Relative Uniform Convergence of $p$-absolutely Summable Sequence of Functions

Diksha Debbarma, Kshetrimayum Renubebeta Devi, Binod Chandra Tripathy

ABSTRACT: In this article, we introduce the notion of relative uniform convergence of $p$-absolutely summable sequence of functions. We have investigated its different properties such as solidness, symmetric, convergence-free, monotonic, etc. We have established some inclusion relations involving this sequence space.

Key Words: Relative uniform convergent, scale function, completeness, convergence free, solidness, monotone, symmetric.

Contents

1 Introduction 1
2 Preliminaries 2
3 Main results 3
4 Acknowledgment 6
5 Declarations 6

1. Introduction

Throughout $\omega$, $\ell_\infty$, c, $c_0$ and $\ell_p$ denote the classes of all, bounded, convergent, null, and $p$-absolutely summable sequences of complex numbers. The class of sequences $\ell_p$ was further studied by Mikail et al. ([6], [7]), Mikail et al. [8], Muhammed et al. [10], Tripathy [12], Tripathy and Borgohain [13], and many other researchers.

Moore [9] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. The aim of this paper is to study this type of convergence of a sequence of functions related to $p$-absolutely summable sequence of functions.

Chittenden [1] formulated the definition of the notion of relative uniform convergence of the sequence of functions as follows.

Definition 1.1. A sequence of function $(f_n(x))$, defined on a compact domain $D$, converges relatively uniformly to a limit function $f(x)$ such that for every small positive number $\varepsilon$, there is an integer $n_\varepsilon$ such that for every $n \geq n_\varepsilon$ the inequality

$$|f(x) - f_n(x)| \leq \varepsilon |\sigma(x)|,$$

holds uniformly in $x$ on the interval $D$. The function $\sigma(x)$ of the definition above is called a scale function.

The notion was further studied by many others researchers like Demirci et al. [2], Demirci and Orhan [3], Sahin and Dirik [11], Devi and Tripathy ([4],[5]) and others.

Example 1.2. Consider, $[0, 1]$ be a compact domain and consider the sequence of functions $(f_n(x))$, $f_n : [0, 1] \to \mathbb{R}$, for $n \in \mathbb{N}$, and $\sigma : [0, 1] \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{nx}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

2010 Mathematics Subject Classification: 40A05, 46B45, 46A45.
Submitted March 29, 2022. Published November 19, 2022
and
\[ \sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & x = 0. \end{cases} \]

Then, for every \( \varepsilon > 0 \), we have
\[
\left| \frac{1}{nx} - 0 \right| < \varepsilon \left| \frac{1}{x} \right|
\]
\[
\Rightarrow \left| \frac{1}{nx} \right| < \varepsilon \left| \frac{1}{x} \right|
\]
\[
\Rightarrow \frac{1}{n} < \varepsilon, \text{ for all } n > \frac{1}{x}
\]
Choosing a positive integer \( k > \frac{1}{x} \), we have
\[
|f_n(x) - f(x)| < \varepsilon, \text{ for all } n > k, x \in [0,1].
\]
This implies, \((f_n(x))\) is relatively uniformly convergent w.r.t. the scale function \( \sigma(x) \).

Chittenden [1] studied different properties of relative uniform convergence of a sequence of real-valued functions.

**Definition 1.3.** A sequence \((f_n(x))\) of the single real valued function \(f_n\) of variable \(x\) defined on a compact domain \(D \subseteq \mathbb{R}\) to a limit \(f(x)\) is said to be bounded if there exists a real constant \(M > 0\) such that
\[
\sum_{n=1}^{\infty} |f_n(x)| \leq M|\sigma(x)|,
\]
holds for every \(x \in D\). Then, the function \(\sigma\) of the definition above is called the scale function.

**2. Preliminaries**

In this section we procure all relevant definitions, discuss examples and preliminary discussions on the basics of this article.

**Definition 2.1.** A sequence of function \((f_n(x))\) defined on a compact domain \(D\) converges pointwise to a pointwise limit function \(f(x)\) such that for each \(x \in D\) and to each \(\varepsilon > 0\) there corresponds an integer \(m\) such that \(\forall n \geq m\)
\[
|f_n(x) - f(x)| < \varepsilon.
\]

**Definition 2.2.** A sequence of function \((f_n(x))\) defined on a compact domain \(D\) is uniformly convergent, if there exist a real valued function \(f(x)\), with domain \(D\) real-valued \(\varepsilon > 0\) there corresponds an integers \(m\) such that \(\forall n \geq m\),
\[
|f_n(x) - f(x)| < \varepsilon.
\]
Also in this case the function \(f(x)\) is the uniform limit of the sequence of function \((f_n(x))\).

**Definition 2.3.** Let \(K = \{k_1 < k_2 < k_3 < \ldots < k_n, \ldots\} \subseteq \mathbb{N}\) and \((x_n) \in \omega\). Then the \(K\)-step space of the sequence space \(E\) is defined by
\[
\lambda_K^E = \{(x_{k_n}) \in \omega : (x_n) \in E\}.
\]

**Definition 2.4.** A canonical pre-image \((y_n)\) of a sequence \((x_n) \in E\), where \(K\)-step space \(\lambda_K^E\) is considered, is defined by
\[
y_n = \begin{cases} x_n, n \in K; \\ 0, \text{ otherwise.} \end{cases}
\]

**Definition 2.5.** A sequence space \(E\) is said to be monotone if it contains all canonical pre-images of all its step spaces.
Definition 2.6. A subset $E \subset \omega$ is said to be solid if $(x_n) \in E \implies (y_n) \in E$, for all sequences $(y_n)$ such that, $|y_n| \leq |x_n|$, for all $n \in \mathbb{N}$.

Remark 2.7. A subset $E \subset \omega$ is solid implying it is monotone but not necessarily conversely in general.

Definition 2.8. A subset $E \subset \omega$ is said to be convergence free, if $(x_n) \in E$ then, $x_n = 0 \Rightarrow y_n = 0$ at a point, and if $(x_n) \neq 0 \Rightarrow (y_n)$ can be any thing, then $(y_n) \in E$.

Definition 2.9. A subset $E$ of $\omega$ is said to be symmetric if $(x_n) \in E$ implies $(x_{\pi(n)}) \in E$, where $\pi$ is a permutation of $\mathbb{N}$.

Definition 2.10. Let $E$ be a sequence space. Then $E$ is said to be a sequence algebra if there is defined a product $*$ on $E$ such that, $x_n, y_n \in E \Rightarrow x_n * y_n \in E$.

3. Main results

In this section, first, we introduce the main definition of this article. Then establish the results of the article.

Definition 3.1. Let $p > 0$ be a real number. Then a sequence of real-valued functions $(f_n(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ is said to be relatively uniformly $p$-absolutely summable if there exists a real constant $M > 0$ such that

$$\sum_{n=1}^{\infty} |f_n(x)|^p \leq M|\sigma(x)|,$$

uniformly on $D$, where $\sigma(x)$ is the scale function.

The class of all relatively uniformly $p$-absolutely summable sequences of real-valued functions is denoted by $\ell_p(\text{ru})$.

Example 3.2. Let us consider a sequence of functions $(f_n(x))$, $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} \frac{1}{n^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, it can be verified that $(f_n(x))$ does not converge relatively uniformly but converges uniformly to zero function with respect to a scale function $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} \frac{x}{x^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Definition 3.3. A sequence of function $(f_n(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ is said to be a null relative uniformly convergence sequence of function if it converges to zero.

The class of all null relative uniformly convergence sequence of function is denoted by, $c_0(\text{ru})$, and it is defined by,

$$c_0(\text{ru}) = \left\{ (f_n) \in \omega : \lim_{n \to \infty} \frac{f_n(x)}{\sigma(x)} = 0 \right\}$$

where, $\sigma(x)$ is the scale function which is defined on $D$.

Theorem 3.4. The space $\ell_p(\text{ru})$ is a linear space.

The proof of this theorem can be established following the linearity of the class $p$-absolutely summable sequences of real or complex terms.

Theorem 3.5. The space $\ell_p(\text{ru})$ is a normed space for $(1 \leq p < \infty)$ with respect to the norm $\{\|\cdot\|_p, D, \ell_p(\text{ru})\}$ defined by

$$\|f\|_p = \sup_{x \in D} \left[ \sum_{n=1}^{\infty} \left( \frac{|f_n(x)|^p}{|\sigma(x)|^p} \right) \right]^{\frac{1}{p}}. \quad (3.1)$$
**Proof.** Consider a sequence \((f_n)\) in \(\ell_p(\mathbb{R})\), for \(1 \leq p < \infty\). Then, it can be verified that

1. \(\|f\|_p = 0\) if and only if \(f = 0\).
2. \(\|\lambda f\|_p = \sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|\lambda f_n(x)|^p}{|f_n(x)|} \right]^{\frac{1}{p}} \leq \|f\|_p\).
3. \(\|f + g\|_p \leq \|f\|_p + \|g\|_p\).

Thus, the space \(\ell_p(\mathbb{R})\) is normed by (3.1).

\[\square\]

**Lemma 3.1.** \(c_0(\mathbb{R})\) is a Banach space.

It can be established following the technique that \(c_0\) is a Banach space.

**Theorem 3.6.** The sequence of functions \(\ell_p(\mathbb{R})\) is a Banach space w.r.t. the norm defined by Eq. (3.1).

**Proof.** Let \((f^i(x))\) be a Cauchy sequence in \(\ell_p(\mathbb{R})\). Then, for a given \(\varepsilon > 0\) there exists a positive integer \(n_0 \in \mathbb{N}\) such that for all \(x \in D\),

\[\|f^i(x) - f^j(x)\| < \varepsilon, \text{ for all } i, j \geq n_0,\]

where, \(f^i = (f^i_n) \in \ell_p(\mathbb{R})\) and \(f^j = (f^j_n) \in \ell_p(\mathbb{R})\).

Then,

\[\sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|f^i_n(x) - f^j_n(x)|^p}{|\sigma(x)|} \right]^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0.\]

\[\Rightarrow \sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|f^i_n(x) - f^j_n(x)|^p}{|\sigma(x)|} \right] < \varepsilon^p, \text{ for all } i, j \geq n_0.\]

\[\Rightarrow \sup_{x \in D} \frac{|f^i_n(x) - f^j_n(x)|}{|\sigma(x)|} < \varepsilon, \text{ for all } i, j \geq n_0.\]

\(\Rightarrow (f^i_n(x))\) is a Cauchy sequence in \(\mathbb{R}\) or \(\mathbb{C}\).

Therefore, \(f^i_n \to f_n \implies \frac{|f^i_n - f_n|}{|\sigma(x)|} \to 0\) as \(n \to \infty\). (Since, \(\mathbb{R}\) or \(\mathbb{C}\) both are complete.)

Let, \(f = (f_1, f_2, \ldots, f_n\ldots)\).

Now,

\[\|f_n - f\| = \sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|} \right]^{\frac{1}{p}} \to 0, \text{ as } n \to \infty.\]

Finally we will prove that, \(f \in \ell_p(\mathbb{R})\).

Since, \(f^i_n \to f_n\) then for a given \(\varepsilon > 0\), there exist \(n_0 \in \mathbb{N}\) such that,

\[\sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|} \right]^{\frac{1}{p}} < \varepsilon.\]

\[\sup_{x \in D} \left[ \sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|} \right] < \varepsilon^p < r < \infty.\]
Hence, \( f_n^i - f_n \in \ell_p(\rho u) \implies f_n - f_n^i \in \ell_p(\rho u) \). (Since, \( \ell_p(\rho u) \) is a linear space).

Now, \( f_n = (f_n - f_n^i) + f_n^i \).

Since, \( (f_n - f_n^i) \in \ell_p(\rho u) \) and \( (f_n^i) \in \ell_p(\rho u) \).

Hence, \( f_n \in \ell_p(\rho u) \).

\[ \text{Result 3.7. The space } \ell_p(\rho u) \text{ is not convergence free.} \]

For the above Result 1 consider the following example.

\[ \text{Example 3.8. Consider a sequence of function } (f_n(x)), f_n(x) : [0, 1] \to \mathbb{R} \text{ defined by,} \]

\[ f_n(x) = \begin{cases} \frac{x^n}{n^\rho}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases} \]

For \( x \neq 0 \),
\[ \sum_{n=1}^{\infty} |\frac{x^n}{n^\rho}|^p \to 0, \text{ as } n \to \infty. \]

For \( x = 0 \), we have \( \sum_{n=1}^{\infty} |f_n(x)|^p = 0. \)

Now, consider another sequence of function \( (g_n(x)) \) defined by

\[ g_n(x) = \begin{cases} x, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases} \]

For \( x \neq 0 \), \( \sum_{n=1}^{\infty} |x|^p \to \infty \text{ as } n \to \infty. \)

Therefore, \( (g_n(x)) \notin \ell_p(\rho u) \).

\[ \text{Theorem 3.9. The class of sequences } \ell_p(\rho u) \text{ is solid.} \]

\[ \text{Proof. Let } (f_n) \in \ell_p(\rho u). \text{ Consider the sequence of scalars } (\alpha_n) \text{ with } |\alpha_n| \leq 1, \text{ for all } n \in \mathbb{N}. \]

Thus the result follows from the following inequality

\[ \sum_{n=1}^{\infty} |\alpha_n f_n(x)|^p \leq \sum_{n=1}^{\infty} |f_n(x)|^p \]

where, \( \sigma(x) \) is the scale function and \( M > 0 \).

Now, consider a sequence of scalars \( (\alpha_n) \) such that \( |\alpha_n| \leq 1. \)

\[ \implies ||(\alpha_n f_n(x))||_p = \sup_{x \in D} \left[ \sum_{n=1}^{\infty} |\alpha_n f_n(x)|^p / |\sigma(x)|^p \right]^{1/p} \]

\[ \implies (\alpha_n f_n) \in \ell_p(\rho u). \text{ Hence, } \ell_p(\rho u) \text{ is solid.} \]

In view of the Theorem 3.9 and Remark 2.7, we state the following result without proof. \( \square \)

\[ \text{Result 3.10. The sequence space } \ell_p(\rho u) \text{ is monotone.} \]

We have stated the following theorem without proof since it can be easily established.

\[ \text{Theorem 3.11. The sequence space } \ell_p(\rho u) \text{ is sequence algebra.} \]

\[ \text{Theorem 3.12. Let } 1 < p < q < \infty, \text{ then the inclusion relation } \ell_p(\rho u) \subset \ell_q(\rho u) \text{ holds and is strict.} \]

\[ \text{Proof. Let us consider the sequence of functions } (f_n) \in \ell_p(\rho u). \]

Then, \( \sum_{n=1}^{\infty} |f_n(x)|^p < M|\sigma(x)|, \) where \( \sigma(x) \) is a scale function and \( M > 0 \) is a constant. For \( p < q, \) we have,

\[ \sum_{n=1}^{\infty} |f_n(x)|^q \leq \sum_{n=1}^{\infty} |f_n(x)|^p. \]
Hence, \((f_n) \in \ell_q(\text{ru})\). Therefore we get, \(\ell_p(\text{ru}) \subset \ell_q(\text{ru})\).

The inclusion is strict following the example below.

**Example 3.13.** Consider the sequence of functions \((f_n(x))\), \(f_n : [1,2] \rightarrow \mathbb{R}\) defined by

\[
f_n(x) = \frac{1}{x^2}, \text{ for all } x \in [1,2].
\]

It can be verified that, \((f_n(x)) \in \ell_2(\text{ru})\) w.r.t. the scale function \(\sigma(x) = 1\), for all \(x \in [1,2]\), but not in \(\ell_1(\text{ru})\).

Hence, the inclusion is strict. \(\square\)

**Theorem 3.14.** The space \(\ell_p(\text{ru})\) is symmetric.

**Proof.** Let \((f_n) \in \ell_p(\text{ru})\) and \(S = \{(f_{\pi(n)}(x)) : \pi(n) \text{ is a permutation on } \mathbb{N}\}\).
Here, \((f_{\pi(n)}(x))\) is the rearrangement of \((f_n(x))\). It is known that a series is absolutely convergent implying its every rearrangement series is also convergent.

Therefore, \((f_{\pi(n)}) \in \ell_p(\text{ru})\).

Hence, the sequence space \(\ell_p(\text{ru})\) is symmetric. \(\square\)

4. Acknowledgment

The authors thank the reviewers for the comments which helped in improving the presentation of the article.

5. Declarations

**Funding.** Not Applicable.

**Conflicts of interest/Competing interests (include appropriate disclosures).** We declare that the article is free from Conflicts of interest and Competing interests.

**Availability of data and material (data transparency).** Not Applicable.

**Code availability (software application or custom code).** Not Applicable.

**Authors’ contributions.** Both the authors have equal contribution in the preparation of this article.

**References**


Diksha Debbarma,
Department of Mathematics,
Tripura University, Agartala-799022,
India.
E-mail address: dikshadebbarma9495@gmail.com

and

Kshetrimayum Renubeta Devi,
Department of Mathematics,
St. Joseph’s College (Autonomous),
Kohima-797001, Nagaland,
India.
E-mail address: renu.ksh11@gmail.com

and

Binod Chandra Tripathy,
Department of Mathematics,
Tripura University, Agartala-799022,
India.
E-mail address: binodtripathy@tripurauniv.in, tripathybc@gmail.com