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Relative Uniform Convergence of p-absolutely Summable Sequence of Functions

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ABSTRACT: In this article, we introduce the notion of relative uniform convergence of p-absolutely summable sequence of functions. We have investigated its different properties such as solidness, symmetric, convergence-free, monotonic, etc. We have established some inclusion relations involving this sequence space.

Key Words: Relative uniform convergent, scale function, completeness, convergence free, solidness, monotone, symmetric.

Contents

1. Introduction

Throughout ω, ℓ_∞, c, c_0 and ℓ_p denote the classes of all, bounded, convergent, null, and p-absolutely summable sequences of complex numbers. The class of sequences ℓ_p was further studied by Mikail et al. ([6], [7]), Mikail et al. [8], Muhammed et al. [10], Tripathy [12], Tripathy and Borgohain [13], and many other researchers.
Moore [9] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. The aim of this paper is to study this type of convergence of a sequence of functions related to p-absolutely summable sequence of functions.
Chittenden [1] formulated the definition of the notion of relative uniform convergence of the sequence of functions as follows.

Definition 1.1. *A sequence of function $(f_n(x))$, defined on a compact domain D, converges relatively uniformly to a limit function $f(x)$ such that for every small positive number ε, there is an integer n_ε such that for every $n \geq n_\varepsilon$ the inequality*

$$|f(x) - f_n(x)| \leq \varepsilon|\sigma(x)|, \tag{1.1}$$

holds uniformly in x on the interval D. The function $\sigma(x)$ of the definition above is called a scale function.

The notion was further studied by many others researchers like Demirci et al. [2], Demirci and Orhan [3], Sahin and Dirik [11], Devi and Tripathy ([4],[5]) and others.

Example 1.2. *Consider, $[0,1]$ be a compact domain and consider the sequence of functions $(f_n(x)), f_n : [0,1] \to \mathbb{R}$, for $n \in \mathbb{N}$, and $\sigma : [0,1] \to \mathbb{R}$ be defined by*

$$f_n(x) = \begin{cases} \frac{1}{nx}, & \text{for } x \neq 0; \\ 0, & x = 0. \end{cases}$$

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and

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & x = 0. \end{cases}$$

Then, for every $\varepsilon > 0$, we have

$$\left|\frac{1}{nx} - 0\right| < \varepsilon\left|\frac{1}{x}\right|$$

$\Rightarrow \left|\frac{1}{nx}\right| < \varepsilon\left|\frac{1}{x}\right|$
$\Rightarrow \frac{1}{n} < \varepsilon$, for all $n > \frac{1}{\varepsilon}$
Choosing a positive integer $k > \frac{1}{\varepsilon}$, we have

$$|f_n(x) - f(x)| < \varepsilon, \text{ for all } n > k, x \in [0,1].$$

This implies, $(f_n(x))$ is relatively uniformly convergent w.r.t. the scale function $\sigma(x)$.

Chittenden [1] studied different properties of relative uniform convergence of a sequence of real-valued functions.

Definition 1.3. *A sequence $(f_n(x))$ of the single real valued function f_n of variable x defined on a compact domain $D \subseteq \mathbb{R}$ to a limit $f(x)$ is said to be bounded if there exists a real constant $M > 0$ such that*

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M|\sigma(x)|,$$

holds for every $x \in D$. Then, the function σ of the definition above is called the scale function.

2. Preliminaries

In this section we procure all relevant definitions, discuss examples and preliminary discussions on the basics of this article.

Definition 2.1. *A sequence of function $(f_n(x))$ defined on a compact domain D converges pointwise to a pointwise limit function $f(x)$ such that for each $x \in D$ and to each $\varepsilon > 0$ there corresponds an integer m such that $\forall n \geq m$*

$$|f_n(x) - f(x)| < \varepsilon.$$

Definition 2.2. *A sequence of function $(f_n(x))$ defined on a compact domain D is uniformly convergent, if there exist a real valued function $f(x)$, with domain D real-valued $\varepsilon > 0$ there corresponds an integers m such that $\forall n \geq m$,*

$$|f_n(x) - f(x)| < \varepsilon.$$

Also in this case the function $f(x)$ is the uniform limit of the sequence of function $(f_n(x))$.

Definition 2.3. *Let $K = \{k_1 < k_2 < k_3 < ... < k_n ...\} \subset \mathbb{N}$ and $(x_n) \in \omega$. Then the K-step space of the sequence space E is defined by*

$$\lambda_K^E = \{(x_{k_i}) \in w : (x_n) \in E\}.$$

Definition 2.4. *A canonical pre-image (y_n) of a sequence $(x_n) \in E$, where K-step space λ_K^E is considered, is defined by*

$$y_n = \begin{cases} x_n, n \in K; \\ 0, \ \text{otherwise}. \end{cases}$$

Definition 2.5. *A sequence space E is said to be monotone if it contains all canonical pre-images of all its step spaces.*

Definition 2.6. *A subset $E \subset \omega$ is said to be solid if $(x_n) \in E \implies (y_n) \in E$, for all sequences (y_n) such that, $|y_n| \leq |x_n|$, for all $n \in \mathbb{N}$.*

Remark 2.7. *A subset $E \subset \omega$ is solid implying it is monotone but not necessarily conversely in general.*

Definition 2.8. *A subset $E \subset \omega$ is said to be convergence free, if $(x_n) \in E$ then, $x_n = 0 \Rightarrow y_n = 0$ at a point, and if $(x_n) \neq 0 \Rightarrow (y_n)$ can be any thing, then $(y_n) \in E$.*

Definition 2.9. *A subset E of ω is said to be symmetric if $(x_n) \in E$ implies $(x_{\pi(n)}) \in E$, where π is a permutation of $\mathbb{N}$.*

Definition 2.10. *Let E be a sequence space. Then E is said to be a sequence algebra if there is defined a product $*$ on E such that $x_n, y_n \in E \Rightarrow x_n * y_n \in E$.*

3. Main results

In this section, first, we introduce the main definition of this article. Then establish the results of the article.

Definition 3.1. *Let $p > 0$ be a real number. Then a sequence of real-valued functions $(f_n(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ is said to be relatively uniformly p-absolutely summable if there exists a real constant $M > 0$ such that*

$$\sum_{n=1}^{\infty} |f_n(x)|^p \leq M|\sigma(x)|,$$

uniformly on D, where $\sigma(x)$ is the scale function.
The class of all relatively uniformly p-absolutely summable sequences of real-valued functions is denoted by $\ell_p(ru)$.

Example 3.2. *Let us consider a sequence of functions $(f_n(x))$, $f_n : [0,1] \to \mathbb{R}$ defined by*

$$f_n(x) = \begin{cases} \frac{1}{n^2 x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, it can be verified that $(f_n(x))$ does not converge relatively uniformly but converges uniformly to zero function with respect to a scale function $\sigma(x)$ defined by

$$\sigma(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Definition 3.3. *A sequence of function $(f_n(x))$ defined on a compact domain $D \subseteq \mathbb{R}$ is said to be a null relative uniformly convergence sequence of function if it converges to zero.*
The class of all null relative uniformly convergence sequence of function is denoted by, $c_0(ru)$, and it is defined by,

$$c_0(ru) = \left\{ (f_n) \in \omega : \lim_{n \to \infty} \frac{f_n(x)}{\sigma(x)} = 0 \right\}$$

where, $\sigma(x)$ is the scale function which is defined on D.

Theorem 3.4. *The space $\ell_p(ru)$ is a linear space.*

The proof of this theorem can be established following the linearity of the class p-absolutely summable sequences of real or complex terms.

Theorem 3.5. *The space $\ell_p(ru)$ is a normed space for $(1 \leq p < \infty)$ with respect to the norm $\{||.||_p, D, \ell_p(ru)\}$ defined by*

$$||f||_p = \begin{matrix} \sup \\ x \in D \end{matrix} \left[\sum_{n=1}^{\infty} \left[\frac{|f_n(x)|^p}{|\sigma(x)|} \right] \right]^{\frac{1}{p}}. \tag{3.1}$$

Proof. Consider a sequence (f_n) in $\ell_p(ru)$, for $1 \leq p < \infty$. Then, it can be verified that

(1) $||f||_p = 0$ if and only if $f = 0$.
(2) $||\lambda f||_p = \sup_{x\in D} \left[\sum_{n=1}^{\infty} \left[\frac{|\lambda f_n(x)|^p}{|\sigma(x)|}\right]\right]^{\frac{1}{p}} \leq \lambda ||f||_p$.
(3) $||f+g||_p \leq ||f||_p + ||g||_p$.

Thus, the space $\ell_p(ru)$ is normed by (3.1). □

Lemma 3.1. *$c_0(ru)$ is a Banach space.*

It can be established following the technique that c_0 is a Banach space.

Theorem 3.6. *The sequence of functions $\ell_p(ru)$ is a Banach space w.r.t. the norm defined by Eq. (3.1).*

Proof. Let $(f^i(x))$ be a Cauchy sequence in $\ell_p(ru)$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0 \in N$ such that for all $x \in D$,

$$||f^i(x) - f^j(x)|| < \varepsilon, \text{ for all } i, j \geq n_0,$$

where, $f^i = (f^i_n) \in \ell_p(ru)$ and $f^j = (f^j_n) \in \ell_p(ru)$.

Then,

$$\sup_{x \in D} \left[\sum_{n=1}^{\infty} \frac{|f^i_n(x) - f^j_n(x)|^p}{|\sigma(x)|}\right]^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0.$$

$$\Rightarrow \sup_{x \in D} \left[\sum_{n=1}^{\infty} \frac{|f^i_n(x) - f^j_n(x)|^p}{|\sigma(x)|}\right] < \varepsilon^p, \text{ for all } i, j \geq n_0.$$

$$\Rightarrow \sup_{x \in D} \frac{|f^i_n(x) - f^j_n(x)|^p}{|\sigma(x)|} < \varepsilon^p, \text{ for all } i, j \geq n_0.$$

$$\Rightarrow \sup_{x \in D} \frac{|f^i_n(x) - f^j_n(x)|}{|\sigma(x)|} < \varepsilon, \text{ for all } i, j \geq n_0.$$

$\Rightarrow (f^i_n(x))$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$.

Therefore, $f^i_n \to f_n \implies \frac{|f^i_n - f_n|}{|\sigma(x)|} \to 0$ as $n \to \infty$. $\big($Since, $\mathbb{R}$ or $\mathbb{C}$ both are complete.$\big)$

Let, $f = (f_1, f_2, ...f_n...)$.

Now,

$$||f_n - f|| = \sup_{x \in D} \left[\sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|}\right]^{\frac{1}{p}} \to 0, \text{ as } n \to \infty.$$

Finally we will prove that, $f \in \ell_p(ru)$.
Since, $f^i_n \to f_n$ then for a given $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that,

$$\sup_{x \in D} \left[\sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|}\right]^{\frac{1}{p}} < \varepsilon.$$

$$\sup_{x \in D} \left[\sum_{n=1}^{\infty} \frac{|f^i_n(x) - f_n(x)|^p}{|\sigma(x)|}\right] < \varepsilon^p < r < \infty.$$

Hence, $f_n^i - f_n \in \ell_p(ru) \implies f_n - f_n^i \in \ell_p(ru)$. (Since, $\ell_p(ru)$ is a linear space).
Now, $f_n = (f_n - f_n^i) + f_n^i$.
Since, $(f_n - f_n^i) \in \ell_p(ru)$ and $(f_n^i) \in \ell_p(ru)$.
Hence, $f_n \in \ell_p(ru)$. □

Result 3.7. *The space $\ell_p(ru)$ is not convergence free.*

For the above Result 1 consider the following example.

Example 3.8. *Consider a sequence of function $(f_n(x))$, $f_n(x) : [0,1] \to \mathbb{R}$ defined by,*

$$f_n(x) = \begin{cases} \frac{x}{n^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

For $x \neq 0$,
$\sum\limits_{n=1}^{\infty} |\frac{x}{n^2}|^p \to 0$, *as $n \to \infty$.*
For $x = 0$, we have $\sum\limits_{n=1}^{\infty} |f_n(x)|^p = 0$.
Now, consider another sequence of function $(g_n(x))$ defined by

$$g_n(x) = \begin{cases} x, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

For $x \neq 0$, $\sum\limits_{n=1}^{\infty} |x|^p \to \infty$ as, $n \to \infty$.
Therefore, $(g_n(x)) \notin \ell_p(ru)$.

Theorem 3.9. *The class of sequences $\ell_p(ru)$ is solid.*

Proof. Let $(f_n) \in \ell_p(ru)$. Consider the sequence of scalars (α_n) with $|\alpha_n| \leq 1$, for all $n \in \mathbb{N}$.
Thus the result follows from the following inequality

$$\sum_{n=1}^{\infty} |\alpha_n f_n(x)|^p \leq \sum_{n=1}^{\infty} |f_n(x)|^p$$

where, $\sigma(x)$ is the scale function and $M > 0$.
Now, consider a sequence of scalars (α_n) such that $|\alpha_n| \leq 1$.
$\implies ||(\alpha_n f_n(x))||_p = \sup\limits_{x \in D} \left[\sum\limits_{n=1}^{\infty} \frac{|\alpha_n f_n(x)|^p}{|\sigma(x)|} \right]^{\frac{1}{p}}$
$\implies (\alpha_n f_n) \in \ell_p(ru)$. Hence, $\ell_p(ru)$ is solid.
In view of the Theorem 3.9 and Remark 2.7, we state the following result without proof □

Result 3.10. *The sequence space $\ell_p(ru)$ is monotone.*

We have stated the following theorem without proof since it can be easily established.

Theorem 3.11. *The sequence space $\ell_p(ru)$ is sequence algebra.*

Theorem 3.12. *Let $1 < p < q < \infty$, then the inclusion relation $\ell_p(ru) \subset \ell_q(ru)$ holds and is strict.*

Proof. Let us consider the sequence of functions $(f_n) \in \ell_p(ru)$.
Then, $\sum\limits_{n=1}^{\infty} |f_n(x)|^p < M|\sigma(x)|$, where $\sigma(x)$ is a scale function and $M > 0$ is a constant. For $p < q$, we have,

$$\sum_{n=1}^{\infty} |f_n(x)|^q \leq \sum_{n=1}^{\infty} |f_n(x)|^p.$$

Hence, $(f_n) \in \ell_q(ru)$. Therefore we get, $\ell_p(ru) \subset \ell_q(ru)$.
The inclusion is strict following the example below.

Example 3.13. *Consider the sequence of functions $(f_n(x)), f_n : [1, 2] \to \mathbb{R}$ defined by*

$$f_n(x) = \frac{1}{x^2}, \text{ for all } x \in [1, 2].$$

It can be verified that, $(f_n(x)) \in \ell_2(ru)$ w.r.t. the scale function $\sigma(x) = 1$, for all $x \in [1, 2]$, but not in $\ell_1(ru)$.

Hence, the inclusion is strict. □

Theorem 3.14. *The space $\ell_p(ru)$ is symmetric.*

Proof. Let $(f_n) \in \ell_p(ru)$ and $S = \{(f_{\pi(n)}(x)) : \pi(n) \text{ is a permutation on } \mathbb{N}\}$.
Here, $(f_{\pi(n)}(x))$ is the rearrangement of $(f_n(x))$. It is known that a series is absolutely convergent implying its every rearrangement series is also convergent.
Therefore, $(f_{\pi(n)}) \in \ell_p(ru)$.
Hence, the sequence space $\ell_p(ru)$ is symmetric. □

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