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Renormalized solutions for some boundary value elliptic problem with L^1- data in generalized sobolev spaces

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ABSTRACT: The aim of this paper is to investigate an existence of renormalized solutions for some boundary value elliptic problem of the form $-\operatorname{div}(a(x,u,\nabla u)+\Phi(x,u))+g(x,u,\nabla u)=f$ in Ω, in the framework of Musielak-Orlicz spaces, where the term Φ satisfies the natural growth condition, the function g has a natural growth with respect to its third argument and without sign condition, no Δ_2-condition is assumed on the Musielak function, and $f\in L^1(\Omega)$.

Key Words: Musielak-Orlicz-Sobolev spaces, Elliptic equation, Renormalized solutions, Truncations, Boundary value problems.

Contents

1. Introduction and Basic Hypothesis

In this note we give an existence result of renormalized solutions for strongly nonlinear boundary value problem of the type:

$$\begin{cases} A(u)-div\Big(\Phi(x,u)\Big)+g(x,u,\nabla u)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω be a bounded domain of $\mathbb{R}^N, N\geq 2,\ A(u)=-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator defined from the space $W_0^1L_\varphi(\Omega)$ into its dual $W^{-1}L_{\overline{\varphi}}(\Omega)$, with φ and $\overline{\varphi}$ are two complementary Musielak-Orlicz functions and where a is a function satisfying the following conditions:

$$a:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N \text{ is a Carathéodory function.} \tag{1.2}$$

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There exist two Musielak-Orlicz functions φ and P such that $P \prec\prec \varphi$, a positive function $d(x) \in E_{\overline{\varphi}}(\Omega)$, $\alpha > 0$ and $k_i > 0$ for $i = 1, \cdots, 4$, such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ and all $\xi,\ \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|a(x,s,\xi)| \leq k_1\Big(d(x) + \overline{\varphi}_x^{-1}\left(P\left(x,k_2|s|\right)\right) + \overline{\varphi}_x^{-1}\left(\varphi\left(x,k_3|\xi|\right)\right)\Big) \tag{1.3}$$

$$(a(x,s,\xi) - a\left(x,s,\xi'\right))\left(\xi - \xi'\right) > 0, \tag{1.4}$$

$$a(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) + \varphi(x,|s|). \tag{1.5}$$

Furthermore, let $g:\ \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$:

$$|g(x,s,\xi)| \leq h(x) + \rho(s)\varphi(x,|\xi|), \tag{1.6}$$

where $\rho : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous positive function which belongs to $L^\infty(\mathbb{R})$ and $h(x) \in L^1(\Omega)$. The lower order term Φ is a Carathéodory function satisfying, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, the following condition:

$$|\Phi(x,s)| \leq c(x)\overline{\varphi}_x^{-1}\varphi(x,\alpha_0|s|), \tag{1.7}$$

where $c(.) \in L^\infty(\Omega)$ such that $\|c(.)\|_{L^\infty(\Omega)} < \frac{\alpha}{2}$ and $0 < \alpha_0 < \min\left(1,\frac{1}{\alpha}\right)$. The right hand side of (1.1) is assumed to satisfy

$$f \in L^1(\Omega). \tag{1.8}$$

In the paper [13], the authors were introduced the notion of renormalized solution in the sobolev spaces, this concept have been used by the authors in [8] where they have studied the problem (1.1), when the right hand side is in $W^{-1,p'}(\Omega)$ and in the case where the nonlinearity g depends only on x and u, this work was then studied by Rakotoson in [32] when the right hand side is in $L^1(\Omega)$, and finally by DalMaso et al. in [26] for the case in which the right hand side is general measure data.

However, on Orlicz-Sobolev spaces,the paper [5] investigate the problem (1.1) where the authors have been considered that $\Phi(x,u) \equiv \Phi(u)$, and the function g depends only on x and u under the restriction that the N-function is supposed satisfying the Δ_2-condition, On the other hand, in [3] the authors have been treated the same problem for N-function not satisfying necessarily the Δ_2-condition and $\Phi(x,u) \equiv \Phi(u)$.

In the note [4], the authors prove the existence and uniqueness of a renormalized solution for a suitable elliptic problem (1.1) in variable exponent Sobolev spaces, where $a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$, $\Phi \equiv 0$, $g \equiv 0$ and where $f \in L^1(\Omega)$.

In [7], the authors have shown an existence result for (1.1) in Musielak-Orlicz-Sobolev spaces, where $g \equiv 0$ and $\Phi \equiv 0$, and where the non-linearity g depends only on x and u. If g depends also on ∇u, the problem (1.1) has been solved in [25] where the authors supposed that $\Phi(x,u) \equiv \Phi(u)$.

Many papers deals with the existence and uniqueness solution of elliptic and parabolic problems under different hypotheses, either in Sobolev spaces and in generalized Sobolev spaces (see [21,25,22,23,24,15, 16,17,18,19,9,11,10,20,14,12] for more details).

The paper is organized as follows: In section 2, we present some preliminary and knowledge. Section 3 contains some technical lemmas which will be nedded to prove our result. Section 4, is the object of our main result and in section5 we prove an existence solution for the problem (1.1).

2. Some Preliminaries and Background

This part is devoted to the preliminary results and properties that concern Musielak-Orlicz spaces (see [30]). Let Ω be an open subset of $\mathbb{R}^N$, a Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}^+$ such that

a) $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t} = \infty$).

b) $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

For a Musielak-Orlicz function φ, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the nonnegative reciprocal function with respect to t, i.e. the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x,\varphi_x^{-1}(t)\right) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a nonnegative function h, integrable in Ω, we have

$$\varphi(x, 2t) \leq k\varphi(x, t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity. Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$:

$$\gamma(x, t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \text{ (resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0, \quad \text{(resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0).$$

Remark 2.1 If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists a nonnegative integrable function h, such that

$$\gamma(x, t) \leq \varphi(x, \varepsilon t) + h(x) \quad \text{for all } t \geq 0 \text{ and for a.e. } x \in \Omega. \tag{2.2}$$

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define :

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}/\ \rho_{\varphi,\Omega}(u) < \infty\right\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable}/\ \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}$$

For a Musielak-Orlicz function φ we pose :

$$\overline{\varphi}(x, s) = \sup_{t>0}\{st - \varphi(x, t)\},$$

$\overline{\varphi}$ is the Musielak-Orlicz function conjugate of φ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we present the two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0/\int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}$$

which is named the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}}\leq 1}\int_\Omega |u(x)v(x)|\, dx$$

where $\overline{\varphi}$ is the Musielak-Orlicz function conjugate to φ. These two norms are equivalent (see [30]). The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [30], Theorem 7.10).

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed $m > 0$ we note:

$$W^m L_\varphi(\Omega) = \left\{u \in L_\varphi(\Omega)/\ \forall|\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega)\right\}$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) / \ \forall |\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega) \Big\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integers $\alpha_i, |\alpha| = |\alpha_1| + \dots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is named the Musielak-Orlicz Sobolev space. Let for $u \in W^m L_\varphi(\Omega)$:

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \leq m} \rho_{\varphi,\Omega}\left(D^\alpha u\right) \ \text{ and } \ \|u\|_{\varphi,\Omega}^m = \inf\left\{ \lambda > 0 / \ \overline{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$$

these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|.\|_{\varphi,\Omega}^m\right)$ is a Banach space if φ verifies the following hypothesis (see [30]):

$$\text{There exist a constant } c_0 > 0 \text{ such that } \inf_{x \in \Omega} \varphi(x, 1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ can be identified to a subspace of the product $\prod_{|\alpha| \leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, and the space $W_0^m E_\varphi(\Omega)$ as the closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces be used latter :

$$W^{-m} L_{\overline{\varphi}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) / \ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_{\overline{\varphi}}(\Omega) \Big\}$$

and

$$W^{-m} E_{\overline{\varphi}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) / \ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_{\overline{\varphi}}(\Omega) \Big\}.$$

We can say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists $k > 0$ such that

$$\lim_{n \to \infty} \overline{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

We remember that

$$\varphi(x, t) \leq t \overline{\varphi}^{-1}(\varphi(x, t)) \leq 2\varphi(x, t) \quad \text{ for all } t \geq 0, \text{ a.e. } x \in \Omega. \tag{2.4}$$

For φ and her conjugate function $\overline{\varphi}$, the following inequality is named the Young inequality (see [30]):

$$ts \leq \varphi(x, t) + \overline{\varphi}(x, s), \quad \forall t, s \geq 0, \text{ a.e. } x \in \Omega. \tag{2.5}$$

Which implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1 \tag{2.6}$$

In $L_\varphi(\Omega)$ we have the follwing relation

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} > 1 \tag{2.7}$$

and

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} \leq 1 \tag{2.8}$$

For two complementary Musielak-Orlicz functions φ and $\overline{\varphi}$, let $u \in L_\varphi(\Omega)$ and $v \in L_{\overline{\varphi}}(\Omega)$, thus we have:

$$\left| \int_\Omega u(x) v(x)\, dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\overline{\varphi},\Omega} \qquad (\text{ the Hölder inequality (see [30])}) \tag{2.9}$$

3. Some Technical Lemmas

This section concern some technical lemmas that will be used in our main result.

Definition 3.1 *We say that a Musielak function φ verifies the log-Hölder continuity hypothesis on Ω if there exists $A > 0$ such that*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)$$

$\forall t \geq 1$ *and* $\forall x, y \in \Omega$ *with* $|x-y| \leq \frac{1}{2}$

Lemma 3.1 *[2] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N(N \geq 2)$ and let φ be a Musielak function verifying the log-Hölder continuity such that*

$$\bar{\varphi}(x,1) \leq c_1 \qquad a.e \text{ in } \Omega \text{ for some } c_1 > 0 \tag{3.1}$$

Then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1L_\varphi(\Omega)$ for the modular convergence.

Remark 3.1 Note that if $\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$, then (3.1) holds (see [2]).
Proof : (see [2]).

Lemma 3.2 *[2] (Poincare's inequality: Integral form) Let Ω be a bounded Lipschitz domain of $R^N(N \geq 2)$ and let φ be a Musielak function satisfying the hypothesis of Lemma 3.1. Then there exists $\beta, \eta > 0$ and $\lambda > 0$ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|v|)dx \leq \beta + \eta\int_\Omega \varphi(x,\lambda|\nabla v|)dx \text{ for all } v \in W_0^1L_\varphi(\Omega). \tag{3.2}$$

Corollary 3.1 *[2] (Poincare's inequality) Let Ω be a bounded Lipchitz domain of $\mathbb{R}^N(N \geq 2)$ and let φ be a Musielak function satisfying the same hypothesis of Lemma 3.2. Then there exists $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1L_\varphi(\Omega).$$

Lemma 3.3 *([31]) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$.*

Hawever, if the set D of discontinuity points of F' is finite, we obtain

$$\frac{\partial F(u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \in D\} \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.4 *[1] (Poincare's inequality). Let φ a Musielak-Orlicz function which satisfies the hypothesis of Lemma 3.1, let $\varphi(x,t)$ decreases with respect of one of coordinate of x, then, that exists $c > 0$ depends only of Ω such that*

$$\int_\Omega \varphi(x,|v|)\,dx \leq \int_\Omega \varphi(x,c|\nabla v|)\,dx \quad \forall u \in W_0^1L_\varphi(\Omega).$$

Lemma 3.5 *[6] Let Ω satisfies the segment property and suppose that $u \in W_0^1L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1L_\varphi(\Omega).$$

In addition to this, if $u \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 3.6 *Suppose that (g_n), $g \in L^1(\Omega)$ such that*
i) $g_n \geq 0$ a.e in Ω,
ii) $g_n \longrightarrow g$ a.e in Ω,
iii) $\displaystyle\int_\Omega g_n(x)\,dx \longrightarrow \int_\Omega g(x)\,dx$.
Then $g_n \longrightarrow g$ strongly in $L^1(\Omega)$.

Lemma 3.7 *[7] If a sequence $h_n \in L_\varphi(\Omega)$ converges in measure to a measurable function h and if h_n remains bounded in $L_\varphi(\Omega)$, then $h \in L_\varphi(\Omega)$ and $h_n \rightharpoonup h$ for $\sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right)$.*

Lemma 3.8 *[7] Let $v_n,\ v \in L_\varphi(\Omega)$. If $v_n \to v$ with respect to the modular convergence, then $v_n \to v$ for $\sigma\left(L_\varphi(\Omega), L_{\overline{\varphi}}(\Omega)\right)$.*

Lemma 3.9 *[27] If $\gamma \prec \varphi$ and $u_n \to u$ for the modular convergence in $L_\varphi(\Omega)$ then $u_n \to u$ strongly in $E_\gamma(\Omega)$.*

Lemma 3.10 *(The Nemytskii Operator). Suppose that Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Suppose that $g : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|g(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi\left(x, k_2|s|\right)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_g defined by $N_g(u)(x) = g(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod\left\{u \in L_M(\Omega) : d\left(u, E_M(\Omega)\right) < \frac{1}{k_2}\right\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence. However if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_g is strongly continuous from $\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$.

4. Main Result

Before we state our main result let us giving the definition of a renormalized solution of (1.1).

Definition 4.1 *A measurable function $u : \Omega \to \mathbb{R}$ is named a renormalized solution of (1.1) if:*

$$T_k(u) \in W_0^1 L_\varphi(\Omega) \quad \text{and} \quad a(x,u,\nabla u) \in \left(L_{\overline{\varphi}}(\Omega)\right)^N, \tag{4.1}$$

$$g(x,u,\nabla u) \in L^1(\Omega) \quad \text{and} \quad g(x,u,\nabla u)u \in L^1(\Omega), \tag{4.2}$$

$$\lim_{m\to+\infty} \int_{\{x\in\Omega:\quad m\leq|u(x)|\leq m+1\}} a(x,u,\nabla u)\nabla u\, dx = 0, \tag{4.3}$$

and $\forall h \in C_c^1(\mathbb{R})$:

$$\begin{aligned} -\operatorname{div}\Big(a(x,u,\nabla u)h(u)\Big) - \operatorname{div}\Big(\Phi(x,u)h(u)\Big) + h'(u)\Phi(x,u)\nabla u + g(x,u,\nabla u)h(u) \\ = fh(u) + h'(u)F\nabla u \quad \text{in } \mathcal{D}'(\Omega). \end{aligned} \tag{4.4}$$

Remark 4.1 Every term in equation (4.4) is meaningful in the distributional sense. Indeed, for $h \in C_c^1(\mathbb{R})$ and $u \in W_0^1 L_\varphi(\Omega)$, then $h(u) \in W^1 L_\varphi(\Omega)$ and for V in $\mathcal{D}(\Omega)$ the function $Vh(u) \in W_0^1 L_\varphi(\Omega)$. Since $\operatorname{div}\Big(a(x,u,\nabla u)\Big) \in W^{-1}L_{\overline{\varphi}}(\Omega)$, we have $\forall V \in \mathcal{D}(\Omega)$:

$$\Big\langle \operatorname{div}\Big(a(x,u,\nabla u)\Big)h(u)\ ;\ V\Big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \Big\langle \operatorname{div}\Big(a(x,u,\nabla u)\Big)\ ;\ Vh(u)\Big\rangle_{W^{-1}L_{\overline{\varphi}}(\Omega), W_0^1 L_\varphi(\Omega)}$$

Finally, $\Phi(x,u)h(u) \in (L^\infty(\Omega))^N$, $\Phi(x,u)h'(u) \in (L^\infty(\Omega))^N$, $\operatorname{div}\Big(\Phi(x,u)h(u)\Big) \in W^{-1}L_{\overline{\varphi}}(\Omega)$ and $\Phi(x,u)h'(u)\nabla u \in L_\varphi(\Omega)$.

Our goal is to show the follwing main result

Theorem 4.1 *If the hypothesis (1.2) − (1.8) are verified, the Problem (1.1) admit at least a renormalized solution .*

5. Proof of Theorem 4.1

Throughout this note, let us define the truncation function T_k:

$$T_k(s) = \max(-k, \min(k, s))$$

5.1. Step 1: Approximate problem

For $n \in \mathbb{N}^*$, we have the approximations of f, Φ and g. Let f_n be a sequence of $L^1(\Omega)$ functions that converge strongly to f in $L^1(\Omega)$, and $\|f_n\|_{L^1} \leq \|f\|_{L^1}$. Let $\Phi_n(x,s) = \Phi(x, T_n(s))$ and $g_n(x,s,\xi) = \dfrac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|}$.
For fixed $n \geq 1$, it's obvious to observe that

$$|g_n(x,s,\xi)| \leq |g(x,s,\xi)| \text{ and } |g_n(x,s,\xi)| \leq n \text{ a.e. in } \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N.$$

Then we consider the approximate equation (1.1) for $n \geq 1:\ u_n \in W_0^1 L_\varphi(\Omega)$

$$-\operatorname{div}\Big(a(x,u_n,\nabla u_n)\Big) + \operatorname{div}\Big(\Phi_n(x,u_n)\Big) + g_n(x,u_n,\nabla u_n) = f_n \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

Since g_n is bounded for any fixed $n > 0$, the approximate problem (5.1) admit at last one solution $u_n \in W_0^1 L_\varphi(\Omega)$ (see [28]).

5.2. Step 2: A Priori Estimates

Choosing $\exp(G(u_n))T_k(u_n)$ as a test function in (5.1), where $G(s) = \displaystyle\int_0^s \frac{\rho(r)}{\alpha'}\,dr$ and $\alpha' > 0$, we get

$$\begin{aligned}\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)\Big)\,dx &+ \int_\Omega \Phi_n(x,u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)\Big)\,dx\\ &+ \int_\Omega g(x,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)\,dx \\ &\leq k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\|f_n\|_{L^1(\Omega)}.\end{aligned} \tag{5.2}$$

By (1.7), Lemma 3.4 and Young inequality, one has:

$$\begin{aligned}&\int_\Omega \Phi_n(x,u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)\Big)\,dx\\ &\leq \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\Big[\alpha_0\int_\Omega \varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)\,dx\\ &\quad+\int_\Omega \varphi(x,|\nabla u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n)\,dx\Big]\\ &\quad+\|c(\cdot)\|_{L^\infty(\Omega)}\alpha_0\int_{Q_\tau}\varphi(x,|u_n|)\exp(G(u_n))\,dx\\ &\quad+\|c(.)\|_{L^\infty(\Omega)}\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\exp(G(u_n))\,dx.\end{aligned} \tag{5.3}$$

However, we have

$$\begin{aligned}&\int_\Omega g_n(x,u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)\,dx\\ &\leq k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\|h\|_{L^1(\Omega)} + \int_\Omega \rho(u_n)\exp(G(u_n))\varphi(x,|\nabla u_n|)T_k(u_n)\,dx.\end{aligned} \tag{5.4}$$

Thanks to (5.3), (5.4) and (1.5) one has

$$\frac{1}{\alpha'}\int_\Omega \varphi(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)\,dx \tag{5.5}$$

$$
\begin{aligned}
&+\frac{\alpha}{\alpha'}\int_\Omega \rho(u_n)\exp(G(u_n))\varphi\left(x,|\nabla T_k\left(u_n\right|\right)T_k\left(u_n\right)\,dx\\
&+\int_\Omega a\left(x,u_n,\nabla u_n\right)\exp\left(G\left(u_n\right)\right)\nabla\left(T_k\left(u_n\right)\right)\,dx\\
\leq\ &\frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha'}\left[\alpha_0\int_\Omega \varphi\left(x,|u_n|\right)\rho\left(u_n\right)\exp\left(G\left(u_n\right)\right)T_k\left(u_n\right)\,dx\right.\\
&\left.+\int_\Omega \varphi\left(x,|\nabla u_n|\right)\rho\left(u_n\right)\exp\left(G\left(u_n\right)\right)T_k\left(u_n\right)\,dx\right]\\
&+\|c(\cdot)\|_{L^\infty(\Omega)}\alpha_0\int_\Omega \varphi\left(x,|u_n|\right)\exp\left(G\left(u_n\right)\right)\chi_{\{|u_n|\leq k\}}\,dx\\
&+\|c(\cdot)\|_{L^\infty(\Omega)}\int_\Omega \varphi\left(x,|\nabla T_k\left(u_n\right)|\right)\exp\left(G\left(u_n\right)\right)\,dx\\
&+\int_\Omega \rho\left(u_n\right)\exp\left(G\left(u_n\right)\right)\varphi\left(x,|\nabla u_n|\right)T_k\left(u_n\right)\,dx\\
&+k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\left[\|f\|_{L^1(\Omega)}+\|h\|_{L^1(\Omega)}\right].
\end{aligned}
$$

By using (1.5) in (5.5) we obtain

$$
\begin{aligned}
&\left(\frac{1-\alpha_0\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\right)\int_\Omega \varphi\left(x,|u_n|\right)\rho\left(u_n\right)\exp\left(G\left(u_n\right)T_k\left(u_n\right)dx\right.\\
&+\left[\frac{\alpha}{\alpha'}-\frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}-1\right]\int_\Omega \rho\left(u_n\right)\exp\left(G\left(u_n\right)\varphi\left(x,|\nabla u_n|\right)T_k\left(u_n\right)dx\right.\\
&+\int_\Omega a\left(x,u_n,\nabla u_n\right)\exp\left(G\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx\\
&\leq\|c(\cdot)\|_{L^\infty(\Omega)}\alpha_0\int_\Omega \varphi\left(x,|u_n|\right)\exp\left(G\left(u_n\right)\right)\chi_{\{|u_n|\leq k\}}\,dx\\
&+\|c(\cdot)\|_{L^\infty(\Omega)}\int_\Omega \varphi\left(x,|\nabla T_k\left(u_n\right)|\right)\exp\left(G\left(u_n\right)\right)\,dx+kc_1
\end{aligned}
$$

By choosing α' such that $\alpha'=\dfrac{\alpha}{2}$ we get

$$
\begin{aligned}
&\int_\Omega a\left(x,u_n,\nabla u_n\right)\exp\left(G\left(u_n\right)\right)\nabla\left(T_k\left(u_n\right)\right)dx\\
\leq\ &\left(\frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha}\right)\left[\alpha_0\alpha\int_\Omega \varphi\left(x,|u_n|\right)\exp\left(G\left(u_n\right)\right)\chi_{T_k(u_n)}dx\right.\\
&\left.+\alpha\int_\Omega \varphi\left(x,|\nabla T_k\left(u_n\right)|\right)\exp\left(G\left(u_n\right)\right)dx\right]+kc_1,
\end{aligned}
$$

since $\alpha_0\alpha<1$ and using (1.5) we get

$$
\left[1-\left(\frac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha}\right)\right]\int_\Omega a\left(x,u_n,\nabla u_n\right)\exp\left(G\left(u_n\right)\right)\nabla\left(T_k\left(u_n\right)\right)\,dx\leq kc_1,
$$

we deduce

$$
\int_\Omega a\left(x,u_n,\nabla u_n\right)\exp\left(G\left(u_n\right)\right)\nabla\left(T_k\left(u_n\right)\right)dx\leq kc'c_1=k\,C_1, \tag{5.6}
$$

where $\dfrac{1}{c'}=1-\left(\dfrac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha}\right)$. By (1.5) and (5.6) we obtain

$$
\int_\Omega \varphi\left(x,|\nabla T_k\left(u_n\right)|\right)dx\leq\frac{kc'c_1}{\alpha}=kC_2. \tag{5.7}
$$

Furthermore, by using Lemma 3.4, there exists $\lambda > 0$ such that

$$\int_\Omega \varphi(x,v)dx \leq \int_\Omega \varphi(x,\lambda|\nabla v|)dx \quad \text{for all } v \in W_0^1 L_\varphi(\Omega). \tag{5.8}$$

Taking $v = \frac{1}{\lambda}\left|T_k\left(u_n\right)\right|$ in (5.8) and using (5.7) we can get

$$\int_\Omega \varphi\left(x, \frac{1}{\lambda}\left|T_k\left(u_n\right)\right|\right) dx \leq \int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) dx \leq C_2 k$$

which implies that

$$\begin{aligned}
\operatorname{meas}\left\{\left|u_n\right|>k\right\} & \leq \frac{1}{\inf_{x \in \Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(x, \frac{k}{\lambda}\right) dx \\
& \leq \frac{1}{\inf_{x \in \Omega} \varphi\left(x, \frac{k}{\lambda}\right)} \int_\Omega \varphi\left(x, \frac{1}{\lambda}\left|T_k\left(u_n\right)\right|\right) dx \\
& \leq \frac{C_2 k}{\inf_{x \in \Omega} \varphi\left(x, \frac{k}{\lambda}\right)}, \quad \forall n, \forall k>0 .
\end{aligned}$$

For any $\beta > 0$, one has

$$\operatorname{meas}\left\{\left|u_n-u_m\right|>\beta\right\} \leq \operatorname{meas}\left\{\left|u_n\right|>k\right\}+\operatorname{meas}\left\{\left|u_m\right|>k\right\}+\operatorname{meas}\left\{\left|T_k\left(u_n\right)-T_k\left(u_m\right)\right|>\beta\right\},$$

and so that

$$\operatorname{meas}\left\{\left|u_n-u_m\right|>\beta\right\} \leq \frac{2 C_2 k}{\inf_{x \in \Omega} \varphi\left(x, \frac{k}{\lambda}\right)}+\operatorname{meas}\left\{\left|T_k\left(u_n\right)-T_k\left(u_m\right)\right|>\beta\right\}. \tag{5.9}$$

By using (5.7) and Corollary 3.1, we deduce that $\left(T_k\left(u_n\right)\right)$ is bounded in $W_0^1 L_\varphi(\Omega)$, then we can suppose that $T_k\left(u_n\right)$ is a cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, then by (5.9) and the fact that $\frac{2C_2 k}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)} \to 0$ as $k \to +\infty$, there exists some $k(\varepsilon) > 0$ such that

$$\operatorname{meas}\left\{\left|u_n-u_m\right|>\beta\right\} \leq \varepsilon, \quad \text{for all } n, m \geq n_0(k(\varepsilon), \beta).$$

Then (u_n) is a cauchy sequence in measure, consequently, u_n converges almost everywhere to u.

Now, by using Lemma 4.4 of [29], we can have for all $k > 0$,

$$T_k\left(u_n\right) \rightharpoonup T_k(u) \text{ weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right) \text{ strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \tag{5.10}$$

5.3. Step 3: Boundedness of $\left(a\left(x, u_n, \nabla u_n\right)\right)_n$ in $\left(L_{\overline{\varphi}}(\Omega)\right)^N$

Suppose that $\vartheta \in \left(E_\varphi(\Omega)\right)^N$ such that $\|\vartheta\|_{\varphi,\Omega} = 1$, we get

$$\int_\Omega\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k\left(u_n\right)-\frac{\vartheta}{k_3}\right] dx \geq 0 .$$

This implies that

$$\begin{aligned}
& \int_\Omega \frac{1}{k_3} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta \, dx \qquad (5.11)\\
& \leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx-\int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\left(\nabla T_k\left(u_n\right)-\frac{\vartheta}{k_3}\right) dx \\
& \leq k C_1-\int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \nabla T_k\left(u_n\right) dx+\frac{1}{k_3} \int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \vartheta \, dx .
\end{aligned}$$

By Young's inequality and (5.7) we obtain

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\vartheta\,dx \tag{5.12}$$

$$\leq k_3kC_1+3k_1\left(1+k_3\right)\int_\Omega \overline{\varphi}\left(x,\frac{\left|a\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right)dx$$

$$+3k_1k_3\int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)dx+3k_1\int_\Omega \varphi(x,|\vartheta|)\,dx$$

$$\leq k_3kC_1+3k_1k_3kC_1+3k_1+3k_1\left(1+k_3\right)\int_\Omega \overline{\varphi}\left(x,\frac{\left|a\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right)dx$$

By (1.3) and taking into account to the convexity of $\overline{\varphi}$ we obtain

$$\overline{\varphi}\left(x,\frac{\left|a\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right)\leq\frac{1}{3}\Big(\overline{\varphi}(x,d(x))+P\left(x,k_2\left|T_k\left(u_n\right)\right|\right)+\varphi(x,|\vartheta|)\Big). \tag{5.13}$$

By using Remark 2.1 there exists $h\in L^1(\Omega)$ such that

$$P\left(x,k_2\left|T_k\left(u_n\right)\right|\right)\leq P\left(x,k_2k\right)\leq\varphi(x,1)+h(x)$$

then by integrating over Ω we conclude that

$$\int_\Omega \overline{\varphi}\left(x,\frac{\left|a\left(x,T_k\left(u_n\right),\frac{v}{k_3}\right)\right|}{3k_1}\right)dx \tag{5.14}$$

$$\leq\frac{1}{3}\left(\int_\Omega \overline{\varphi}(x,c(x))\,dx+\int_\Omega h(x)\,dx+\int_\Omega \varphi(x,1)\,dx+\int_\Omega \varphi(x,|\vartheta|)\,dx\right)\leq c_k',$$

where c_k' is a constant depending on k, then $\forall\vartheta\in\left(E_\varphi(\Omega)\right)^N$ with $\|\vartheta\|_{\varphi,\Omega}=1$ we have

$$\int_\Omega a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\vartheta dx\leq c_k',$$

and thus $\left\|a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\right\|_{\overline{\varphi},\Omega}\leq c_k'$, which implies that

$$\Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\Big)_n \text{ is bounded in } L_{\overline{\varphi}}(\Omega)^N. \tag{5.15}$$

5.4. Step 4: Renormalization identity for the approximate solutions

Consider the function $Z_m\left(u_n\right)=T_1\left(u_n-T_m\left(u_n\right)\right)^-$ and by taking $\exp\left(-G\left(u_n\right)\right)Z_m\left(u_n\right)$ as test function in (5.1) we get

$$\int_\Omega a_n\left(x,u_n,\nabla u_n\right)\nabla\Big(\exp\left(-G\left(u_n\right)\right)Z_m\left(u_n\right)\Big)\,dx \tag{5.16}$$

$$+\int_\Omega \Phi_n\left(x,u_n\right)\nabla\Big(\exp\left(-G\left(u_n\right)\right)Z_m\left(u_n\right)\Big)\,dx+\int_\Omega g_n\left(x,u_n,\nabla u_n\right)\exp\left(-G\left(u_n\right)\right)Z_m\left(u_n\right)\,dx$$

$$=\int_\Omega f_n\exp\left(-G\left(u_n\right)\right)Z_m\left(u_n\right)dx.$$

By the same argument used in a priori estimates, we get

$$\int_\Omega \varphi\left(x, |\nabla Z_m\left(u_n\right)|\right)\, dx \tag{5.17}$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \left[\int_\Omega f_n Z_m\left(u_n\right) dx + \int_\Omega h(x) Z_m\left(u_n\right)\, dx\ \right]$$

where $\dfrac{1}{C} = \left[1 - \left(\dfrac{\|c(\cdot)\|_{L^\infty(\Omega)}}{\alpha}\right)\right]$.
So as to pass to the limit in (5.17) as $n \to +\infty$,
we can use the convergence of u_n and strongly convergence in $L^1(\Omega)$ of f_n, we get

$$\lim_{n\to+\infty} \int_\Omega \varphi\left(x, |\nabla Z_m\left(u_n\right)|\right)\, dx \tag{5.18}$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \left[\int_\Omega f Z_m(u) dx + \int_\Omega h(x) Z_m(u) dx\ \right]$$

Thanks to Lebesgue's theorem and passing to the limit as $m \to +\infty$, in every term of the previous inequalities, we obtain

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \int_\Omega \varphi\Big(x, |\nabla Z_m\left(u_n\right)|\Big)\, dx = 0. \tag{5.19}$$

Using (1.7) and Young inequality, for $n > m+1$ we have

$$\int_\Omega \left|\Phi_n\left(x, u_n\right) \exp\left(-G\left(u_n\right)\right) \nabla Z_m\left(u_n\right)\right| dx \tag{5.20}$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) \left[\int_{\{-(m+1)\leq u_n \leq -m\}} \varphi\left(x, \alpha_0 \left|T_{m+1}\left(u_n\right)\right|\right) dx + \int_\Omega \varphi\left(x, |\nabla Z_m\left(u_n\right)|\right) dx\right].$$

Thanks to Lebesgue's theorem, and by convergence of u_n we can have

$$\lim_{n\to+\infty} \int_\Omega \Big|\Phi_n\left(x, u_n\right) \exp\left(-G\left(u_n\right)\right) \nabla Z_m\left(u_n\right)\Big|\, dx \tag{5.21}$$

$$\leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) \left[\int_{\{-(m+1)\leq u \leq -m\}} \varphi\left(x, \alpha_0 \left|T_{m+1}(u)\right|\right) dx + \lim_{n\to+\infty} \int_\Omega \varphi\left(x, |\nabla Z_m(u)|\right)\, dx\right],$$

Passing to the limit in (5.21) as $m \to +\infty$, we obtain

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \int_\Omega \Phi_n\left(x, u_n\right) \exp\left(-G\left(u_n\right)\right) \nabla Z_m\left(u_n\right)\, dx = 0.$$

and passing to the limit in (5.17), we get

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \int_{\{-(m+1)\leq u_n \leq -m\}} a_n\left(x, u_n, \nabla u_n\right) \nabla u_n\, dx = 0.$$

By the same way we take $Z_m\left(u_n\right) = T_1\left(u_n - T_m\left(u_n\right)\right)^+$ and testing the equation (5.1) by the function $\exp\left(G\left(u_n\right)\right) Z_m\left(u_n\right)$ and we obtain

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \int_{\{m\leq u_n \leq m+1\}} a_n\left(x, u_n, \nabla u_n\right) \nabla u_n dx = 0.$$

Finally we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m\leq |u_n| \leq m+1\}} a\left(x, u_n, \nabla u_n\right) \nabla u_n\, dx = 0. \tag{5.22}$$

5.5. Step 5: Almost everywhere convergence of the gradients

Suppose $v_j \in \mathcal{D}(\Omega)$ be a sequence such that $v_j \to u$ in $W_0^1 L_\varphi(\Omega)$ for the modular convergence. For $m \geq k$, we define the function ϱ_m by

$$\varrho_m(s) = \begin{cases} 1 & \text{if} \quad |s| \leq m \\ m+1-|s| & \text{if} \quad m \leq |s| \leq m+1 \\ 0 & \text{if} \quad |s| \geq m+1 \end{cases}$$

We define $\epsilon(n, \eta, j, m)$ all quantities (possibly different) such that

$$\lim_{m \to +\infty} \lim_{j \to +\infty} \lim_{\eta \to +\infty} \lim_{n \to +\infty} \epsilon(n, \eta, j, m) = 0.$$

For fixed $k \geq 0$, let $W_\eta^{n,j} = T_\eta\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right)^+$ and $W_\eta^j = T_\eta\left(T_k(u) - T_k\left(v_j\right)\right)^+$. Multiplying the approximating equation by $\exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m\left(u_n\right)$, we obtain

$$\begin{aligned} &\int_\Omega a_n\left(x, u_n, \nabla u_n\right) \exp\left(G\left(u_n\right)\right) \nabla W_\eta^{n,j} \varrho_m\left(u_n\right)\, dx \\ &+\int_\Omega a_n\left(x, u_n, \nabla u_n\right) \nabla u_n \exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m'\left(u_n\right)\, dx \\ &-\int_\Omega \Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \nabla W_\eta^{n,j} \varrho_m\left(u_n\right)\, dx \\ &-\int_\Omega \Phi_n\left(x, u_n\right) \nabla u_n \exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m'\left(u_n\right)\, dx \\ &\leq \int_\Omega f_n \exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m\left(u_n\right) dx + \int_\Omega h(x) \exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m\left(u_n\right)\, dx \end{aligned} \tag{5.23}$$

Remark that if $n > m+1$, we obtain

$$\Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \varrho_m\left(u_n\right) = \Phi\left(x, T_{m+1}\left(u_n\right)\right) \exp\left(G\left(T_{m+1}\left(u_n\right)\right)\right) \varrho_m\left(T_{m+1}\left(u_n\right)\right),$$

then $\Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \varrho_m\left(u_n\right)$ is bounded in $L_{\overline{\varphi}}(\Omega)$, thus, by using the convergence of u_n and thanking to Lebesgue's theorem one has $\Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \varrho_m\left(u_n\right)$ converges to $\Phi(x,u)\exp(G(u))\varrho_m(u)$ with the modular convergence as $n \to +\infty$, then

$$\Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \varrho_m\left(u_n\right) \longrightarrow \Phi(x,u)\exp(G(u))\varrho_m(u) \text{ for } \sigma\left(\Pi L_\varphi, \Pi L_\varphi\right).$$

In the other hand for $0 \leq T_k\left(u_n\right) - T_k\left(v_j\right) \leq \eta$ then $\nabla W_\eta^{n,j} = \nabla\Big(T_k(u_n) - T_k(v_j)\Big)$ converges to $\nabla\Big(T_k(u) - T_k(v_j)\Big)$ weakly in $\left(L_\varphi(\Omega)\right)^N$ as n tends to $+\infty$, then

$$\lim_{n \to +\infty} \int_\Omega \Phi_n\left(x, u_n\right) \exp\left(G\left(u_n\right)\right) \varrho_m\left(u_n\right) \nabla W_\eta^{n,j}\, dx = \int_\Omega \Phi(x,u)\varrho_m(u)\exp(G(u))\nabla W_\eta^j\, dx.$$

By using the modular convergence of W_η^j as $j \to +\infty$ and letting μ tends to infinity, we get

$$\int_\Omega \Phi_n\left(x, u_n\right) \varrho_m\left(u_n\right) \exp\left(G\left(u_n\right)\right) \nabla W_\eta^{n,j}\, dx = \epsilon(n,j) \quad \text{ for any } m \geq 1. \tag{5.24}$$

Furthermore for $n > m+1 > k$, we get $\nabla u_n \varrho_m'\left(u_n\right) = \nabla T_{m+1}\left(u_n\right)$ a.e. in Ω. By the almost every where convergence of u_n we have $\exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \to \exp(G(u))W_\eta^j$ in $L^\infty(\Omega)$ weak- * and since the sequence $\left(\Phi_n\left(x, T_{m+1}\left(u_n\right)\right)\right)_n$ converge strongly in $E_{\overline{\varphi}}(\Omega)$ then

$$\Phi_n\left(x, T_{m+1}\left(u_n\right)\right) \exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \to \Phi\left(x, T_{m+1}(u)\right) \exp(G(u)) W_\eta^j$$

converge strongly in $E_{\overline{\varphi}}(\Omega)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_\varphi(\Omega))^N$ as $n \to +\infty$ we have

$$\lim_{n\to+\infty} \int_{\{m\le|u_n|\le m+1\}} \Phi_n(x, T_{m+1}(u_n))\nabla u_n \varrho'_m(u_n) \exp(G(u_n))W_\eta^{n,j}\, dx \tag{5.25}$$

$$= \int_{\{m\le|u|\le m+1\}} \Phi(x,u)\nabla u \exp(G(u))W_\eta^j\, dx$$

with the modular convergence of W_η^j as $j \to +\infty$, we obtain

$$\int_\Omega \Phi_n(x,u_n)\nabla u_n \varrho'_m(u_n)\exp(G(u_n))W_\eta^{n,j}\, dx = \epsilon(n,j) \quad \text{for any } m \ge 1 \tag{5.26}$$

Concerning the first term of (5.23) we have

$$\int_\Omega a_n(x,u_n,\nabla u_n)\varrho'_m(u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j}\, dx \tag{5.27}$$

$$\le \eta C \int_{\{m\le|u_n|\le m+1\}} a_n(x,u_n,\nabla u_n)\nabla u_n\, dx$$

Using (5.22), we get

$$\int_\Omega a_n(x,u_n,\nabla u_n)\varrho'_m(u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j} dx \le \epsilon(n,m). \tag{5.28}$$

The weakly convergence of $T_k(u_n)$ to $T_k(v_j)$ in $W^{0,1}L_\varphi(\Omega)$ as n tends to $+\infty$, the bounded character of ϱ_m and $W_\eta^{n,j}$, we obtain

$$\int_\Omega f_n \varrho_m(u_n)\exp(G(u_n))W_\eta^{n,j}\, dx = \epsilon(n,\eta), \tag{5.29}$$

$$\int_\Omega h(x)\exp(G(u_n))W_\eta^{n,j}\varrho_m(u_n)\, dx = \epsilon(n,\eta), \tag{5.30}$$

In the other hand we have

$$\int_\Omega a_n(x,u_n,\nabla u_n)\varrho_m(u_n)\exp(G(u_n))\nabla W_\eta^{n,j}\, dx \tag{5.31}$$

$$= \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} a_n(x,T_k(u_n),\nabla T_k(u_n))\varrho_m(u_n)\exp(G(u_n)) \times(\nabla T_k(u_n) - \nabla T_k(v_j))\, dx$$

$$- \int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} a_n(x,u_n,\nabla u_n)\varrho_m(u_n)\exp(G(u_n))\nabla T_k(v_j)\, dx.$$

Since $a_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{\varphi}}(\Omega))^N$, there exist some $\varpi_{k+\eta} \in (L_{\overline{\varphi}}(\Omega))^N$ such that $a_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \rightharpoonup \varpi_{k+\eta}$ weakly in $(L_{\overline{\varphi}}(\Omega))^N$. Thus:

$$\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} a_n(x,u_n,\nabla u_n)\varrho_m(u_n)\exp(G(u_n))\nabla T_k(v_j)\, dx$$
$$= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j))\le\eta\}} \varrho_m(u)\exp(G(u))\varpi_{k+\eta}\nabla T_k(v_j)\, dx + \epsilon(n), \tag{5.32}$$

By letting $j \to +\infty$, we get

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j))\le\eta\}} \varrho_m(u)\exp(G(u))\nabla T_k(v_j)\varpi_{k+\eta}\, dx \tag{5.33}$$

$$= \int_{\{|u|>k\}\}} \varrho_m(u) \exp(G(u)) \nabla T_k(u) \varpi_{k+\eta} dx + \epsilon(n,j)$$
$$= \epsilon(n,j).$$

Thanks to (5.24)-(5.33), one has

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j))\leq\eta\}} a_n\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \varrho_m\left(u_n\right) \exp\left(G\left(u_n\right)\right) \times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right) dx \leq C\eta+\epsilon(n,j,m). \tag{5.34}$$

Since $\exp\left(G\left(u_n\right)\right) \geq 1$ and $\varrho_m\left(u_n\right)=1$ for $|u_n| \leq k$ then

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a_n\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right) dx \leq C\eta+\epsilon(n,j,m). \tag{5.35}$$

Finally we show that,

$$\int_\Omega \Big(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\Big)\Big(\nabla T_k\left(u_n\right)-\nabla T_k(u)\Big) dx \to 0. \tag{5.36}$$

For $s>0$, denoting by $\Omega^s=\{x \in \Omega:|\nabla T_k(u)| \leq s\}$ and $\Omega_j^s=\{x \in \Omega:|\nabla T_k(v_j)| \leq s\}$ then by χ^s and χ_j^s the characteristic functions of Ω^s and Ω_j^s respectively, letting $0<\delta<1$, define

$$\Theta_{n,k}=\Big(a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u)\right)\Big)\Big(\nabla T_k\left(u_n\right)-\nabla T_k(u)\Big).$$

For $s>0$, we have

$$0 \leq \int_{\Omega^s} \Theta_{n,k}^\delta \, dx=\int_{\Omega^s} \Theta_{n,k}^\delta \chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} dx+\int_{\Omega^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)>\eta\}} \, dx.$$

Concerning the first term of the right-side hand, by using the Hölder inequality we obtain

$$\int_{\Omega^s} \Theta_{n,k}^\delta \chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \, dx \leq\left(\int_{\Omega^*} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} dx\right)^\delta\left(\int_{\Omega^*} dx\right)^{1-\delta} \leq C_1\left(\int_{\Omega^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \, dx\right)^\delta. \tag{5.37}$$

Concerning the second term of the right-side hand, thanking to the Hölder inequality we get

$$\int_{\Omega^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)>\eta\}} dx \leq\left(\int_{\Omega^s} \Theta_{n,k} \, dx\right)^\delta\left(\int_{\{T_k(u_n)-T_k(v_j)>\eta\}} dx\right)^{1-\delta}, \tag{5.38}$$

since $a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)$ is bounded in $\left(L_{\overline{\varphi}}(\Omega)\right)^N$, while $\nabla T_k\left(u_n\right)$ is bounded in $\left(L_{\varphi}(\Omega)\right)^N$ then

$$\int_{\Omega^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)>\eta\}} \, dx \leq C_2 \operatorname{meas}\left\{x \in \Omega: T_k\left(u_n\right)-T_k\left(v_j\right)>\eta\right\}^{1-\delta} \tag{5.39}$$

We obtain

$$\int_{\Omega^g} \Theta_{n,k}^\delta dx \leq C_1\left(\int_{\Omega^s} \Theta_{n,k} \chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \, dx\right)^\delta+C_2 \operatorname{meas}\left\{x \in \Omega: T_k\left(u_n\right)-T_k\left(v_j\right)>\eta\right\}^{1-\delta} \tag{5.40}$$

On the other hand

$$\int_{\Omega^s} \Theta_{n,k}\chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}}\,dx \tag{5.41}$$
$$\leq \int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi_s\right)\Big)$$
$$\times\Big(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi_s\Big)\,dx.$$

For each $s,\ r\in\mathbb{R}^+$ with $s>r$ one has

$$0\leq \int_{\Omega^r\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta)\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\right)\Big) \tag{5.42}$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right)\,dx$$
$$\leq \int_{\Omega^s\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta)\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\right)\Big)$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right)\,dx$$
$$= \int_{\Omega^s\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi_s\right)\Big)$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi_s\right)\,dx$$
$$\leq \int_{\Omega\cap\{0\leq T_k(u_n)-T_k(v_j)\leq\eta)\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\Big)$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\chi^s\right)\,dx$$
$$= \int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\Big)$$
$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\,dx$$
$$+\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(u)\chi^s\right)\,dx$$
$$+\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} \left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)-a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\right)\nabla T_k\left(u_n\right)\,dx$$
$$-\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\nabla T_k\left(v_j\right)\chi_j^s\,dx$$
$$+\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\nabla T_k(u)\chi^s\,dx$$
$$= I_1+I_2+I_3+I_4+I_5.$$

Now by passing to the limit in I_i when $n,\ j,\ \mu$, and $s\to+\infty$. one has

$$I_1 = \int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right)dx$$
$$-\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\Big(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\Big)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)\right)dx$$
$$-\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)dx$$

Thanks to (5.35), the first term of the right hand side in I_1, we get

$$\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right)dx$$

$$\leq C\eta+\epsilon(n,m,j,s)-\int_{\{|u|>k\cap 0\leq T_k(u)-T_k(v_j)\leq\eta\}} a\left(x,T_k(u),0\right)\nabla T_k\left(v_j\right)\,dx$$

$$\leq C\eta+\epsilon(n,m,j).$$

Since $a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)$ is bounded in $\left(L_{\overline{\varphi}}(\Omega)\right)^N$, there exist some $\varpi_k\in\left(L_{\overline{\varphi}}(\Omega)\right)^N$ such that (for a subsequence still denoted by u_n):

$$a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\to\varpi_k \quad \text{in} \quad \left(L_{\varphi}(\Omega)\right)^N \quad \text{for} \quad \sigma\left(\Pi L_{\varphi},\Pi E_{\varphi}\right)$$

By using in the fact
$\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)\right)\chi_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}}$ strongly converges to
$\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)\right)\chi_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}}$ in $\left(E_{\varphi}(\Omega)\right)^N$ as $n\to+\infty$.
The second term of the right hand side of I_1 tends to

$$\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\Big(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\Big)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)\right)\,dx$$

$$=\int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}}\varpi_k\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)\right)\,dx+\epsilon(n).$$

The third term of the right-hand side tends to

$$\int_{\{0\leq T_k(u_n)-T_k(v_j)\leq\eta\}} a\left(x,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\,dx$$

$$=\int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}} a\left(x,T_k(u),\nabla T_k\left(v_j\right)\chi_j^s\right)\left(\nabla T_k(u)-\nabla T_k\left(v_j\right)\chi_j^s\right)\,dx+\epsilon(n),$$

Letting $j\to+\infty$ and $\mu\to+\infty$ of I_1, it possible to conclude that

$$I_1\leq C\eta+\epsilon(n,j,s).$$

Concerning I_2, by letting $n\to+\infty$, we obtain

$$I_2\to\int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}}\varpi_k\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(u)\chi^s\right)\,dx.$$

Since $a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\to\varpi_k$ in $\left(L_{\overline{\varphi}}(\Omega)\right)^N$, for $\sigma\left(\Pi L_{\overline{\varphi}},\Pi E_{\varphi}\right)$ while

$$\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(u)\chi^s\right)\chi_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}}\to\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(u)\chi^s\right)\chi_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}}$$

strongly in $\left(E_{\varphi}(\Omega)\right)^N$. Now, letting $j\to+\infty$, and thanks to Lebesgue's theorem, we obtain

$$I_2=\epsilon(n,j),$$

$$I_3=\epsilon(n,j),$$

$$I_4=\int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}} a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx+\epsilon(n,j,s,m),$$

and

$$I_5=\int_{\{0\leq T_k(u)-T_k(v_j)\leq\eta\}} a\left(x,T_k(u),\nabla T_k(u)\right)\nabla T_k(u)dx+\epsilon(n,j,s,m).$$

Consequently, we obtain

$$\int_{\Omega^s} \Theta_{n,k} dx \leq C_1(C\eta + \epsilon(n,\eta,m))^\delta + C_2(\epsilon(n,))^{1-\delta}.$$

Which leads to

$$\int_{\{T_\eta(T_k(u_n)-T_k(v_j))\geq 0\}\cap\Omega^r} \Big[\Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big) \times (\nabla T_k(u_n) - \nabla T_k(u))\Big]^\delta dx = \epsilon(n). \tag{5.43}$$

By taking $W_\eta^{n,j} = T_\eta\left(T_k\left(u_n\right) - T_k\left(v_j\right)\right)^-$ and $W_\eta^j = T_\eta\left(T_k(u) - T_k\left(v_j\right)\right)^-$, then testing the approximating equation by $\exp\left(G\left(u_n\right)\right) W_\eta^{n,j} \varrho_m\left(u_n\right)$, we obtain

$$\int_{\{T_\eta(T_k(u_n)-T_k(v_j))\leq 0\}\cap\Omega^r} \Big[\Big(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) - a\left(x,T_k\left(u_n\right),\nabla T_k(u)\right)\Big) \times \left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right)\Big]^\delta dx = \epsilon(n). \tag{5.44}$$

Thanks to (5.43) and (5.44) we have

$$\int_{\Omega^r} \Big[\left(a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) - a\left(x,T_k\left(u_n\right),\nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right) - \nabla T_k(u)\right)\Big]^\delta dx = \epsilon(n)$$

As a consequence, since r is arbitrary:

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega, \tag{5.45}$$

and $\forall k \geq 0$, we get

$$a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) \rightharpoonup a\left(x,T_k(u),\nabla T_k(u)\right) \text{ weakly in } \left(L_\psi(\Omega)\right)^N, \tag{5.46}$$

$$\varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) \to \varphi\left(x,\left|\nabla T_k(u)\right|\right) \text{ strongly in } L^1(\Omega). \tag{5.47}$$

5.6. Step 6: Equi-integrability of the non-linearities.

Defining $g_0\left(u_n\right) = \displaystyle\int_0^{u_n} \rho(s)\chi_{\{s>h\}} ds$ and choosing $\exp\left(G\left(T_k\left(u_n\right)\right)\right) g_0\left(u_n\right)$ as a test in the approximate problem then by the same technique used in a priori estimates we can have

$$\int_{\{u_n>h\}} \rho\left(u_n\right)\varphi\left(x,\nabla u_n\right) dx \leq \left(\int_h^{+\infty} \rho(s)dx\right) \exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'}\right)\left[\|f\|_{L^1(\Omega)} + \|h(x)\|_{L^1(\Omega)}\right] \tag{5.48}$$

Since $\rho \in L^1(\mathbb{R})$, we get

$$\lim_{h\to 0}\sup_{n\in\mathbb{N}} \int_{\{u_n>h\}} \rho\left(u_n\right)\varphi\left(x,\nabla u_n\right) dx = 0. \tag{5.49}$$

Similarly, Considering $g_0\left(u_n\right) = \displaystyle\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}} dx$ and choosing $\exp\left(G\left(T_k\left(u_n\right)\right)\right) g_0\left(u_n\right)$ as a test in the approximate problem, then by the same technique used in a priori estimates we can get

$$\lim_{h\to 0}\sup_{n\in\mathbb{N}} \int_{\{u_n<-h\}} \rho\left(u_n\right)\varphi\left(x,\nabla u_n\right) dx = 0. \tag{5.50}$$

As a consequence of (5.49) and (5.50), it follows that

$$\lim_{h\to 0}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}}\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)\,dx=0. \tag{5.51}$$

Let $D\subset\Omega$ then

$$\int_D\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)\,dx \tag{5.52}$$

$$\leq\max_{\{|u_n|\leq h\}}(\rho(x))\int_{D\cap\{|u_n|\leq h\}}\varphi\left(x,\nabla u_n\right)dx+\int_{D\cap\{|u_n|>h\}}\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)\,dx.$$

It follows that $\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)$ is equi-integrable, then $\rho\left(u_n\right)\varphi\left(x,\nabla u_n\right)$ converges to $\rho(u)\varphi(x,\nabla u)$ strongly in $L^1(\mathbb{R})$. This proves that $g_n\left(x,u_n,\nabla u_n\right)$ is equi-integrable, Consequently, by Vitali's theorem one has $g(x,u,\nabla u)\in L^1(\Omega)$, and

$$g_n\left(x,u_n,\nabla u_n\right)\longrightarrow g(x,u,\nabla u)\quad\text{ strongly in } L^1(\Omega). \tag{5.53}$$

5.7. Step 7: Renormalization identity for the solutions.

We show that The limit u of the solution u_n of (5.1) satisfies:

$$\lim_{m\to\infty}\int_{\{m\leq|u|\leq m+1\}}a(x,u,\nabla u)\nabla u dx=0. \tag{5.54}$$

For this, note that for any $m>0$ we have

$$\int_{\{m\leq|u_n|\leq m+1\}}a\left(x,u_n,\nabla u_n\right)\nabla u_n\,dx \tag{5.55}$$

$$=\int_\Omega a\left(x,u_n,\nabla u_n\right)\left(\nabla T_{m+1}\left(u_n\right)-\nabla T_m\left(u_n\right)\right)\,dx$$

$$=\int_\Omega a\Big(x,T_{m+1}\left(u_n\right),\nabla T_{m+1}\left(u_n\right)\Big)\nabla T_{m+1}\left(u_n\right)\,dx-\int_\Omega a\Big(x,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\Big)\nabla T_m\left(u_n\right)\,dx.$$

According to (5.46), (5.47) we can pass to the limit as n tends to infinity for fixed m and to obtain

$$\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}}a\left(x,u_n,\nabla u_n\right)\nabla u_n\,dx \tag{5.56}$$

$$=\int_\Omega a\Big(x,T_{m+1}\left(u\right),\nabla T_{m+1}\left(u\right)\Big)\nabla T_{m+1}\left(u\right)\,dx-\int_\Omega a\Big(x,T_m\left(u\right),\nabla T_m\left(u\right)\Big)\nabla T_m\left(u\right)\,dx.$$

$$=\int_{\{m\leq|u|\leq m+1\}}a(x,u,\nabla u)\nabla u\,dx$$

By letting m tends to $+\infty$ and by (5.22) prove that u satisfies (5.54).

5.8. Step 8: Passing to the limit

Let $h\in\mathcal{C}_c^1(\mathbb{R})$ and $V\in\mathcal{D}(\Omega)$. Using the test function $h\left(u_n\right)V$ in (5.1) leads to

$$\begin{aligned}&\int_\Omega a\left(x,u_n,\nabla u_n\right)\nabla u_n h'\left(u_n\right)V dx+\int_\Omega a\left(x,u_n,\nabla u_n\right)\nabla V h\left(u_n\right)dx\\&\quad+\int_\Omega\Phi_n\left(x,u_n\right)\nabla\left(h\left(u_n\right)V\right)dx+\int_\Omega g_n\left(x,u_n,\nabla u_n\right)h\left(u_n\right)V dx\\&\quad=\int_\Omega f_n h\left(u_n\right)V dx.\end{aligned} \tag{5.57}$$

We shall pass to the limit in each term in (5.57), for this, we can see that since h and h' have a compact support in h, there exists $K>0$ such that $supp(h)\subset[-K,K]$. For n large enough, one has:

$$\Phi_n(x,t)h(t)=\Phi_n\left(x,T_n(t)\right)h(t)=\Phi\left(x,T_K(t)\right)h(t)$$
$$\Phi_n(x,t)h'(t)=\Phi_n\left(x,T_n(t)\right)h'(t)=\Phi\left(x,T_K(t)\right)h'(t)$$

Let us start by the third integral of the left-hand side and the right hand-side of (5.57). Since $h\in C_c^1(\mathbb{R})$ and $V\in\mathcal{D}(\Omega)$, then there exists two positive constant c_1 and c_1' such that $\|h\left(T_K\left(u_n\right)\right)\nabla V\|_\infty\le c_1$ and $\|h'(t)\left(T_K\left(u_n\right)V\nabla T_K\left(u_n\right)\|_\infty\le c_1'\right.$ Now since $T_K\left(u_n\right)$ is bounded in $W_0^1L_\varphi(\Omega)$, then there exists two positive constant λ_0 and λ such that

$$\int_\Omega\varphi\left(x,\frac{\left|\nabla T_K\left(u_n\right)\right|}{\lambda}\right)dx\le\lambda_0.$$

Using the convexity and monotonicity of φ, for η large enough, we get

$$\begin{aligned}
&\int_\Omega\varphi\left(x,\frac{\nabla\left(h\left(T_K\left(u_n\right)\right)V\right)}{\eta}\right)dx\\
&\quad=\int_\Omega\varphi\left(x,\frac{h\left(T_K\left(u_n\right)\right)\nabla V+h'(t)\left(T_K\left(u_n\right)V\left|\nabla T_K\left(u_n\right)\right|\right.}{\eta}\right)dx\\
&\le\int_\Omega\varphi\left(x,\frac{c_1+c_1'\lambda\frac{\left|\nabla T_K\left(u_n\right)\right|}{\lambda}}{\eta}\right)dx\\
&\quad\le\int_\Omega\varphi\left(x,\frac{c_1}{\eta}\right)dx+\frac{c_1'\lambda}{\eta}\int_\Omega\varphi\left(x,\frac{\left|\nabla T_K\left(u_n\right)\right|}{\lambda}\right)dx\\
&\quad\le C_{\eta,c_1}+\frac{c_1'\lambda\lambda_0}{\eta}\qquad\text{where } C_{\eta,c_1}=\int_\Omega\varphi\left(x,\frac{c_1}{\eta}\right)dx<\infty.
\end{aligned}$$

Then the sequence $\{\nabla\left(h\left(T_K\left(u_n\right)\right)V\right)\}$ is bounded in $\left(L_\varphi(\Omega)\right)^N$, as a consequence, we deduce

$$h\left(u_n\right)V\rightharpoonup h(u)V\text{ weakly in }W_0^1L_\varphi(\Omega)\text{ for }\sigma\left(\Pi L_\varphi,\Pi E_{\overline{\varphi}}\right).\tag{5.58}$$

Moreover, since $\Phi\left(x,T_K\left(u_n\right)\right)$ is bounded in $L_{\overline{\varphi}}(\Omega)$, we have from Lemma 3.9

$$\Phi\left(x,T_K\left(u_n\right)\right)\to\Phi\left(x,T_K(u)\right)\qquad\text{strongly in }E_{\overline{\varphi}}(\Omega).$$

By (5.58), we get

$$\lim_{n\to\infty}\int_\Omega\Phi_n\left(x,u_n\right)\nabla\left(h\left(u_n\right)V\right)dx=\int_\Omega\Phi\left(x,T_K(u)\right)\nabla(h(u)V)dx.$$

Moreover we have

$$\lim_{n\to\infty}\int_\Omega f_nh\left(u_n\right)Vdx=\int_\Omega fh(u)Vdx,$$

For the first integral of (5.57), while supp $h'\subset[-K,K]$, we get

$$h'\left(u_n\right)Va\left(x,u_n,\nabla u_n\right)\nabla u_n=h'\left(u_n\right)Va\left(x,T_K\left(u_n\right),\nabla T_K\left(u_n\right)\right)\nabla T_K\left(u_n\right)\qquad\text{a.e. in }\Omega.$$

The convergence of u_n to u, the bounded character of $h'V$, (5.46) and (5.47) imply that

$$h'\left(u_n\right)Va\left(x,u_n,\nabla u_n\right)\nabla u_n\rightharpoonup h'(u)Va\left(x,T_K(u),\nabla T_K(u)\right)\nabla T_K(u)\text{ weakly in }L^1(\Omega).$$

The term $h'(u)Va\left(x,T_K(u),\nabla T_K(u)\right)\nabla T_K(u)$ is identified with $h'(u)Va\left(x,u,\nabla u\right)\nabla u$.

Now since $h\left(u_n\right)Va\left(x,u_n,\nabla u_n\right)=h\left(u_n\right)Va\left(x,T_K\left(u_n\right),\nabla T_K\left(u_n\right)\right)$ a.e. in Ω, and using the strongly convergence of $h\left(u_n\right)\nabla V$ to $h(u)\nabla V$ in $\left(E_\varphi(\Omega)\right)^N$, and using the weakly convergence of $a\left(x,T_K\left(u_n\right),\nabla T_K\left(u_n\right)\right)$ to $a\left(x,T_K(u),\nabla T_K(u)\right)$ in $\left(L_\psi(\Omega)\right)^N$ for $\sigma\left(\Pi L_\psi,\Pi E_\varphi\right)$, then

$$\lim_{n\to\infty}\int_\Omega a\left(x,u_n,\nabla u_n\right)\nabla Vh\left(u_n\right)dx=\int_\Omega a(x,u,\nabla u)\nabla Vh(u)\,dx.$$

The fact that $h(u_n)V$ converges to $h(u)V$ weakly in $L^\infty(\Omega)$ for $\sigma^*\left(L^\infty, L^1\right)$ and (5.53) enable us to pass to the limit in the fourth term of (5.57) to get

$$\lim_{n\to\infty}\int_\Omega g_n(x,u_n,\nabla u_n)\,h(u_n)\,V\,dx=\int_\Omega g(x,u,\nabla u)h(u)V\,dx.$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in (5.57) and to conclude that u satisfies (4.4).

6. Conclusion

In view of **Step 1** to **Step 8**, we can deduce that u is a renormalized solution of the problem(1.1). This completes the proof of Theorem 4.1.

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