a subgroup of G for all $x \in A$, the proof is clear.
b) Since (F, A) is a soft p-group over G, then by Definition 3.2 there exists a constant prime number p such that $F(x)$ is a p-group for all $x \in A$. By Theorem 5.1, since $(F, A) \widetilde{\rightarrow}_s (F, A)$, then there exist a $g_x \in G$ such that $g_x F(x) g_x^{-1} = F(x)$ for each $x \in A$. Let $x_0 \in A$. Since $F(x_0)$ is a subgroup of G and there exist $g_{x_0} \in G$ such that $g_{x_0} F(x_0) g_{x_0}^{-1} = F(x_0)$, then the normalizer of $F(x_0)$ in G is the subgroup $N_{F(x_0)} = \{g \in G | gF(x_0)g^{-1} = F(x_0)\}$ of G such that $N_{F(x_0)}$ is the largest subgroup of G having $F(x_0)$ as a normal subgroup. Then by Lemma 5.1, for all $x \in A$,

$$(N_{F(x)} : F(x)) \equiv (G : F(x))(mod\ p)$$

is obtained. The proof of c) is obtained similar to the proof of b) by using Definition 3.1, hence omitted. □

Example 5.2 Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$, $A = \{e_1, e_2\}$. Firstly note that $|S_A(G)| = 6^2 = 36$, since there are 6 different subgroups of G, . Now, assume that $(F, A) \in S_A(G)$ given by $F(e_1) = \langle (12) \rangle$ and $F(e_2) = \langle (123) \rangle$. Since $F(e_1)$ is a 2-group and $F(e_2)$ is a 3-group, then (F, A) is a soft prime group over G. It is easily checked that $N_{F(e_1)} = F(e_1)$ and then $(N_{F(e_1)} : F(e_1)) = 1$. Therefore,

$$(N_{F(e_1)} : F(e_1)) = 1 \equiv 3 = (G : F(e_1))(mod\ 2).$$

Similarly, it is seen that $N_{F(e_2)} = S_3 = G$, then $(N_{F(e_2)} : F(e_2)) = 2$. Therefore,

$$(N_{F(e_2)} : F(e_2)) = 2 \equiv 2 = (G : F(e_2))(mod\ 3)$$

.

Definition 5.2 *([2]) Let (F, A) and (H, B) be two soft groups over the groups G and K, respectively. The product of soft groups (F, A) and (H, B) is defined as $(F, A) \times (H, B) = (U, A \times B)$, where $U(x, y) = F(x) \times H(y)$ for all $(x, y) \in A \times B$.*

Theorem 5.2 *([2]) Let (F, A) and (H, B) be two soft groups over the groups G and K, respectively. Then the product $(F, A) \times (H, B)$ is a soft group over $G \times K$.*

Lemma 5.2 *Let G be a group, (F, A), (H, B) soft groups over G and $g_1, g_2 \in G$. Then, for each $x_1 \in A$ and $x_2 \in B$, $(g_1, g_2)(F(x_1) \times H(x_2))(g_1, g_2)^{-1} = g_1 F(x_1) g_1^{-1} \times g_2 H(x_2) g_2^{-1}$.*

Proof: Let $(h_1, h_2) \in (g_1, g_2)(F(x_1) \times H(x_2))(g_1, g_2)^{-1}$. Then there exist $(y_1, y_2) \in F(x_1) \times H(x_2)$ such that $(h_1, h_2) = (g_1, g_2)(y_1, y_2)(g_1, g_2)^{-1}$. Therefore $(h_1, h_2) = (g_1 y_1 g_1^{-1}, g_2 y_2 g_2^{-1})$. Thus $(h_1, h_2) \in g_1 F(x_1) g_1^{-1} \times g_2 H(x_2) g_2^{-1}$.
On the other hand let $(h_1, h_2) \in g_1 F(x_1) g_1^{-1} \times g_2 H(x_2) g_2^{-1}$. Then there exist $y_1 \in F(x_1)$ and $y_2 \in H(x_2)$ such that $(h_1, h_2) = (g_1 y_1 g_1^{-1}, g_2 y_2 g_2^{-1})$. Thus $(h_1, h_2) = (g_1, g_2)(y_1, y_2)(g_1, g_2)^{-1}$ and hence $(h_1, h_2) \in (g_1, g_2)(F(x_1) \times H(x_2))(g_1, g_2)^{-1}$. □

Proposition 5.3 *Let G be a group, (F,A), (H,B), (T,C) and (S,D) soft groups over G. If $(F,A)\widetilde{\rightrightarrows}(T,C)$ and $(H,B)\widetilde{\rightrightarrows}(S,D)$, then $(F,A)\times(H,B)\widetilde{\rightrightarrows}(T,C)\times(S,D)$.*

Proof: Since $(F,A)\widetilde{\rightrightarrows}(T,C)$ and $(H,B)\widetilde{\rightrightarrows}(S,D)$, there exist $g_1 \in G$ and $y_1 \in C$ such that $g_1F(x_1)g_1^{-1} = T(y_1)$ for each $x_1 \in A$ and $g_2 \in G$ and $y_2 \in D$ such that $g_2H(x_2)g_2^{-1} = S(y_2)$ for each $x_2 \in B$. By Lemma 5.2 for each $x_1 \in A$ and $x_2 \in B$, $(g_1,g_2)(F(x_1)\times H(x_2))(g_1,g_2)^{-1} = g_1F(x_1)g_1^{-1}\times g_2H(x_2)g_2^{-1} = T(y_1)\times S(y_2)$. Thus $(F,A)\times(H,B)\widetilde{\rightrightarrows}(T,C)\times(S,D)$. □

Corollary 5.2 *Let G be a group, (F,A), (H,A), (T,A) and (S,A) soft groups over G. If $(F,A)\widetilde{\rightrightarrows}_s(T,A)$ and $(H,A)\widetilde{\rightrightarrows}_s(S,A)$, then $(F,A)\times(H,A)\widetilde{\rightrightarrows}_s(T,A)\times(S,A)$.*

Corollary 5.3 *Let G be a group, (F,A), (H,A), (T,A) and (S,A) soft groups over G. If $(F,A)\widetilde{\rightrightarrows}_s(T,A)$ and $(H,A)\widetilde{\rightrightarrows}_s(S,A)$, then $(F,A)\times(H,A)\widetilde{\leftrightarrows}(T,A)\times(S,A)$.*

Proof: The proof is obvious by Proposition 5.1 and Proposition 5.3.

□

6. Applications of Sylow Theorems to Soft Groups

Sylow theorems are used to understand the structure of an arbitrary finite group. Hence, it is appropriate to start by defining the Sylow p-subgroup structure in soft set theory in order to investigate the structures of soft groups over a finite group.

Definition 6.1 *Let G be a finite group and p be a prime number. A soft group (P,A) over G is said to be a **soft Sylow p-group** over G if $P(x)$ is a Sylow p-subgroup of G for all $x \in A$.*

Lemma 6.1 *([13])**(First Sylow Theorem)** Let G be a finite group, p be a prime and let $|G| = p^n m$ where $n \geq 1$ and p does not divide m.*

a) G contains a subgroup of order p^i for each i where $1 \leq i \leq n$,

b) Every subgroup H of order p^i is a normal subgroup of a subgroup of order p^{i+1} for $1 \leq i < n$.

Theorem 6.1 *Let G be a finite group, p be a prime and let $|G| = p^n m$ where $n \geq 1$ and p does not divide m. Then,*

a) There exists an $(F,A) \in S_A(G)$ such that $o(F(x)) = p^i$ for all $x \in A$ and for each i, where $1 \leq i \leq n$,

b) For $(F,A) \in S_A(G)$ such that $o(F(x)) = p^i$ for all $x \in A$ and for each i, where $1 \leq i < n$, there exists an $(H,A) \in S_A(G)$ such that $(F,A)\tilde{\lhd}(H,A)$ where $o(H(x)) = p^{i+1}$ for all $x \in A$ and for each i, where $1 \leq i < n$.

Proof: a) Under assumptions, G contains a subgroup K_i of order p^i for each i, where $1 \leq i \leq n$ by Lemma 6.1. For $A = \{x_1, x_2, ..., x_n\}$ the soft set (F,A) over G is given as $F(x_i) = K_i$ for each i, where $1 \leq i \leq n$. In this case, $(F,A) \in S_A(G)$ becomes a soft group over G containing the desired conditions.
b) Under assumptions, there exists subgroups K_i and T_i of G such that $o(K_i) = p^i$, $o(T_i) = p^{i+1}$ and K_i is a normal subgroup of T_i for each i, where $1 \leq i < n$, by Lemma 6.1. For $A = \{x_1, x_2, ..., x_{n-1}\}$ the soft sets (F,A) and (H,A) over G are given respectively as $F(x_i) = K_i$ and $H(x_i) = T_i$ for each i, where $1 \leq i < n$. Then, $(F,A),(H,A) \in S_A(G)$ are such that $(F,A)\tilde{\lhd}(H,A)$ by Definition 2.6. □

Lemma 6.2 *([15])**(Second Sylow Theorem)** If H is a p-subgroup of a finite group G, and P is any Sylow p-subgroup of G, then there exists $g \in G$ such that H is a subgroup of gPg^{-1}. In particular, any two Sylow p-subgroups are conjugate.*

Theorem 6.2 *Let G be a finite group, p be a prime number, (P_1,A) be a soft Sylow p-subgroup over G and (H,A) be a soft p-group over G. Then,* *a) There exists a $g_x \in G$ such that $H(x)$ is a subgroup of $g_x P_1(x) g_x^{-1}$ for each $x \in A$.*

b) If (P_2, A) is a soft Sylow p-subgroup over G, then $(P_1, A) \widetilde{\rightarrow}_s (P_2, A)$.

Proof: a) Since (H, A) is a soft p-group over G and (P_1, A) is a soft Sylow p-subgroup over G, then $H(x)$ is a p-group and $P_1(x)$ is a Sylow p-subgroup of G for all $x \in A$ by Definition 3.2 and Definition 6.1. Then,there exists a $g_x \in G$ such that $H(x)$ is a subgroup of $g_x P_1(x) g_x^{-1}$ for each $x \in A$ by Lemma 6.2.
b) Since (P_1, A) and (P_2, A) are soft Sylow p-subgroups over G, then $P_1(x)$ and $P_2(x)$ are Sylow p-subgroups of G for all $x \in A$ by Definition 6.1. By Lemma 6.2, $P_1(x)$ and $P_2(x)$ are conjugate for all $x \in A$, then, there exists a $g_x \in G$ such that $g_x P_1(x) g_x^{-1} = P_2(x)$ for each $x \in A$. Therefore, $(P_1, A) \widetilde{\rightarrow}_s (P_2, A)$ by Definition 5.1. □

Lemma 6.3 *([13])***(Third Sylow Theorem)** *If G is a finite group and p divides $|G|$, then the number of Sylow p-subgroups is congruent to 1 modulo p and divides $|G|$.*

Theorem 6.3 *Let G be a finite group and let p be a prime number such that p divides $|G|$. Then, there exists a soft Sylow p-subgroup (P, A) over G such that $\sum_{i=1}^{|A|} 1 \equiv 1 (mod\ p)$ and $\sum_{i=1}^{|A|} 1$ divides $|G|$.*

Proof: Assume that the number of Sylow p-subgroups is s. Let the soft set (P, A) over G be such that $A = \{x_1, x_2, ..., x_s\}$, $P(x_i)$ is a Sylow p-subgroup and $P(x_i) \neq P(x_j)$ if $i \neq j$ for all $i, j \in \{1, 2, ..., s\}$. By Definition 6.1, (P, A) is a Sylow p-subgroup over G. Since $P(x_i) \neq P(x_j)$ if $i \neq j$ for all $i, j \in \{1, 2, ..., s\}$, then $s = \sum_{i=1}^{|A|}$. Therefore, the proof is obtained, since $s \equiv 1 (mod\ p)$ and s divides $|G|$ by Lemma 6.3. □

7. Conclusion

In this study, our aim has been to transfer the concepts p-groups, conjugate groups, Cauchy Theorem and Sylow Theorems, which are all the basic and fundamental concepts and theorems in group theory, to soft group theory. We have defined soft prime group, conjugate group, soft conjugate group and strong conjugate group. We have investigated their properties in detailed under group mappings, group homomorphisms and soft homomorphisms and obtain the transformations where these structures are preserved. Also, the relationship between the conjugation and strong conjugation has been derived. We show that being a soft prime group is preserved under the soft conjugations. It is also obtained that strong conjugation is an equivalence relation on the set of all soft groups over G with the parameter set A. Moreover, since Cauchy's Theorem is related to Lagrange's theorem, we have conveyed this theorem to soft groups over a finite group. As Sylow theorems are used to understand the structure of an arbitrary finite group, we have defined soft Sylow p-subgroup for the soft set theory and obtain the corresponding Sylow Theorems in soft group theory with the concept of soft Sylow p-subgroups. By this way, we have expanded the soft set theory. As a future study, one can focus on examining the partitions of the set $S_A(G)$ using a strong congruence relation and expanding the results obtained to all soft groups defined on the group G will lead us to many remarkable results. The authors hope that this paper inspires the scientists working in this field.

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