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Some Innovations for Hardy-type Inequalities on Time Scales

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ABSTRACT: In this article, we introduce some innovations for the validity of a generalized two-weighted and variable exponent Hardy-type inequality on time scales via diamond$-\alpha$ integral. The corresponding continuous case is given when $T = R$. At the end of our study, some applications are added that prove the validity of our main result for some continuous results that are well-known in the literature.

Key Words: Hardy-type inequality, time scales, diamond$-\alpha$ integral.

Contents

1. Introduction

For the last fifty years, operators, dynamic integral equations have gained a very important place in harmonic analysis and time scales. In [1] the discrete Hardy inequality was proved by G.H. Hardy.

$$\sum_{m=1}^{\infty}\left(\frac{b_1+b_2+...+b_m}{m}\right)^c < \left(\frac{c^2}{c-1}\right)^c \sum_{m=1}^{\infty} b_m^c$$

where $c > 1$,$c \in R$ and (b_m) is a sequence of non-negative real numbers.
In [2] the classical Hardy inequality was proved by G.H. Hardy.

$$\int_0^{\infty}\left(\frac{1}{s}\int_0^s g(\tau)d\tau\right)^c ds \leq \left(\frac{c^2}{c-1}\right)^c \int_0^{\infty} g^c(s)ds$$

where g, g^c are an integrable functions over non negative interval for $c > 1$. Because of the important role of this integral inequality in mathematical analysis, this inequality has been studied by many authors. We recommend that the reader refer to the papers and books [3-11].
In [12] Muckenhoupt proved that the inequality

$$\left[\int_0^{\infty} \mid v(t)\int_0^t g(s)ds \mid^p dt\right]^{1/p}$$

holds if and only if the following conditions hold:

$$D := \sup_{x>0}\left(\int_x^{\infty} \mid v(t) \mid^p dt\right)^{1/p}\left(\int_0^x \mid u(t) \mid^{-p'} dt\right)^{1/p'} < \infty,$$

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$D \leq C \leq D(p)^{1/p}(p')^{1/p'}$, where $1/p + 1/p' = 1$ for $1 \leq p \leq \infty$.
In [13] Gurka proved that the inequality

$$\left(\int_a^b \mid v(\tau) \mid^q w(\tau)d\tau\right)^{1/q} \leq C_M \left(\int_a^b \mid v'(\tau) \mid^p u(\tau)d\tau\right)^{1/p} \tag{1.1}$$

holds for every $v \in AC(a,b)$ such that $v(a) = 0$ if and only if

$$K_M := \sup_{a<\tau<b} \left(\int_\tau^b w(x)dx\right)^{1/q} \left(\int_a^\tau u^{1-p'}(x)dx\right)^{1/p'} < \infty,$$

and for the best possible constant C_M, the estimate $K_M \leq C_M \leq K_M(p)^{1/q}(p')^{1/q'}$ was satisfied.
In [7, Theorem 2.57] the classical Wirtinger inequality is given by

$$\int_x^y v^2(\tau)d\tau \leq \int_x^y (v'(\tau))^2 d\tau \tag{1.2}$$

for any $v \in C^1([x,y])$ with $v(x) = v(y) = 0$. We recommend that the reader refer to the papers [13-15].
In [16] Hinton and Lewis proved inequality (1.2) by using the Schwarz inequality that

$$\frac{1}{4}\int_a^b v^2(\tau) \mid M'(\tau) \mid d\tau \leq \int_a^b (v'(\tau))^2 \frac{M^2(\tau)}{\mid M'(\tau) \mid} d\tau \tag{1.3}$$

for any $M \in C^1([a,b])$ with $M'(\tau) \neq 0, v \in C^2([a,b])$, and $v(a) = v(b) = 0$.
In [17] Stepanov proved that for $1 < p \leq p' < \infty$ and $1 \leq c$,

$$\left(\int_0^\infty \mid u(\tau) \mid^{p'} v(\tau)d\tau\right)^{1/p'} \leq C \left(\int_0^\infty \mid u^{(c)}(\tau) \mid^p w(\tau)d\tau\right)\right)^{1/p} \tag{1.4}$$

holds for all u, u^{c-1} are integrable on $[0,\infty)$ and satisfies the condition $u(0) = u'(0)) = ... = u^{(c-1)}(0) = 0$ if and only if the following conditions hold:

$$\sup_{0<\tau<\infty} \left(\int_\tau^\infty (x-\tau)^{(c-1)p'} v(x)dx\right)^{1/p'} \left(\int_0^\tau w^{1-q}(x)dx\right)^{1/q} < \infty,$$

and

$$\sup_{0<\tau<\infty} \left(\int_\tau^\infty v(x)dx\right)^{1/p'} \left(\int_0^\tau w^{1-q}(x)(x-\tau)^{(c-1)q}dx\right)^{1/q} < \infty.$$

In [18] Kufner et al. interested inequality (1.4) when $c = t + n$, and considered the inequality

$$\left(\int_0^\infty \mid w(\tau) \mid^q u(\tau)d\tau\right)^{1/q} \leq C \left(\int_0^\infty \mid w^{(t+n)}(\tau) \mid^p v(\tau)d\tau\right)\right)^{1/p} \tag{1.5}$$

for any finite constant C and $t, n \geq 1$ under the following conditions:

$$w(0) = w'(0)) = ... = w^{(t-1)}(0) = 0$$

$$w^{(t)}(\infty) = w^{(t+1)}(\infty) = ... = w^{(t+n-1)}(\infty) = 0$$

and they proved that inequality (1.5) holds if and only if

$$K_1 := \sup_{0<\tau<\infty} \left(\int_\tau^\infty u(x)x^{(t-1)p'}dx\right)^{1/p'} \left(\int_0^\tau v^{1-q}(x)x^{nq}dx\right)^{1/q} < \infty,$$

and

$$K_2 := \sup_{0<\tau<\infty} \left(\int_0^\tau u(x) x^{tp'} dx \right)^{1/p'} \left(\int_\tau^\infty v^{1-q}(x) x^{(n-1)q} dx \right)^{1/q} < \infty.$$

For more details, we refer to the papers [17-21].
In recent years, the dynamic integral inequalities on time scales have been studied by many authors [22-34]. Now let's give some of these studies that will motivate us about the aim of this study. In [35] Saker et al. proved that the inequality

$$\left(\int_x^y u(\tau) \mid \int_x^{\sigma(\tau)} g(t)\Delta t \mid^{p'} \Delta t \right)^{1/p'} \leq C \left(\int_x^y g^p(\tau) v(\tau) \Delta\tau \right)^{1/p} \tag{1.6}$$

holds if and only if

$$K := \sup_{0<\tau<\infty} \left(\int_\tau^y u(t) dt \right)^{1/p'} \left(\int_x^{\sigma(\tau)} v^{1-q}(t) dt \right)^{1/q} < \infty,$$

where $K \leq C \leq k(p,p')K$, and $k(p,p')$ is defined by $k(p,p') = ((p'+q)/q)^{1/p'}((p'+q)/p')^{1/q}$ with $1 < p \leq p' < \infty$.

The rest of the paper is organized as follows: In Section 2, we present some preliminaries about time scales and variable exponent spaces which is the cornerstone of our main proof. In Section 3, we prove the main result of this paper which is some necessary and sufficient conditions for the accuracy of a generalization of the weighted and variable exponent Hardy-type inequality on time scales via diamond$-\alpha$ integral. At the end of our study, some applications are added that prove the accuracy of our main result for some continuous results that are well-known in the literature. In section 4, we give the conclusion.

2. Mathematical background and preliminaries

The development of the theory of time scales was initiated by Hilger [36] in 1988, since then many authors have studied the theory of certain dynamic inequalities and differential equations on time scales. A time scale T is a nonempty closed subset of R. Let $[x,y]$ be an arbitrary closed interval on time scale T. The time scale interval $[x,y]_T$ is denoted by $[x,y] \cap T$. We refer to the references [37-39] for more details.

Definition 2.1 *[37] The mappings $\sigma, \rho : T \longrightarrow T$ are defined by $\sigma(t) = \inf s \in T : s > t$, $\rho(t) = \sup s \in T : s > t$ for $t \in T$. $\sigma(t)$ is forward jump operator and $\rho(t)$ is backward jump operator, respectively. If $\sigma(t) > t$, then t is right-scattered and if $\sigma(t) = t$, then t is called right-dense. If $\rho(t) < t$, then t is left-scattered and if $\rho(t) = t$, then t is called left-dense.*

Definition 2.2 *[37] Let two mappings $\mu, v : T \longrightarrow R^+$ such that $\mu(t) = \sigma(t) - t, v(t) = t - \rho(t)$. The mappings $\mu(t)$ and $v(t)$ are called graininess mappings. If T has a left-scattered maximum m, then $T^k = T - m$. Otherwise $T^k = T$. In [37-39] T^k is defined as follows*

$$T^k = \begin{cases} T \ \ (\rho \sup T, \sup T), & \text{if} \quad \sup T < \infty \\ T, & \text{if} \quad \sup T = \infty \end{cases}$$

and

$$T_k = \begin{cases} T \ \ [\inf T, \sigma(\inf T)], & \mid \inf T \mid < \infty \\ T, & \inf T = -\infty \end{cases}$$

Assume that $h : T \longrightarrow R$ is a function and let t be right-dense.
i) If h is $\Delta-$differentiable at point $t(t \in T^k(t \neq \min T))$, then h is continuous at point t.
ii) If h is left continuous at point t and t is right-scattered, then h is $\Delta-$ differentiable at point t,

$$h^\Delta(t) = \frac{h^\sigma(t) - h(t)}{\mu(t)}$$

iii) If h is $\Delta-$ differentiable at point t and $\lim_{s\to t}\frac{h(t)-h(s)}{t-s}$, then

$$h^{\Delta}(t)=\lim_{s\to t}\frac{h(t)-h(s)}{t-s}.$$

iv) If h is $\Delta-$ differentiable at point t, then $h^{\sigma}(t)=h(t)+\mu(t)h^{\Delta}(t)$.

Remark 2.1 *[39] If $T=R$, then $h^{\Delta}(t)=h'(t)$, and if $T=Z$, then $h^{\Delta}(t)$ reduces to $\Delta h(t)$.*

Definition 2.3 *[38] If $H:T\to R$ is defined a $\Delta-$antiderivative of $h:T\to R$, then $H^{\Delta}=h(t)$ holds for all $t,s\in T$ and we define the $\Delta-$integral of h by*

$$\int_s^t h(\tau)\Delta\tau=H(t)-H(s).$$

Let us now give similar definitions for the ∇ (nabla) operator.

Definition 2.4 *[38] Let $h:T_k\to R$ is called $\nabla-$ differentiable for all $t\in T_k$. If $\varepsilon>0$, then there exists a neighborhood V of t such that*

$$\mid h(\rho(t))-h(s)-h^{\nabla}(t)(\rho(t)-s)\mid\leq\varepsilon\mid\rho(t)-s\mid$$

for all $s\in V$.

Definition 2.5 *[38] If $H:T\to R$ is called a $\nabla-$antiderivative of $h:T\to R$, then then we define*

$$\int_s^t h(\tau)\nabla\tau=H(t)-H(s).$$

for $s,t\in T$.

Let $f(t)$ be $\diamond_\alpha-$differentiable on T for all $\alpha,t\in T$. Then, we define $f^{\diamond_\alpha}(t)$ by

$$f^{\diamond_\alpha}(t)=\alpha f^{\Delta}(t)-(1-\alpha)f^{\nabla}(t)$$

for $0\leq\alpha\leq1$.

Proposition 2.1 *[38] If $f,h:T\to R$ are $\diamond_\alpha-$differentiable for all $\alpha,t\in T$, then*

i) $(f+h):T\to R$ is $\diamond_\alpha-$ differentiable for $t\in T$ with

$$(f+h)^{\diamond_\alpha}(t)=f^{\diamond_\alpha}(t)+h^{\diamond_\alpha}(t).$$

ii) Let $k\in R$, $kf:T\to R$ is $\diamond_\alpha-$differentiable for $\alpha,t\in T$ with

$$(kf)^{\diamond_\alpha}(t)=kf^{\diamond_\alpha}(t).$$

iii) $f,h:T\to R$ is $\diamond_\alpha-$ differentiable for all $\alpha,t\in T$ with

$$(fh)^{\diamond_\alpha}(t)=f^{\diamond_\alpha}(t)h(t)+\alpha f^{\sigma}(t)h^{\Delta}(t)+(1-\alpha)f^{\rho}(t)h^{\nabla}(t).$$

Definition 2.6 *[38] If $f:T\to R$ is $\diamond_\alpha-$integrable for all $\alpha,b,t\in T$, then*

$$\int_b^t f(\delta)\diamond_\alpha\delta=\int_b^t f(\delta)\Delta\delta+(1-\alpha)\int_b^t f(\delta)\nabla\delta$$

for $0\leq\alpha\leq1$.

Definition 2.7 *[40,41] If $f \in C_{rd}(T, R)$, $t \in T^k$ and $f : T \to R$ is $\diamond_\alpha-$integrable, then*

$$\int_t^{\sigma(t)} f(\tau) \diamond_\alpha \tau = \mu(t) f(t).$$

In this work, we will assume that the functions in the theorems are non-negative, $rd-$continuous, and integrable.

Theorem 2.1 *[42] (Fubini's theorem) Let A and B be two time scales with $0 \leq \infty \leq 1$. If $g : A \times B \to R$ is $\diamond_\alpha-$integrable function and we define the μ, θ functions*

$$\theta(s) = \int_B g(s,t) \diamond_\alpha (t), \qquad s \in A$$

$$\mu(s) = \int_A g(s,t) \diamond_\alpha (s), \qquad t \in B,$$

then μ, θ are $\diamond_\alpha-$ integrable on A, B respectively and

$$\int_A \diamond_\alpha s \int_B g(s,t) \diamond_\alpha t = \int_B \diamond_\alpha t \int_A g(s,t) \diamond_\alpha s \tag{2.1}$$

Lemma 2.1 *[43] If $k, t \geq 1$, $0 \leq \infty \leq 1$ and*

$$m_1(y, d) = \int_0^d (y-x)^{k-1} (d-x)^{t-1} \diamond_\alpha x \tag{2.2}$$

then

$$\frac{t d^t y^{k-1}}{k+t-1} \leq m_1(y,d) t \leq y^{k-1} d^t \tag{2.3}$$

where $0 < x \leq \sigma(x) < d < y$.

Lemma 2.2 *[43] If $k, t \geq 1$, $0 \leq \infty \leq 1$ and*

$$m_2(y, d) = \int_0^y (y-x)^{k-1} (d-x)^{t-1} \diamond_\alpha x \tag{2.4}$$

then

$$\frac{k y^k d^{t-1}}{k+t-1} \leq m_2(y,d) k \leq y^k d^{t-1} \tag{2.5}$$

where $0 < x \leq \sigma(x) < y < d$.

Now let's talk about the concept of variable exponent, which will help us to prove our main results and has an important place in harmonic analysis. The idea of variable exponent $L^{p(.)}$ space was popularized by Orlicz. Scientists in this field have extensively analyzed integral equations, integral operators, and inequalities in variable exponent Lebesgue space $L^{p(.)}$. Moreover, these works have been stimulated by problems of elasticity, fluid dynamics, electrorheological fluids, and calculus of variations $[44-54]$. Inspired by the relationships with non-standard growth and variation integrals associated with electrorheological fluids and liquids design. $[55-62]$.
The variable exponent Lebesgue space $L^{p(.)}(\Omega)$ denote space of measurable and integrable functions on Ω such that for $\lambda > 0$,

$$\int_\Omega \left| \frac{f(t)}{\lambda} \right|^{p(t)} dt < \infty$$

with norm

$$\|f\|_{p(t),\Omega} = \inf\{\lambda > 0 : \int_\Omega \left(\frac{|f(t)|}{\lambda} \right)^{p(t)} dt \leq 1\},$$

where $p(.) : \Omega \to [1, \infty)$ a measurable function.

3. Necessary and sufficient conditions for Hardy-type inequality

Now, we will state and prove our results, which demonstrate the accuracy of the generalized two-weighted and variable exponent Hardy-type inequality on time scales. Before coming to the proof of our results, we will use the following notations to help us

$$K_1 := \sup_{s\in[0,\infty)\cap T} \left(\int_s^\infty v(x)x^{(\theta-1)q} \diamond_\alpha x\right)^{1/q} \left(\int_0^{\sigma(s)} w^{1-p'}(x)x^{\varepsilon p'} \diamond_\alpha x\right)^{1/p'} < \infty, \tag{3.1}$$

$$K_2 := \sup_{s\in[0,\infty)\cap T} \left(\int_0^{\sigma(s)} v(x)x^{\theta q} \diamond_\alpha x\right)^{1/q} \left(\int_s^\infty w^{1-p'}(x)x^{(\varepsilon-1)p'} \diamond_\alpha x\right)^{1/p'} < \infty, \tag{3.2}$$

and

$$\begin{cases} u(0) = u^{\diamond_\alpha}(0) = ... = u^{\diamond_\alpha{}^{(\theta-1)}}(0) = 0 \\ u^{\diamond_\alpha{}^{(\theta)}}(\infty) = u^{\diamond_\alpha{}^{(\theta+1)}}(\infty) = ... = u^{\diamond_\alpha{}^{(\varphi-1)}}(\infty) = 0 \end{cases} \tag{3.3}$$

where $\varphi = \theta + \varepsilon$, θ and ϑ are nonnegative integers.

Theorem 3.1 *Let $1 < p \leq q < \infty$, $0 \leq \infty \leq 1$, $c > 1$, $u \in C_{rd}([0,\infty)\cap T, R^+)$, and δ, ϑ are $\diamond_\alpha-$integrable and positive $rd-$continuous functions on $[0,\infty)\cap T$.*

$$\left[\int_0^\infty \delta(s) \mid u(s) \mid^q \diamond_\alpha s\right]^{1/q} \leq \left(\frac{c}{c-1}\right)^c \left[\int_0^\infty \vartheta(s) \mid u^{\diamond_\alpha{}^{(\varphi)}}(s) \mid^p \diamond_\alpha s\right]^{1/p} \tag{3.4}$$

holds for all $u \in C_{rd}^{(\varphi)}([0,\infty)\cap T, R^+)$ if and only if $K_1 < \infty$ and $K_2 < \infty$.

Proof: We can write the (3.4) inequalities as follows:

$$\left[\int_0^\infty \delta(s) \mid (Bg)(s) \mid^q \diamond_\alpha s\right]^{1/q} \leq \left(\frac{c}{c-1}\right)^c \left[\int_0^\infty \vartheta(s) \mid g(s) \mid^p \diamond_\alpha s\right]^{1/p} \tag{3.5}$$

where $u = Bg$ and $g = u^{\diamond_\alpha{}^{(\varphi)}}$. Now, we will integer $\theta, \varepsilon \geq 1$ and take $(Bg)(s)$ as the form

$$(Bg)(s) = \frac{1}{(\theta-1)!(\varepsilon-1)!}\int_0^{\sigma(s)} (s-x)^{(\theta-1)} \left[\int_x^\infty (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t\right] \diamond_\alpha x. \tag{3.6}$$

Set $C_{\theta,\varepsilon} = (\theta-1)!(\varepsilon-1)!$, then we have

$$C_{\theta,\varepsilon}(Bg)(s) = \int_0^{\sigma(s)} (s-x)^{(\theta-1)} \left[\int_x^\infty (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t\right] \diamond_\alpha x$$

$$= \int_0^{\sigma(s)} (s-x)^{(\theta-1)} \left[\int_x^s (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t + \int_s^\infty (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t\right] \diamond_\alpha x$$

$$= \int_0^{\sigma(s)} (s-x)^{(\theta-1)} \left[\int_x^s (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t\right] \diamond_\alpha x + \int_0^s (s-x)^{(\theta-1)} \left[\int_s^\infty (t-x)^{(\varepsilon-1)} g(t) \diamond_\alpha t\right] \diamond_\alpha x.$$

By using Fubini's theorem (2.1), we obtain

$$C_{\theta,\varepsilon}(Bg)(s) = \int_0^{\sigma(s)} g(t) \left[\int_0^t (s-x)^{(\theta-1)}(t-x)^{(\varepsilon-1)} \diamond_\alpha x\right] \diamond_\alpha t$$

$$+ \int_s^\infty g(t) \left[\int_0^s (s-x)^{(\theta-1)}(t-x)^{(\varepsilon-1)} \diamond_\alpha x\right] \diamond_\alpha t,$$

then

$$C_{\theta,\varepsilon}(Bg)(s) = (I_1 g)(s) + (I_2 g)(s) \tag{3.7}$$

where

$$(I_1 g)(s) = \int_0^{\sigma(s)} d_1(s,t) g(t) \diamond_\alpha t \tag{3.8}$$

and

$$(I_2 g)(s) = \int_0^{\sigma(s)} d_2(s,t) g(t) \diamond_\alpha t. \tag{3.9}$$

Due to the inequality (2.3), the function in (3.8) is equivalent to the function $\int_0^{\sigma(s)} s^{(\theta-1)} t^{\varepsilon} g(t) \diamond_\alpha t$. Replacing $t^{\varepsilon} g(t)$ with $\overline{g}(t)$, $s^{(\theta-1)q(.)}\delta(s)$ with $\overline{\delta}(s)$, $s^{-\varepsilon p(.)}\vartheta(s)$ with $\overline{\vartheta}(s)$, and $s^{\varepsilon} g(s)$ with $\overline{g}(s)$, we obtain that

$$\left(\int_0^{\infty}\left(\int_0^{\sigma(s)} \overline{g}(t) \diamond_\alpha t\right)^q \overline{\delta}(s) \diamond_\alpha s\right)^{1/q} = \left(\int_0^{\infty} \delta(s)\left(\int_0^{\sigma(s)} s^{(\theta-1)} t^{\varepsilon} g(t) \diamond_\alpha t\right)^q \diamond_\alpha s\right)^{1/q}$$

$$= \left(\int_0^{\infty}\left(\int_0^{\sigma(s)} t^{\varepsilon} g(t) \diamond_\alpha t\right)^q s^{(\theta-1)q}\delta(s) \diamond_\alpha s\right)^{1/q}$$

$$\leq \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} g^p(s)\vartheta(s) \diamond_\alpha (s)\right)^{1/p} = \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} g^p(s) s^{\varepsilon p} s^{-\varepsilon p}\vartheta(s) \diamond_\alpha (s)\right)^{1/p}$$

$$= \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} (s^{\varepsilon} g(s))^p \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p} = \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} (\overline{g}(s))^p \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p}.$$

Then

$$\left[\int_0^{\infty}\left(\int_0^{\sigma(s)} \overline{g}(t) \diamond_\alpha t\right)^q \overline{\delta}(s) \diamond_\alpha s\right]^{1/q} \leq \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} (\overline{g}(s))^p \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p}. \tag{3.10}$$

Now, inequality (3.10) holds if and only if

$$K_1 = \sup_{s\in(0,\infty)} \left(\int_s^{\infty} \overline{\delta}(x) \diamond_\alpha x\right)^{1/q} \left(\int_0^{\sigma(s)} \left(\overline{\vartheta}(x)\right)^{1-p'} \diamond_\alpha x\right)^{1/p'}$$

$$= \sup_{s\in(0,\infty)} \left(\int_s^{\infty} \delta(s) x^{(\theta-1)q} \diamond_\alpha x\right)^{1/q} \left(\int_0^{\sigma(s)} \left(x^{-\varepsilon p}\vartheta(x)\right)^{(1-p')} \diamond_\alpha x\right)^{1/p'}$$

$$= \sup_{s\in(0,\infty)} \left(\int_s^{\infty} \delta(s) x^{(\theta-1)q} \diamond_\alpha x\right)^{1/q} \left(\int_0^{\sigma(s)} x^{\varepsilon p'} \left(\vartheta(x)\right)^{(1-p')} \diamond_\alpha x\right)^{1/p'} < \infty$$

where $1/p + 1/p' = 1$. Due to the inequality (2.5) the function in (3.9) is equivalent to the function $\int_0^{\sigma} s^{(\theta-1)} t^{\varepsilon} g(t) \diamond_\alpha t$. Replacing $t^{\varepsilon-1} g(t)$ with $\overline{g}(t)$,$s^{(\theta)q(.)}\delta(s)$ with $\overline{\delta}(s)$,$s^{(1-\varepsilon)p}\vartheta(s)$ with $\overline{\vartheta}(s)$, and $s^{\varepsilon-1}, g(s)$ with $\overline{g}(s)$, we obtain that

$$\left(\int_0^{\infty}\left(\int_s^{\infty} \overline{g}(t) \diamond_\alpha t\right)^q \overline{\delta}(s) \diamond_\alpha s\right)^{1/q} = \left(\int_0^{\infty}\left(\int_s^{\infty} s^{\theta} t^{\varepsilon-1} g(t) \diamond_\alpha t\right)^q \delta(s) \diamond_\alpha s\right)^{1/q}$$

$$= \left(\int_0^{\infty}\left(\int_s^{\infty} t^{\varepsilon-1} g(t) \diamond_\alpha t\right)^q s^{\theta q(s)}\delta(s) \diamond_\alpha s\right)^{1/q}$$

$$\leq \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} g^p(s)\vartheta(s) \diamond_\alpha (s)\right)^{1/p} = \left(\frac{c}{c-1}\right)^c \left(\int_0^{\infty} g^p(s) s^{(1-\varepsilon)p} s^{(\varepsilon-1)p}\vartheta(s) \diamond_\alpha (s)\right)^{1/p}$$

$$= \left(\frac{c}{c-1}\right)^c \left(\int_0^\infty (s^{(\varepsilon-1)} g(s))^p s^{(1-\varepsilon)p} \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p} = \left(\frac{c}{c-1}\right)^c \left(\int_0^\infty (\overline{g}(s))^p \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p}.$$

Then

$$\left[\int_0^\infty \left(\int_s^\infty \overline{g}(t) \diamond_\alpha t\right)^q \overline{\delta}(s) \diamond_\alpha s\right]^{1/q} \leq \left(\frac{c}{c-1}\right)^c \left(\int_0^\infty (\overline{g}(s))^p \overline{\vartheta}(s) \diamond_\alpha (s)\right)^{1/p}. \tag{3.11}$$

Now, inequality (3.11) holds if and only if

$$K_2 = \sup_{s\in(0,\infty)} \left(\int_s^{\sigma(s)} \overline{\delta}(x) \diamond_\alpha x\right)^{1/q} \left(\int_s^\infty (\overline{\vartheta}(x))^{1-p'} \diamond_\alpha x\right)^{1/p'}$$

$$= \sup_{s\in(0,\infty)} \left(\int_s^{\sigma(s)} \delta(x) x^{\theta q} \diamond_\alpha x\right)^{1/q} \left(\int_s^\infty \left(x^{(1-\varepsilon)p} \vartheta(x)\right)^{(1-p')} \diamond_\alpha x\right)^{1/p'}$$

$$= \sup_{s\in(0,\infty)} \left(\int_0^{\sigma(s)} \delta(x) x^{\theta q} \diamond_\alpha x\right)^{1/q} \left(\int_s^\infty x^{(\varepsilon-1)p'} (\vartheta(x))^{(1-p')} \diamond_\alpha x\right)^{1/p'} < \infty$$

Herewith, we demonstrated that conditions (3.1) and (3.2) are necessary and sufficient for the accuracy of inequalities (3.10) and (3.11). Thus, we complete our proof. □

Remark 3.1 *In Theorem* (3.1)*, if we get $T = R$, $c > 1$, $1 < p \leq q < \infty$, and $\alpha = 1$, then we obtain weighted Hardy inequality as mentioned in* [11, 18]*:*

Corollary 3.1 *Let $1 < p \leq q < \infty$, $0 \leq \alpha \leq 1$, $c > 1$, $\theta, \varepsilon \geq 1$, $m, n > 0$, $u \in C_{rd}([0,\infty) \cap T, R^+)$, and u is $\diamond_\alpha-$integrable on $[0,\infty) \cap T$.*

$$\left[\int_0^\infty s^m \mid u(s) \mid^q \diamond_\alpha s\right]_{1/q} \leq \left(\frac{c}{c-1}\right)^c \left[\int_0^\infty s^n \mid u^{(\diamond_\alpha)^\varphi}(s) \mid^p \diamond_\alpha s\right]^{1/p} \tag{3.12}$$

holds for all u if and only if

$$K_3 = \sup_{s\in[0,\infty)\cap T} \left(\int_s^\infty x^{m+(\theta-1)q} \diamond_\alpha x\right)^{1/q} \left(\int_0^{\sigma(s)} x^{n(1-p')+\varepsilon p'} \diamond_\alpha x\right)^{1/p'} < \infty$$

and

$$K_4 = \sup_{s\in[0,\infty)\cap T} \left(\int_0^{\sigma(s)} x^{m+\theta q} \diamond_\alpha x\right)^{1/q} \left(\int_s^\infty x^{n(1-p')+(\varepsilon-1)p'} \diamond_\alpha x\right)^{1/p'} < \infty.$$

Proof: If we take $\delta(s) = s^m$ and $\vartheta(s) = s^n$ in Theorem (3.1), we get the required result. Thus, we complete our proof. □

Remark 3.2 *In inequality* (3.12) *if we get $T = R$, $c > 1$, $m, n > 0$, $1 < p \leq q < \infty$, and $\alpha = 1$, then we obtain*

$$\left[\int_0^\infty s^m \mid u(s) \mid^q ds\right]_{1/q} \leq \left(\frac{c}{c-1}\right)^c \left[\int_0^\infty s^n \mid u^{(\diamond_\alpha)^\varphi}(s) \mid^p ds\right]^{1/p}$$

which holds if and only if

$$K_5 = \sup_{s\in[0,\infty)} \left(\int_s^\infty x^{m+(\theta-1)q} dx\right)^{1/q} \left(\int_0^s x^{n(1-p')+\varepsilon p'} dx\right)^{1/p'} < \infty$$

and

$$K_6 = \sup_{s\in[0,\infty)} \left(\int_0^s x^{m+\theta q} dx\right)^{1/q} \left(\int_s^\infty x^{n(1-p')+(\varepsilon-1)p'} dx\right)^{1/p'} < \infty.$$

Corollary 3.2 *Let* $1 < p \leq q < \infty$, $0 \leq \alpha \leq 1$, $c > 1$, $m > 0$, $u \in C_{rd}([0,\infty) \cap T, R^{+})$, *and* u *is* $\diamond_\alpha-$*integrable on* $[0,\infty) \cap T$ *with* $u(0) = u^{\diamond_\alpha}(\infty) = 0$.

$$\int_0^\infty s^{m-2p} \mid u(s) \mid^p \diamond_\alpha s \leq \left(\frac{c}{c-1}\right)^c \int_0^\infty s^m \mid u^2(s) \mid^p \diamond_\alpha s \tag{3.13}$$

holds for all function u *if and only if*

$$K_7 = \sup_{s\in[0,\infty)\cap T} \left(\int_s^\infty x^{m-2p} \diamond_\alpha x\right)^{1/p} \left(\int_0^{\sigma(s)} x^{m+p'(1-m)} \diamond_\alpha x\right)^{1/p'} < \infty$$

and

$$K_8 = \sup_{s\in[0,\infty)\cap T} \left(\int_0^{\sigma(s)} x^{m-p} \diamond_\alpha x\right)^{1/p} \left(\int_s^\infty x^{m(1-p')} \diamond_\alpha x\right)^{1/p'} < \infty.$$

Proof: If we take $\delta(s) = s^{m-2p}$, $\vartheta(s) = s^m$, $\alpha = \theta = \varepsilon = 1$ for $p = q$ in inequality (3.4), we obtain the required result. Thus, we complete our proof. □

Remark 3.3 *In inequality* (3.13) *if we get* $T = R$, $c > 1$, $m > 0$, $1 < p < \infty$, *and* $\alpha = 1$ *for some* $u(0) = u'(\infty) = 0$ *then we obtain continuous weighted inequality as mentioned in* [10].

4. Conclusion

In this study, we have determined some necessary and sufficient conditions to prove the accuracy of the generalized two weighted and variable exponent Hardy-type inequality on time scales via $\diamond_\alpha-$ integral. We have given some applications that confirm our results. Moreover, we plan to carry these studies to the more general variable exponent grand Lebesgue spaces and create necessary and sufficient conditions for this.

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