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Proximity equitability Colouring in graphs

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ABSTRACT: Let G be a simple, finite, undirected and connected graph. Let S be the set of all vertices of maximum degree in G. The proximity of a vertex $u \in V(G)$ is the shortest distance of u from S. Two vertices of G are said to be proximity equitable if their proximity difference is at most 1. In this paper, a study of proximity equitable proper colouring is initiated.

Key Words: Maximum Degree, Shortest distance, Proximity Equitable Coloring.

Contents

1. Introduction

W.Meyer [4] introduced the concept of cardinality equitability among colour classes in proper colouring. E.Sampathkumar gave the concept of degree equitability in graphs. Several papers [1,5,6,7,8,9,10, 11,12,13] were published involving the degree equitability of vertices. A graph is said to be a a chordal graph if every cycle of length four or more has a chord. That is, there is an edge between two non consecutive vertices of the cycle. In his paper, a new type of equitability called proximity equitability in connected graphs is introduced. In our society, persons who are some what close to people in power become highly influential than those who are far from power group. So, the influence of a person can be measured by his proximity to power group. A graph model can be created to study this situation. Consider a connected graph. The power of a vertex can be measured either by labels attached to the vertices or by its degree. Here we consider power measured by degree. A vertex with Maximum degree is considered to be a powerful vertex. The distance of a vertex from the set of powerful vertices is a measure the influence of a vertex and this distance is called the proximity to powerful set of vertices. Two vertices are proximity equitable if the difference between their proximities is less than or equal to one. In the following proximity equitablity theory is initiated.

2. Equitable Proximity

Definition 2.1 *Let G be a connected simple, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let S be the set of all vertices of maximum degree of G namely $\Delta(G)$. The proximity of a vertex say u of G , denoted by $pr(u)$, is defined as the shortest distance of u from S. That is, $pr(u) = d(u, S)$. Two vertices u and v of G sare said to be equitable proximity vertices if $|d(u, S) - d(v, S)| \leq 1$.*

Remark 2.1 If u and v are adjacent, then $pr(u)$ and $pr(v)$ differ by at most one. Hence if $deg_{pr}(u)$ is defined as the number of neighbours of u whose pr value differ by at most one from $pr(u)$, then $deg_{pr}(u)$ coincides with $deg(u)$. Where as degree equitable domination is different from domination, proximity equitable domination coincides with domination.

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3. Proximity equitable colouring

Definition 3.1

Let $E_{pr} = max\{|S| : \text{any two elements of } S \text{ are proximity equitable}\}$.
$epr = min\{|S| : \text{any two elements of } S \text{ are proximity equitable}\}$.
$Epri = max\{|S| : \text{any two elements of } S \text{ are proximity equitable and independent and } S \text{ is maximal}\}$.
$epri = min\{|S| : \text{any two elements of } S \text{ are proximity equitable and independent and } S \text{ is maximal}\}$.

Definition 3.2 *Let $\pi = \{V_1, V_2, \cdots, V_k\}$ be a partition of $V(G)$ into independent sets such that for any i, any two vertices of V_i are proximity equitable. Each V_i is called a pre-colour class. The trivial partition of $V(G)$ into singletons is such a partition. The minimum cardinality of such a partition is called the proximity equitable independent partition of G and is denoted by $\chi_{pre}(G)$.*

Remark 3.1 Let $\pi = \{V_1, V_2, \cdots, V_k\}$ be a $\chi_{pre}(G)$ proximity equitable independent partition of $V(G)$. Then in any two distinct pre-classes V_i and V_j, there exist vertices $u \in V_i$ and $v \in V_j$ such that either u and v are adjacent or $|pr(u) - pr(v)| \geq 2$.

Remark 3.2 Let G be a connected graph. Then $2 \leq \chi(G) \leq \chi_{pre}(G) \leq n$.

Proposition 3.1 *Let G be a connected graph. Then $\frac{n}{Epri(G)} \leq \chi_{pre}(G) \leq n - Epri(G) + 1$.*

Proof. Let $\pi = \{V_1, V_2, \cdots, V_k\}$ be a $\chi_{pre}(G)$ proximity equitable and independent partition of $V(G)$. Since $|V_i| \leq Epri(G)$ for each $i, 1 \leq i \leq k, n = \sum_{i=1}^{k} |V_i| \leq kEpri(G)$. Therefore, $\frac{n}{Epri(G)} \leq k = \chi_{pre}(G)$. Let V_1 be a $Epri$-set of G. Let $\pi_1 = \{V_1, V_2, \cdots, V_t\}$ be a partition of $V(G)$ such that each $V_j, 2 \leq j \leq t$ are the singletons from $V(G) - V_1$ where $t = n - |V_1| = n - Epri(G)$. Clearly, π_1 is a proximity equitable and independent partition of G. Therefore, $\chi_{pre}(G) \leq |\pi_1| = n - Epri(G) + 1$.

Example 3.1 Let G be the complete bipartite graph $K_{m,n}$. Then $Epri(G) = max\{m, n\}$ and $epri(G) = min\{m, n\}$.

Remark 3.3 For $K_n, Epri(K_n) = 1$ and hence the bounds given above for $\chi_{pre}(G)$ are sharp.

Definition 3.3 *A connected graph is pre-complete if $\chi_{pre}(G) = n$. That is, any two vertices u, v of G are either adjacent or $|pr(u) - pr(v)| \geq 2$.*

Theorem 3.1 *$\chi_{pre}(G) = n$ if and only if G is complete.*

Proof. If G is complete, then $\chi_{pre}(G) = n$.

Conversely, let $\chi_{pre}(G) = n$. Let u, v be any two vertices of G. Then either u and v are adjacent or $|pr(u) - pr(v)| \geq 2$. Suppose u and v are not adjacent. Then $|pr(u) - pr(v)| \geq 2$. Let S be the set of all maximum degree vertices of G. Since any two vertices of S have equal proximity namely zero, S is a clique. Let $u \in V(G) - S$. Then u does not have maximum degree. Let $pr(u) = k$. If $k = 1$, then u is adjacent with some vertex of S. If S has a unique vertex say v, then u and v are adjacent. If there exists no other vertex then $G = K_2$. Since v is the unique vertex of maximum degree, any vertex adjacent to v has degree less than that of v. All adjacent vertices of v are mutually adjacent since otherwise, there are two non-adjacent vertices in $N(v)$ such that their proximity difference is 0, a contradiction. Therefore, $N[v]$ is complete. Since v is the unique vertex of maximum degree, there can not be two or more adjacent vertices for v since otherwise, there will be more than one vertex of maximum degree. Also, no neighbour of v can have a neighbour different from v since otherwise v can not be a maximum degree vertex. Hence $G = K_2$. If S has more than one vertex, then uis adjacent with every vertex of S since otherwise, there exists a vertex $v \in S$ such that u and v are non-adjacent and $|pr(u) - pr(v)| = 1$, a contradiction. Therefore, $S \cup \{u\}$ is a clique. Hence, all the neighbours of S together with S form a clique. Hence u has maximum degree in G, a contradiction since u is not in S. If $k \geq 2$, then there exists a vertex $w \in V(G) - S$ such that $prox(w) = 1$ and this leads to contradiction. Hence $S = V(G)$. That is all vertices of V have maximum degree and belong to S. Since S is a clique, G is complete.

Remark 3.4 Degree equitable complete graphs are defined in [1]. In these graphs, any two vertices are either adjacent or their degree difference is at most one. For example, K_4 with a pendent vertex attached with exactly one vertex of K_4 is degree equitable complete but not proximity complete.

Proposition 3.2 *Given a positive integer k, there exists a connected graph G such that $\chi_{pre}(G)-\chi(G)=k$.*

Proof. Consider the subdivision graph G of a star with degree of the centre at least three. Then $\chi_{pre}(G)=3$ and $\chi(G)=2$. Let H be the graph obtained from $K_{1,3}$ by subdividing each edge exactly k times. Then $\chi_{pre}(H)=k+2$ and $\chi(H)=2$.

Proximity sequence in a connected graph: Let G be a simple, finite, undirected and connected graph. Just like degree sequence, one can define proximity sequence. The proximity of a vertex, being the distance of the vertex from the set of maximum degree vertices enables to introduce proximity sequence. The vertices of maximum degree have proximity zero. The neighbours of maximum degree vertices have proximity one etc. For example, in a regular connected graph, the proximity sequence is $\{0^n\}$.(that is, all the n vertices have proximity 0). In $K_{1,n}$, the proximity sequence is $\{0,1^n\}$.

Proposition 3.3 *Let G be a connected simple graph with proximity sequence $\{0^{n0},\cdots,(k-1)^{n(k-1)}\}$. Then $\chi_{pre}(G)\leq k\chi(G)$.*

Proof. Let $V_i=\{v\in V(G):pr(v)=i\}$. Then $\chi_{pre}(G)\leq\sum\limits_{i=0}^{k-1}\chi(V_i)\leq\sum\limits_{i=0}^{k-1}\chi(G)=k\chi(G)$.

Corollary 3.1 *If G is a connected planar graph , then $\chi(G)\leq 4$. Hence, if G has k distinct proximity levels, $\chi_{pre}G)\leq 4k$.*

Corollary 3.2 *If G is a connected chordal graph with k distinct proximity levels and clique number $\omega(G)$, then $\chi_{pre}(G)\leq k\omega(G)$ (since in a chordal graph, $\chi(G)=\omega(G)$).*

Proposition 3.4 *Given any positive integer k, there exists a connected graph G with k distinct proximity levels such that $\chi_{pre}(G)=k\chi(G)$.*

Proof. Let G be the graph obtained from a path P_{2k-1} by attaching t pendent vertices at the initial vertex of the path where $t\geq 3$. Then, $\chi_{pre}(G)=2k$ and $\chi(G)=2$. Hence, $\chi_{pre}(G)=k\chi(G)$.

Remark 3.5 Let G be a caterpillar whose spine is a path of length $2k-1$ with the degree of the initial vertex, strictly greater than the degree of any subsequent vertex on the path. The degree of the initial vertex is strictly greater than the degree of any subsequent vertex on the path. Then, $\chi_{pre}(G)=k\chi(G)$. Note that G is also planar and connected. Another example is as follows: Let G be the unicyclic graph obtained from C_4 with a diagonal by attaching a path of length $3k-1$ to the vertex of degree 3 in C_4 with a diagonal. Then $\chi_{pre}(G)=3k$ and $\chi(G)=3$. Also $\omega(G)=3$ and so, $\chi_{pre}(G)=k\omega(G)$. Clearly G is a chordal graph.

Theorem 3.2 *[3] If G is a graph of order n, then (i) $2\sqrt{n}\leq\chi(G)+\chi(\overline{G})\leq n+1$ (ii) $n\leq\chi(G)\chi(\overline{G})\leq\frac{(n+1)}{2}^2$.*

Theorem 3.3 *For any graph G with G and $\overline{G}$ connected and with the same number of proximity level k, (i) $2\sqrt{n}\leq\chi_{pre}(G)+\chi_{pre}(\overline{G})\leq k(n+1)$ (ii) $n\leq\chi_{pre}(G)\chi_{pre}(\overline{G})\leq k^2\frac{(n+1)}{2}^2$.*

Proof. Since $\chi(G)\leq\chi_{pre}(G)\leq k\chi(G)$, the above result follow from the above Theorem.

Remark 3.6 Let $G=P_4$. Then $\overline{G}=P_4.\chi_{pre}(G)=2=\chi_{pre}(\overline{G}).k=1$. Hence, the left inequalities in (i) and (ii) are sharp. Let $G=C_5$. Then $\overline{G}=C_5.\chi_{pre}(G)=3=\chi_{pre}(\overline{G}).k=1$. So, the right inequalities in (i) and (ii) are sharp.

Theorem 3.4 *[2] For any fixed integer $k\geq 3, k$-colourability is NP- complete.*

Theorem 3.5 *Given a positive integer $k \geq 3$, the problem of deciding whether $\chi_{pre}(G) \geq k$ is NP-complete for any graph G with $\chi(G) \geq 3$.*

Proof. Let G be a connected graph with $\chi(G) \geq 3$. Let G_1 be the graph obtained from G by attaching suitable number of pendent vertices at each vertex so that degree of any vertex of G in G_1 is $\Delta(G) + 1$. (see Theorem 2.18 [1]). Clearly $\chi_{pre}(G) = \chi(G) + 1$. Hence the result follows from the above Theorem.

Theorem 3.6 *Let G be a connected unicyclic graph with cycle C_n. Let P_t be the longest path attached at any vertex of C_n. Then $\chi_{pre}(G) = max\{s, t\}$ where s is the maximum proximity value of any vertex on C_n.*

Proof. Let G a connected unicyclic graph with cycle P_n. Let P_t be the longest path attached at any vertex say u of C_n. Then, the pair of vertices adjacent to you have proximity value 0 or 1. They can be combined with the adjacent point of u in the attached path so as to form an independent proximity equitable set. To this set, vertices on C_n with degree equal to that of u as well as the first vertex on each path attached to the vertices of C_n can be added. Again, those vertices at distance 2 from a vertex of maximum degree can be combined with the distance 2 vertices on the paths attached to the vertices of C_n. The vertices at a distance t on any path P_t as well as those on the cycle with proximity value t on the cycle can be put together. Suppose, there is a unique vertex say u on the cycle with maximum degree. Then there are pairs of vertices on either side of u at with proximity value from 1 to $\lceil \frac{n+1}{2} \rceil$. If $t \leq \lceil \frac{n+1}{2} \rceil$, then $\chi_{pre}(G) = \lceil \frac{n+1}{2} \rceil$. Otherwise, $\chi_{pre}(G) = t$. Suppose there are more than one vertex on C_n with maximum degree. Let s be the maximum proximity value of any vertex on C_n. Then, $\chi_{pre}(G) = max\{s, t\}$.

Example 3.2 Let G be obtained from C_{11} by attaching a path of length 6 at exactly one vertex of C_{11}. $\lceil \frac{n+1}{2} \rceil = 6.t = 7$. Hence $\chi_{pre}(G) = 7$.If P_4 is attached at a unique vertex of C_{11}, then $\chi_{pre}(G) = 6$. Suppose, P_4 is attached at u_1, u_2, u_5, u_7 and u_9. Then $s = 2$. $t = 4$. Hence $\chi_{pre}(G) = 4$.

Definition 3.4 *Let G be a connected graph. G is called pre-bipartite if $\chi_{pre}(G) = 2$.*

Theorem 3.7 *A connected graph is pre- bipartite if and only if G is bipartite and any non- maximum degree vertex is adjacent with a maximum degree vertex.*

Proof. Let G be a connected graph which is pre-bipartite $V(G)$. Then, there are only two proximity values namely 0 for maximum degree vertices and 1 for non-maximum degree vertices. Hence any non-maximum degree vertex is adjacent with a maximum degree vertex. If any non-maximum degree vertex is not adjacent with any maximum degree vertex, then proximity value of that vertex is great than or equal to 2, a contradiction. Hence the theorem.

Example 3.3 (i) Consider G obtained from C_4 by attaching a pendent vertex at exactly one vertex of C_4. Then G is bipartite and there exists a vertex which is not adjacent with a maximum degree vertex. Here,$\chi_{pre}(G) = 3$.

(ii) Let G be the graph which is obtained from K_3 by attaching a pendent vertex at exactly one vertex of K_3. Then there are only two proximity levels 0 and 1. Every non maximum degree vertex is adjacent to a maximum degree vertex. Yet $\chi_{pre}(G) = 3$. It is because G is not bipartite.

Remark 3.7 Any tree can be obtained from a unicyclic graph by removing an edge in the cycle.

Hence $\chi_{pre}(T)$ can be obtained from χ_{pre} of the corresponding unicyclic graph.

Open problems:

1. Edge removal or edge addition and its effect on $\chi_{pre}(G)$.
2. Dominator and colour class domination in independent and proximity equitable partition.
3. Secure independent proximity equitable partition.
4. Colourful dominating set from independent proximity equitable partition.

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