Applications of $L_{gf s}$-closed Sets In Mixed Fuzzy Soft Ideal Topological Spaces

Md Mirazul Hoque, Jayasree Chakraborty, Baby Bhattacharya* and Binod Chandra Tripathy

ABSTRACT: The primary goal of this treatise is to introduce a new kind of generalized closed sets termed as $L_{gf s}$-closed sets in the light of fuzzy soft local function in a mixed fuzzy soft ideal topological space. We also define the notion of fuzzy soft $*$-separated set. In addition, we procure the idea of fuzzy soft regularity, normality and fuzzy soft compactness. We also study their behavior in terms of $L_{gf s}$-closed sets.

Key Words: $L_{gf s}$-closed set, * dense-in-itself, fuzzy soft $*$-separated set, $L_{gf s}$-regularity, $L_{gf s}$-normality, $L_{gf s}$-compactness.

Contents

1 Introduction and preliminaries 1
2 Main results 3
3 Some applications of $L_{gf s}$-closed sets 6
4 Conclusion 9

1. Introduction and preliminaries

The ever great scholar L. A Zadeh [29] introduced the concept of fuzzy set in 1965. By the use of this notion of fuzzy sets, Chang [7] in 1968 and Lowen [13] in 1976 have introduced the theory of fuzzy topology with different objectives. Molodtsov [18] in 1999, initiated the idea of soft sets for handling uncertainties. He additionally carried out the soft set theory in operation research, game theory, theory of probability, and so on. Since then many researchers have worked on soft sets.

Levine [12] presented the concept of a generalised closed set, established its basic characteristics, and demonstrated some of its applications in general topological spaces. Generalized fuzzy closed set concept and its applications in fuzzy topological spaces (fts, for short) were given by Balasubramanian and Sundaram [2] in 1997. Different kind of generalized closed sets are being studied by different authors in various environments, for instance [19,4,3,20].

The notion of soft topological spaces introduced and studied in [24]. Maji et al. [16] initiated the idea of fuzzy soft sets. Indeed, a fuzzy soft set, where soft set is defined over the fuzzy set, is a combination of fuzzy and soft sets. The topological structure of fuzzy soft sets was first developed in 2011 by Tanay et al. [26]. In general topological spaces, the introduction of notion of ideal was given by Kuratowski [11] and then further studied by [28] and so on in fuzzy topological spaces, it had been introduced by Sarkar [23]. In fuzzy soft topological spaces (FSTSs), the fuzzy soft ideal was first introduced by Kandil et al. [9] in 2016.

The idea of mixed fuzzy topology given by Das and Baishya [8] in 1995 and by Tripathy and Ray [27] in 2012 in different ways and using these concepts, Borah and Hazarika [5] introduced mixed fuzzy soft topological spaces. Further, the study involving the notion of mixed fuzzy soft ideal topology was given by Borah and Hazarika [6].

The main objective of this present article is to introduce a new type of generalized closed set via fuzzy soft local function in a mixed fuzzy soft ideal topological space (MFSITS). The collection of such closed sets is found to contain the collection of all generalized fuzzy soft closed sets. As an application of the said closed sets we consider some extended version of fuzzy soft regularity, normality and compactness.
Before moving on to the main part, let’s provide some basic and preliminary concepts about the prevailing definitions and results that play an important role in this study.

In this article, we denote initial universal set by $X$ and the set of all possible parameters for $X$ by $E$.

**Definition 1.1** [22] Let $A \subseteq E$. A pair $(F_A, E)$, denoted by $F_A$ is said to be a fuzzy soft set over $X$ if $F_A: E \rightarrow I^X$ is a mapping defined by $F_A(e) = \mu_{F_A}^e$, where $\mu_{F_A}^e = \bar{1}$ if $e \in E - A$ and $\mu_{F_A}^e \neq \bar{0}$ if $e \in A$, where $\bar{0}(x) = 0, \forall x \in X$. By $FS(X)_E$ we mean the family of all fuzzy soft set over $X$.

**Definition 1.2** [22] The complement of a fuzzy soft set $(F_A, E)$ on $(X, E)$ is a fuzzy soft set $(F_A^C, E)$, denoted by $F_A^C$, where $F_A^C : E \rightarrow I^X$ is defined by $\mu_{F_A^C}^e = \bar{1}$ if $e \in E - A$ and $\mu_{F_A^C}^e = 1 - \mu_{F_A}^e$ if $e \in A$, where $\bar{1}(x) = 1, \forall x \in X$.

**Definition 1.3** [26] A fuzzy soft set $F_E$ over $X$ is defined to be a null fuzzy soft set, denoted by $\bar{0}_E$ if $F_E(e) = \bar{0}, \forall e \in E$.

**Definition 1.4** [26] A fuzzy soft set $F_E$ over $X$ is defined to be an absolute fuzzy soft set, denoted by $\bar{1}_E$ if $F_E(e) = \bar{1}, \forall e \in E$.

**Definition 1.5** [22] Let $F_A, G_B \in FS(X)_E$. Then $F_A$ is fuzzy soft subset of $G_B$, denoted by $F_A \subseteq G_B$ if $\mu_{F_A}^e(x) \leq \mu_{G_B}^e(x), \forall e \in E, \forall x \in X$.

**Theorem 1.6** [22] Let $F_A, G_B \in FS(X)_E$. The union of $F_A, G_B$ over $X$ is a fuzzy soft set $H_C$, denoted by $F_A \cup G_B$ and is defined as $H_C(e) = \mu_{H_C}^e = \mu_{F_A}^e \cup \mu_{G_B}^e$, for all $e \in E$, where $C = A \cup B$.

**Definition 1.7** [22] Let $F_A, G_B \in FS(X)_E$. The intersection of $F_A, G_B$ over $X$ is a fuzzy soft set $H_C$, denoted by $F_A \cap G_B$ and is defined as $H_C(e) = \mu_{H_C}^e = \mu_{F_A}^e \cap \mu_{G_B}^e$, for all $e \in E$, where $C = A \cap B$.

**Definition 1.8** [21] Let $(X, \tau)$ be an FSTS and $F_A \in FS(X)_E$. The fuzzy soft closure of $F_A$ is denoted by $cl(F_A)$, is the intersection of all fuzzy soft closed sets which contains $F_A$.

**Theorem 1.9** [21] Let $(X, \tau)$ be an FSTS and $F_A \in FS(X)_E$. The fuzzy soft interior of $F_A$ is denoted by $int(F_A)$, is the union of all fuzzy soft open sets which contained in $F_A$.

**Definition 1.10** [14] Let $F_A \in FS(X)_E$. Then $F_A$ is called a fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $F_A(a) = x_a$ and $F_A(a) = \bar{0}$ for $a \in A - \{a\}$. This is denoted by $x^e_A$ or $e(F_A)$. We say that $x^e_A \in F_A$, if for $e \in A$ and $x \in X$, $\alpha \leq \mu_{F_A}^e(x)$.

**Definition 1.11** [14] A fuzzy soft point $x^e_A$ is called q-coincident with $F_A \in FS(X)_E$, denoted by $x^e_A qF_A$, if $\alpha + \mu_{F_A}^e(x) > 1$ for some $x \in X$.

**Definition 1.12** [14] Let $F_A, G_B \in FS(X)_E$. Then $F_A$ is q-coincident with $G_B$, denoted by $F_A qG_B$, if there exists $x \in X$ such that $\mu_{F_A}^e(x) + \mu_{G_B}^e(x) > 1$. When $F_A$ and $G_B$ are not q-coincident, we denote it by $F_A q\bar{G}_B$.

**Proposition 1.13** [25] Let $F_A, G_A \in FS(X)_E$. Then $F_A \subseteq G_A$ if and only if $F_A qG_A$.

**Definition 1.14** [9] A collection of non-empty fuzzy soft sets $I$ over $X$ is called a fuzzy soft ideal on $X$ if the following conditions are met:

(i) If $F_A, G_B \in \bar{I}$, then $F_A \cup G_B \in \bar{I}$;
(ii) If $F_A \in \bar{I}$ and $G_B \subseteq F_A$, then $G_B \in \bar{I}$.
Definition 1.15 [5] Let \((X, \tau_1)\) and \((X, \tau_2)\) be two FSTSs. Consider the collection: \(\tau_1(\tau_2) = \{F_A \in FS(X)_E : \text{for any } G_B \in FS(X)_E \text{ with } F_A \cup G_B \text{ then there exists } H_C \in \tau_2 \text{ such that } G_B \cap H_C \in \tau_1\} \). Then this family will form a topology and the topology generated by this is called a mixed fuzzy soft topology.

Definition 1.16 [10] Let \((X, \tau, \tilde{I})\) be fuzzy soft ideal topological space and \(F_A \in FS(X)_E\). Then \(F_A^*(\tilde{I}, \tau)\) for any \(F_A \in FS(X)_E\) then fuzzy soft local function \(F_A^*(\tilde{I}, \tau)\) (or \(F_A^*\)) of \(F_A\) is the union of all fuzzy soft points \(x_e^*\) such that if \(G_B \in N(x_e^*)\) and \(H_C \in \tilde{I}\), then there exist at least one \(x \in X\) and \(e \in E\) for which \(\mu_{F_A}(x) + \mu_{G_B}(x) - 1 > \mu_{H_C}(x)\).

Definition 1.17 [9] Let \((X, \tau, \tilde{I})\) be a fuzzy soft ideal topological space and \(F_A \in FS(X)_E\). Then the fuzzy soft local function \(F_A^*(\tilde{I}, \tau)\) (or \(F_A^*\)) of \(F_A\) is the union of all fuzzy soft points \(x_e^*\) such that if \(G_B \in N(x_e^*)\) and \(H_C \in \tilde{I}\), then there exist at least one \(x \in X\) and \(e \in E\) for which \(\mu_{F_A}(x) + \mu_{G_B}(x) - 1 > \mu_{H_C}(x)\).

Definition 1.18 [6] Let \((X, \tau_1, \tau_2)\) be a mixed fuzzy soft topological space and let \(\tilde{I}\) be a fuzzy soft ideal over \(X\). Then \((X, \tau_1, \tau_2, \tilde{I})\) is called a mixed fuzzy soft ideal topological space (MFSITS).

Theorem 1.19 [6] Let \((X, \tau_1, \tau_2, \tilde{I})\) be an MFSITS. Let \(H_A, K_A \in FS(X)_E\), then

\begin{enumerate}
  \item \(H_A \subseteq K_A \Rightarrow H_A^* \subseteq K_A^*\),
  \item \(H_A^* = cl(H_A^*) \subseteq cl(H_A)\),
  \item \((H_A^*)^* \subseteq H_A^*\),
  \item \((H_A \cap K_A)^* = H_A^* \cup K_A^*\),
  \item \(H_A^*\) is fuzzy soft closed in \(X\).
\end{enumerate}

2. Main results

This particular section deals with the idea of generalized closed set in the light of local function in an MFSITS and we emphasize that this approach is something different from the conventional directions. Nonetheless, we introduce \(gfs\)-closed set in the sense of [2] to find the relationship among the newly defined notion with the same in an MFSITS.

Definition 2.1 A fuzzy soft set \(H_A\) in an MFSITS \(X\) is called generalized fuzzy soft closed set (\(gfs\)-closed, in short) if \(cl(H_A) \subseteq U_A\), whenever \(H_A \subseteq U_A\) and \(U_A\) is fuzzy soft open set in \(X\). \(H_A\) is called generalized fuzzy soft open set (\(gfs\)-open, in short) if its complement, \(H_A^c\) is \(gfs\)-closed set.

The following example shows the existence of the above defined notion.

Example 2.2 Let \(X = \{x, y\}\) and \(E = A = \{e_1, e_2\}\).

Let \(\tau_1 = \{0_E, 1_E, H_A, K_A\}\) and \(\tau_2 = \{0_E, 1_E, F_A, G_A\}\), where \(H_A = \{H(e_1) = \{(x, 0.1), (y, 0.2)\}, H(e_2) = \{(x, 0.3), (y, 0.2)\}\}\), \(K_A = \{K(e_1) = \{(x, 0.8), (y, 0.7)\}, K(e_2) = \{(x, 0.6), (y, 0.7)\}\}\), \(F_A = \{F(e_1) = \{(x, 0.9), (y, 0.8)\}, F(e_2) = \{(x, 0.7), (y, 0.8)\}\}\) and \(G_A = \{G(e_1) = \{(x, 0.2), (y, 0.3)\}, G(e_2) = \{(x, 0.4), (y, 0.3)\}\}\). Let us consider any fuzzy soft ideal \(\tilde{I}\).

Then \(\tau_1(\tau_2) = \{0_E, 1_E, F_A, G_A\}\) and consequently \((X, \tau_1(\tau_2), \tilde{I})\) is an MFSITS.

Let \(Q_A = \{Q(e_1) = \{(x, 0.7), (y, 0.6)\}, Q(e_2) = \{(x, 0.5), (y, 0.2)\}\}\). Here, we claim that \(Q_A\) is \(gfs\)-closed. As \(Q_A \subseteq F_A\) and \(cl(Q_A) \subseteq F_A\).

We now propose the definition of \(L_{gfs}\)-closed set in \((X, \tau_1(\tau_2), \tilde{I})\) as follows.

Definition 2.3. Let \(X\) be an MFSITS. A fuzzy soft set \(H_A\) over \(X\) is called a generalized fuzzy soft closed set with respect to a fuzzy soft local function (in short, \(L_{gfs}\)-closed) if \(H_A^* \subseteq U_A\), whenever \(H_A \subseteq U_A\) and \(U_A\) is fuzzy soft open set. \(H_A\) is called a generalized fuzzy soft open set with respect to a fuzzy soft local function (in short, \(L_{gfs}\)-open) if its complement, \(H_A^c\) is \(L_{gfs}\)-closed set.
Existence of such closed sets are very interesting in this domain and we illustrate the following as ready reference.

**Example 2.4.** Let $X = \{x, y\}$ and $E = A = \{e_1, e_2\}$.

Let $F_A = \{F(e_1) = \{(x, 1), (y, 1)\}, F(e_2) = \{(x, 0.6), (y, 0.6)\}\}, G_A = \{G(e_1) = \{(x, 0), (y, 0)\}, G(e_2) = \{(x, 0.4), (y, 0.4)\}\}.

Let $\tau_1 = \{0_E, 1_E, G_A\}, \tau_2 = \{0_E, 1_E, F_A\}$. Then $\tau_1(\tau_2) = \{0_E, 1_E, F_A\}$.

Let $H_A = \{H(e_1) = \{(x, 0), (y, 0)\}, H(e_2) = \{(x, 0.5), (y, 0.5)\}\}$ and $\bar{I} = \{0_E, K_A\}$ where $K_A = \{K(e_1) = \{(a, 0), (b, 0)\}, K(e_2) = \{(x, 0.7), (y, 0.7)\}\}.

Now, to compute $H_A$, we consider the fuzzy soft point $e_1(X_A) = \{(e_1) = \{(x, 1), (y, 1)\}, (e_2) = \{(x, 0), (y, 0)\}\}$. The open sets in $\tau_1(\tau_2)$ containing $e_1(X_A)$ are $F_A$ and $1_E$. Here, $F_A \cap H_A = H_A \in \bar{I}$ and $1_E \cap H_A = H_A \in \bar{I}$.

Which implies that $e_1(X_A) \not\in H_A^\ast$.

Again, we consider the fuzzy point $e_2(X_A) = \{(e_1) = \{(x, 0), (y, 0)\}, (e_2) = \{(x, 1), (y, 1)\}\}$ and the open sets in $\tau_1(\tau_2)$ containing $e_2(H_A)$ is $1_E$ only. Clearly $1_E \cap H_A = H_A \in \bar{I}$, and consequently we have $e_2(X_A) \not\in H_A^\ast$. Then $H_A^\ast \neq 1_E$ as well as $H_A^\ast \neq 0_E$, so we have $H_A = \{H(e_1) = \{(x, 0), (y, 0)\}, H(e_2) = \{(x, 0.4), (y, 0.4)\}\}$, as $H_A^\ast$ is fuzzy soft closed in $X$. Consequently, we have $H_A \subseteq F_A$ implies $H_A^\ast \subseteq F_A$.

Therefore, $H_A$ is an $L_{gfs}$-closed set in $X$.

**Theorem 2.5.** Every fuzzy soft closed as well as $gfs$-closed set in an MFSTTS $X$ is an $L_{gfs}$-closed.

**Proof:** Let $H_A$ be generalized fuzzy soft closed set ($gfs$-closed) such that $H_A \subseteq U_A$, where $U_A$ is fuzzy soft open set. Then $cl(H_A) \subseteq U_A$ and consequently $H_A \subseteq U_A$ (by Theorem 1.19(ii)) and therefore $H_A$ is $L_{gfs}$-closed.

Again, since every fuzzy soft closed set is $gfs$-closed and hence the theorem is completed.

However, the converses are not necessarily true. The following example demonstrates our claim.

**Example 2.6.** From the above Example 2.4, we have $H_A$ is $L_{gfs}$-closed set but not a $gfs$-closed set and also not a fuzzy soft closed set, since $cl(H_A) = 1_E$ and $H_A \not\subseteq F_A$.

But in particular, if $\bar{I} = \{0_E\}$, then $H_A^\ast = cl(H_A), \forall H_A \in FS(X)_E$ [9], and hence every $L_{gfs}$-closed set is $gfs$-closed set.

**Theorem 2.7.** If $H_A$ and $K_A$ are two $L_{gfs}$-closed sets in an MFSTTS $X$, then $H_A \cup K_A$ is also an $L_{gfs}$-closed set therein.

**Proof:** Suppose $H_A \cup K_A \subseteq U_A$, for some fuzzy soft open set $U_A$. Then $H_A \subseteq U_A$ and $K_A \subseteq U_A$ and hence $H_A^\ast \subseteq U_A$ and $K_A^\ast \subseteq U_A$, as $H_A$ and $K_A$ are two $L_{gfs}$-closed sets. Now, $(H_A \cup K_A)^\ast = H_A^\ast \cup K_A^\ast \subseteq U_A$ (by Theorem 1.19(v)). This shows that $H_A \cup K_A$ is also $L_{gfs}$-closed set.

It is pertinent to note that the arbitrary union of $L_{gfs}$-closed sets may not be an $L_{gfs}$-closed set. This follows from the fact that in an MFSTTS $X$ if we consider $\bar{I} = \{0_E\}$, then $H_A = cl(H_A), \forall H_A \in FS(X)_E$ [9] and the fuzzy soft closure of infinite unions may not be equal to the union of closures. However, even the intersection of two $L_{gfs}$-closed sets may not be an $L_{gfs}$-closed set. To show this we consider the following example.

**Example 2.8.** Let $X = \{x, y\}$ and $E = A = \{e_1, e_2\}$.

Let $F_A = \{F(e_1) = \{(x, 0.6), (y, 0.6)\}, F(e_2) = \{(x, 0.3), (y, 0.3)\}\}, G_A = \{G(e_1) = \{(x, 0.4), (y, 0.4)\}, G(e_2) = \{(x, 0.7), (y, 0.7)\}\}.

Let $\tau_1 = \{0_E, 1_E, G_A\}, \tau_2 = \{0_E, 1_E, F_A\}$. Then $\tau_1(\tau_2) = \{0_E, 1_E, F_A\}$.

We consider the ideal $\bar{I} = \{0_E, K_A\}$ where $K_A = \{K(e_1) = \{(x, 0), (y, 0)\}, K(e_2) = \{(x, 0.3), (y, 0.3)\}\}.

Let $H_A = \{H(e_1) = \{(x, 0.9), (y, 0.9)\}, H(e_2) = \{(x, 0.2), (y, 0.2)\}\}, K_A = \{K(e_1) = \{(x, 0.7), (y, 0.7)\}, K(e_2) = \{(x, 0.2), (y, 0.2)\}\}$. Then we have $H_A^\ast = 1_E$ and $K_A^\ast = 1_E$ and consequently $H_A$ and $K_A$ are $L_{gfs}$-closed sets.
Again, we have $H_A \cap K_A = \{(e_1) = \{(x, 0.3), (y, 0.3)\}, (e_2) = \{(x, 0.2), (y, 0.2)\}\}$ and one can show that $(H_A \cap K_A)^* = 1_E$. Since here, $H_A \cap K_A \subseteq F_A$ and $(H_A \cap K_A)^* \subseteq F_A$, which is showing that $H_A \cap K_A$ is not an $L_gfs$-closed.

Our concern is to find out that required condition with which an $L_gfs$-closed set becomes a $gfs$-closed set. For the purpose, we define the following notion that is of interest in itself.

**Definition 2.9.** A fuzzy soft set $H_A$ in an MFSITS $X$ is said to be $*$ dense-in-itself if $H_A \subseteq H_A^*$.

**Theorem 2.10.** If $H_A$ be an $L_gfs$-closed set and $*$ dense-in-itself in $X$, then it is $gfs$-closed.

**Proof:** Let $U_A \in \tau_1(\tau_2)$ such that $H_A \subseteq U_A$. We have $H_A^* \subseteq U_A$ and $H_A \subseteq H_A^*$. Since $H_A$ is fuzzy soft closed and $cl(H_A)$ is the smallest closed set containing $H_A$, we get $H_A \subseteq cl(H_A) \subseteq H_A^* \subseteq U_A$ and hence $cl(H_A) \subseteq U_A$. Therefore, $H_A$ is $gfs$-closed set.

**Definition 2.11.** Let $(X, \tau_1(\tau_2), I)$ be an MFSITS. The fuzzy soft closure and interior operator of a fuzzy soft set $H_A$ in $X$ is defined as $cl^*(H_A) = H_A \cup H_A^*$, $Int^*(H_A) = H_A \cap (H_A^*)^c$ respectively.

**Theorem 2.12.** Let $H_A$ be any $L_gfs$-closed set and $K_A$ be any fuzzy soft set in an MFSITS $(X, \tau_1(\tau_2), I)$, then
(i) if $H_A \subseteq K_A \subseteq H_A^*$, then $K_A$ is $L_gfs$-closed set.
(ii) if $H_A \subseteq K_A \subseteq cl^*(H_A)$, then $K_A$ is $L_gfs$-closed set.

**Proof:** (i) Suppose $K_A \subseteq U_A$, for some fuzzy soft open set $U_A$. Then $H_A \subseteq U_A$ and $H_A$ being $L_gfs$-closed set, $H_A^* \subseteq U_A$. Now $K_A \subseteq H_A^*$ implies $K_A \subseteq (H_A^*)^c \subseteq H_A^*$. Thus $K_A \subseteq U_A$. Therefore, $K_A$ is $L_gfs$-closed set.
(ii) Suppose $K_A \subseteq V_A$, for some fuzzy soft open set $V_A$. Then $H_A \subseteq V_A$ and $H_A$ being $L_gfs$-closed set, $H_A^* \subseteq V_A$. Now $K_A \subseteq cl^*(H_A) = H_A \cup H_A^*$, we have $K_A \subseteq (H_A \cup H_A^*)^c \subseteq H_A^* \subseteq V_A$. Thus $K_A \subseteq V_A$ and consequently $K_A$ is $L_gfs$-closed set.

**Corollary 2.13** If $H_A$ and $K_A$ are two fuzzy soft sets such that $H_A \subseteq K_A \subseteq H_A^*$ and if $H_A$ is $L_gfs$-closed set. Then both $H_A$ and $K_A$ are $gfs$-closed set.

**Proof.** It follows from Theorem 2.12(i) and Theorem 2.10.

**Theorem 2.14.** A fuzzy soft set $H_A$ is $L_gfs$-open if and only if $K_A \subseteq Int^*(H_A)$ whenever $K_A$ is fuzzy soft closed and $K_A \subseteq H_A$.

**Proof:** Let $H_A$ be $L_gfs$-open set and $K_A$ is fuzzy soft closed set such that $K_A \subseteq H_A$. Then $H_A^c$ is $L_gfs$-closed set and $H_A^c \subseteq K_A^c$. As $H_A^c$ is $L_gfs$-closed and $K_A^c$ is fuzzy soft open, we have $(H_A^c)^* \subseteq K_A^c$ and then $cl^*(H_A^c)^c = Int^*(H_A)$ (it can be proved easily by Definition 2.11 and Theorem 1.19(v)).

Conversely, let $U_A \in \tau_1(\tau_2)$ such that $H_A^c \subseteq U_A$. Then $U_A^c$ is fuzzy soft closed and $U_A^c \subseteq H_A$ and by the given condition, we have $U_A^c \subseteq Int^*(H_A)$. This implies $cl^*(H_A^c)^c = Int^*(H_A)$ and from this we have, $(H_A^c)^* \subseteq U_A$. Hence, $H_A^c$ is $L_gfs$-closed set and accordingly $H_A$ is $L_gfs$-open set.

**Theorem 2.15.** If $H_A$ and $K_A$ are two $L_gfs$-open sets, then $H_A \cap K_A$ is $L_gfs$-open set.

**Proof:** Let $H_A$ and $K_A$ be two $L_gfs$-open sets. Suppose that $(H_A \cap K_A)^c \subseteq U_A$, for some fuzzy soft open set $U_A$. Then $H_A^c \subseteq U_A$ and $K_A^c \subseteq U_A$. This implies $H_A^c \subseteq U_A$ and $K_A^c \subseteq U_A$. Now, both $H_A^c$ and $K_A^c$ being $L_gfs$-closed sets, giving that $(H_A^c)^* \subseteq U_A$, $(K_A^c)^* \subseteq U_A$ and so, $(H_A \cap K_A)^c = (H_A^c)^* \cup (K_A^c)^* \subseteq U_A$ (by Theorem 1.19(v)). This implies $(H_A \cap K_A)^c$ is $L_gfs$-closed set and thus $H_A \cap K_A$ is $L_gfs$-open set.

**Remark 2.16.** In general union of two $L_gfs$-open sets is not an $L_gfs$-open set. It can be easily established in view of the Example 2.8 itself.

**Theorem 2.17.** If $Int^*(H_A) \subseteq K_A \subseteq H_A$ and if $H_A$ is an $L_gfs$-open set. Then $K_A$ is also an $L_gfs$-open set.
Proof: Let $H_A$ be $L_{gfs}$-open set and $Int^*(H_A) \subseteq K_A \subseteq H_A$. Then $(H_A)^c$ is $L_{gfs}$-closed set and $H_A^c \subseteq K_A^c \subseteq (Int^*(H_A))^c = d^*(H_A^c)$. By Theorem 2.12(ii), $K_A^c$ is $L_{gfs}$-closed set and so $K_A$ is $L_{gfs}$-open set.

We conclude this section by showing that condition under which union of two $L_{gfs}$-open sets is again an $L_{gfs}$-open set.

We have seen that union of two $L_{gfs}$-open sets may not be an $L_{gfs}$-open set. However, if we consider the pair $H_A$ and $K_A$ as fuzzy soft Q-separated set in the sense of Abd El-Latif [1], the following holds true.

Proposition 2.18. If $H_A$ and $K_A$ are two Q-separated $L_{gfs}$-open sets, then their union is an $L_{gfs}$-open set.

Proof. It can be easily established and hence omitted.

Now, to define separated set in this context of fuzzy soft local function we need the following preliminary.

Definition 2.19. [17] In an ideal topological space $(X, \tau, I)$ two non-empty subsets $H, K$ of $X$ are said to be $\ast$-separated if $H^* \cap K = H \cap K^* = H \cap K = \emptyset$.

Definition 2.20. Two non-zero fuzzy soft subsets $H_A, K_A$ of an MFSITS $(X, \tau_1(\tau_2), \tilde{I})$ are called fuzzy soft $\ast$-separated if $H_A^* \cap K_A^* = H_A \cap K_A = \emptyset$.

Using the above definition 2.20 of fuzzy soft $\ast$-separated set, the union of two $L_{gfs}$-open sets may not be an $L_{gfs}$-open set. Nevertheless, we searched that required condition under which the union of two $L_{gfs}$-open sets will be again an $L_{gfs}$-open set.

Theorem 2.21. If $F_A$ and $G_A$ are two fuzzy soft $\ast$-separated $L_{gfs}$-open sets in an MFSITS $X$, then $F_A \cup G_A$ is $L_{gfs}$-open set if both $F_A$ and $G_A$ are $\ast$ dense-in-itself in $X$.

Proof: Let both $F_A$ and $G_A$ be fuzzy soft $\ast$-separated $L_{gfs}$-open sets as well as $\ast$ dense-in-itself in $X$. Let $K_A$ be a fuzzy soft closed set such that $K_A^* \subseteq (F_A \cup G_A)$. Now, $K_A^* \subseteq (F_A \cup G_A)$ implies $K_A^* \cap F_A^* \subseteq (F_A \cup G_A) \cap F_A^* = (F_A \cap F_A^*) \cup (G_A \cap G_A^*) = F_A \cup 0_E = F_A$. Since $K_A \cap F_A^*$ is fuzzy soft closed and $K_A \cap F_A^* \subseteq F_A$ and by the Theorem 2.14, we have $K_A^* \cap F_A^* \subseteq Int^*(F_A)$. Similarly, one can show that $K_A \cap G_A^* \subseteq Int^*(G_A)$.

Now, $K_A = K_A^* \cap (F_A \cup G_A) \subseteq (K_A^* \cap F_A^*) \cup (K_A^* \cap G_A^*)$ (by the given condition of dense-in-itself). This implies that $K_A^* \subseteq Int^*(F_A) \cup Int^*(F_A) \subseteq Int^*(F_A \cup G_A)$ (it follows from Definition 2.11 and Theorem 1.19(iv),(v) and then again by Theorem 2.14, $F_A \cup G_A$ is an $L_{gfs}$-open set.

Corollary 2.22. If $H_A$ and $K_A$ are two $L_{gfs}$-closed sets and if $H_A^*$ and $K_A^*$ are $\ast$-separated fuzzy soft sets, then $H_A \cap K_A$ is $L_{gfs}$-closed set.

3. Some applications of $L_{gfs}$-closed sets

In this section, we introduce generalized fuzzy soft regular, fuzzy soft normal spaces and generalized fuzzy soft compactness in the light of $L_{gfs}$-closed sets.

Definition 3.1 [15] An FSTS $(X, \tau)$ is said to be fuzzy soft regular if for every fuzzy soft point $e(F_A)$ and fuzzy soft closed set $G_A$ not containing $e(F_A)$, there exist disjoint fuzzy soft open sets $U_A$ and $V_A$ such that $e(F_A) \subseteq U_A$, $G_A \subseteq V_A$.

Definition 3.2. Let $X$ be an MFSITS. Then $X$ is said to be $L_{gfs}$-regular if for any fuzzy soft point $e(F_A)$ and for each $L_{gfs}$-closed set $G_A$ in $X$ with $e(F_A) \notin G_A$, there exist fuzzy soft open sets $U_A$ and $V_A$ such that $e(F_A) \subseteq U_A$, $G_A \subseteq V_A$ and $U_A \cap V_A = \emptyset$.

We now give some characterizations of $L_{gfs}$-regularity:
Theorem 3.3. The followings are equivalent in an MFSITS $X$:

(a) $X$ is $L_{gfs}$-regular.
(b) For each $L_{gfs}$-closed set $F_A$ in $X$ and for each fuzzy soft point $e(Q_A)$ with $e(Q_A)\tilde{q}F_A$, there exist fuzzy soft open sets $U_A$ and $V_A$ such that $e(G_A)\subseteq U_A$, $F_A\subseteq V_A$, and $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$.
(c) For each fuzzy soft point $e(Q_A)$ in $X$ and for each $L_{gfs}$-open set $\tilde{U}_A$ containing $e(Q_A)$, there exists a fuzzy soft open set $V_A$ such that $e(Q_A)\subseteq V_A \subseteq \text{cl}(V_A)\subseteq U_A$.

Proof. (a) $\Rightarrow$ (b): Let $F_A$ be any $L_{gfs}$-closed set and $e(Q_A)$, any fuzzy soft point in $X$ such that $e(Q_A)\tilde{q}F_A$. Then by (a) there exist $W_A, V_A \subseteq \tau(\tau_2)$ such that $e(Q_A)\subseteq W_A$, $F_A\subseteq V_A$, and $W_A \ng F_A$. Then, we have $W_A \tilde{q}\text{cl}(V_A)$. Since $cl(F_A)$ is a fuzzy soft closed set, it is $L_{gfs}$-closed set. Again, since $X$ is $L_{gfs}$-regular, so there exist $G_A, H_A \subseteq \tau(\tau_2)$ such that $e(Q_A)\subseteq G_A$, $\text{cl}(V_A)\subseteq H_A$ and $G_A\tilde{q}H_A$ and thus $\text{cl}(G_A)\tilde{q}H_A$.

Let $U_A = W_A \tilde{q}G_A$, then both $U_A, V_A \subseteq \tau(\tau_2)$ such that $e(Q_A)\subseteq U_A$, $F_A\subseteq V_A$, and $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$. In fact, $\text{cl}(U_A) = \text{cl}(W_A \tilde{q}G_A)\subseteq \text{cl}(G_A)$ and $\text{cl}(G_A)\tilde{q}H_A$, we have $\text{cl}(U_A)\tilde{q}H_A$ and as $\text{cl}(V_A)\subseteq H_A$, consequently, we obtain $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$.

(b) $\Rightarrow$ (c): Let $e(Q_A)$ be any fuzzy soft point and $U_A$ be any $L_{gfs}$-open set in $X$ with $e(Q_A)\subseteq U_A$. Then we have $e(Q_A)\tilde{q}U_A$. By the given condition (b), there exist $V_A, W_A \subseteq \tau(\tau_2)$ such that $e(Q_A)\subseteq V_A$, $W_A\subseteq V_A$, and $\text{cl}(V_A)\tilde{q}\text{cl}(W_A)$. This gives $e(Q_A)\subseteq V_A$, $W_A\subseteq V_A$, and $\text{cl}(V_A)\subseteq W_A$. Hence, $e(Q_A)\subseteq V_A\subseteq \text{cl}(V_A)\subseteq W_A\subseteq V_A$. Therefore, we get $e(Q_A)\subseteq V_A\subseteq \text{cl}(V_A)\subseteq U_A$.

(c) $\Rightarrow$ (a): Let $F_A$ be any $L_{gfs}$-closed set and $e(Q_A)$ be any fuzzy soft point in $X$ such that $e(Q_A)\tilde{q}F_A$. Then $e(Q_A)\subseteq F_A$. Since $F_A$ is an $L_{gfs}$-open set in $X$, by the condition (c), there exists $V_A \subseteq \tau(\tau_2)$ such that $e(Q_A)\subseteq V_A$, $W_A\subseteq V_A$, and $\text{cl}(V_A)\subseteq W_A$. Then from this we get $F_A\subseteq (\text{cl}(V_A))^c$. Hence, $e(Q_A)\subseteq V_A\subseteq \text{cl}(V_A)\subseteq W_A\subseteq V_A\subseteq U_A$, eventually $X$ is $L_{gfs}$-regular.

Definition 3.4 [15] An FSTS $(X, \tau)$ is said to be fuzzy soft normal if for every pair of disjoint fuzzy soft closed sets $F_A$ and $G_A$, there exist disjoint fuzzy soft open sets $U_A$ and $V_A$ such that $F_A\subseteq U_A$, $G_A\subseteq V_A$.

Definition 3.5. Let $X$ be an MFSITS. Then $X$ is said to be $L_{gfs}$-normal if for any two $L_{gfs}$-closed sets $F_A, G_A$ in $X$ with $F_A\tilde{q}G_A$, then there exist fuzzy soft open sets $U_A$ and $V_A$ such that $F_A\subseteq U_A$, $G_A\subseteq V_A$, and $U_A\n V_A$.

We now give some characterizations of $L_{gfs}$-normality:

Theorem 3.6. The followings are equivalent in an MFSITS $X$:

(a) $X$ is $L_{gfs}$-normal.
(b) For any two $L_{gfs}$-closed sets $F_A, G_A$ in $X$ with $F_A\tilde{q}G_A$, there exist fuzzy soft open sets $U_A$ and $V_A$ such that $F_A\subseteq U_A$, $G_A\subseteq V_A$, and $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$.
(c) For each $L_{gfs}$-closed set $F_A$ and for each $L_{gfs}$-open set $H_A$ containing $F_A$, there exists a fuzzy soft open set $V_A$ such that $F_A\subseteq V_A\subseteq \text{cl}(V_A)\subseteq H_A$.

Proof. (a) $\Rightarrow$ (b): Let $F_A$ and $G_A$ be two $L_{gfs}$-closed sets in $X$ with $F_A\tilde{q}G_A$. Then by (a) there exist $W_A, V_A \subseteq \tau(\tau_2)$ such that $F_A\subseteq W_A$, $G_A\subseteq V_A$, and $W_A \ng F_A$. Then $F_A\tilde{q}\text{cl}(V_A)$ and $W_A\tilde{q}\text{cl}(V_A)$. Now $\text{cl}(V_A)$ is being fuzzy soft closed, it is $L_{gfs}$-closed set. Thus $F_A$ and $\text{cl}(V_A)$ are two $L_{gfs}$-closed sets in $X$ such that $F_A\tilde{q}\text{cl}(V_A)$. Again, since $X$ is $L_{gfs}$-normal, there exist $G_A, H_A \subseteq \tau(\tau_2)$ such that $F_A\subseteq G_A$, $\text{cl}(V_A)\subseteq H_A$, and $G_A\tilde{q}H_A$. Thus $\text{cl}(G_A)\tilde{q}H_A$.

Let $U_A = W_A\tilde{q}G_A$, then $U_A, V_A \subseteq \tau(\tau_2)$ such that $F_A\subseteq U_A$, $G_A\subseteq V_A$, and $W_A\tilde{q}\text{cl}(V_A)$.

(b) $\Rightarrow$ (c): Let $F_A$ and $H_A$ be any $L_{gfs}$-closed and $L_{gfs}$-open set, respectively in $X$ such that $F_A\subseteq H_A$. Then we have $F_A\tilde{q}H_A$ and $H_A$ is $L_{gfs}$-closed set in $X$ and hence by (b), there exist $U_A, V_A \subseteq \tau(\tau_2)$ such that $F_A\subseteq U_A$, $H_A\subseteq V_A$, and $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$. Now from $H_A\subseteq V_A\subseteq \text{cl}(V_A)$, we have $(\text{cl}(V_A))^c\subseteq H_A$ and $\text{cl}(U_A)\tilde{q}\text{cl}(V_A)$ implies $\text{cl}(U_A)\subseteq (\text{cl}(V_A))^c\subseteq H_A$. Thus, we have $U_A \subseteq \tau(\tau_2)$ such that $F_A \subseteq U_A \subseteq \text{cl}(U_A)\subseteq H_A$, and $U_A$ is the required fuzzy soft open set.

(c) $\Rightarrow$ (a): Let $F_A, G_A$ be two $L_{gfs}$-closed sets with $F_A\tilde{q}G_A$. Then $F_A\subseteq G_A$ and $G_A$ is $L_{gfs}$-open set.
By (c), there exists $U_A \in \tau_1(\tau_2)$ such that $F_A \subseteq U_A \subseteq \text{cl}(U_A) \subseteq G_A^c$ that is, $F_A \subseteq U_A$, $G_A \subseteq (\text{cl}(U_A))^c$ and $U_A \subseteq (\text{cl}(U_A))^c$ and consequently $X$ is $L_{gfs}$-normal.

We now introduce $L_{gfs}$-compactness in an MFSITS $(X, \tau_1(\tau_2), I)$ as follows:

**Definition 3.7.** A collection $\{U_A : i \in \Lambda\}$ of $L_{gfs}$-open sets in an MFSITS $X$ is called $L_{gfs}$-open cover of a fuzzy soft set $F_A$ if $F_A \subseteq \bigcup_{i \in \Lambda} U_A$.

**Definition 3.8.** An MFSITS $X$ is called $L_{gfs}$-compact if every $L_{gfs}$-open cover of $X$ has a finite subcover.

**Definition 3.9.** A fuzzy soft set $F_A$ in an MFSITS $X$ is called $L_{gfs}$-compact relative to $X$ if for every collection $\{U_A : i \in \Lambda\}$ of $L_{gfs}$-open sets of $X$ such that $F_A \subseteq \bigcup_{i \in \Lambda} U_A$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $F_A \subseteq \bigcup_{i \in \Lambda_0} U_A$.

**Theorem 3.10.** Let $F_A$ be an $L_{gfs}$-closed subset of an $L_{gfs}$-compact space $X$. Then $F_A$ is $L_{gfs}$-compact.

**Proof:** Let $F_A$ be an $L_{gfs}$-closed subset of $X$ such that $F_A \subseteq \bigcup_{i \in \Lambda} U_A$, where $U_A$’s are $L_{gfs}$-open sets for each $i \in \Lambda$. Then $\{U_A : i \in \Lambda\} \cup \{F_A^c\}$ is an $L_{gfs}$-open cover of $X$. Now, $X$ being $L_{gfs}$-compact space, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $X = \bigcup_{i \in \Lambda_0} \bigcup \{F_A^c\}$. As $F_A$ and $F_A^c$ are disjoint, hence we have $F_A \subseteq \bigcup_{i \in \Lambda_0} U_A$. This shows that $F_A$ is $L_{gfs}$-compact relative to $X$ and thus $F_A$ is $L_{gfs}$-compact.

**Theorem 3.11.** If $F_A$ and $G_A$ are two fuzzy soft subsets of an MFSITS $X$ such that $F_A$ is $L_{gfs}$-compact and $G_A$ is $L_{gfs}$-closed in $X$. Then $F_A \cap G_A$ is $L_{gfs}$-compact.

**Proof:** Let $\{U_A : i \in \Lambda\}$ be a $L_{gfs}$-open cover of $F_A \cap G_A$ in $X$ and as $G_A^c$ is $L_{gfs}$-open set, $\{U_A : i \in \Lambda\} \cup \{G_A^c\}$ is a $L_{gfs}$-open cover of $X$. Now $F_A$ being $L_{gfs}$-compact, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $F_A \subseteq \bigcup_{i \in \Lambda_0} \bigcup \{G_A^c\}$. This implies that $F_A \cap G_A \subseteq \bigcup_{i \in \Lambda_0} \bigcup \{G_A^c\}$. Thus $F_A \cap G_A$ is $L_{gfs}$-compact.

**Theorem 3.12.** Every $L_{gfs}$-closed subset $F_A$ of a fuzzy soft compact space $X$ is fuzzy soft compact if $F_A$ is $*$-dense-in-itself in $X$.

**Proof:** Let $F_A \subseteq \bigcup_{i \in \Lambda} U_A$, where $U_A$ is fuzzy open, for each $i \in \Lambda$. Then $F_A^c \subseteq \bigcup_{i \in \Lambda} U_A$, as $F_A$ is $L_{gfs}$-closed set in $X$. Since $F_A^c$ is fuzzy soft closed in $X$, it is fuzzy soft compact. Then $F_A^c \subseteq \bigcup_{i \in \Lambda} U_A$, and consequently $F_A \subseteq F_A^c \subseteq \bigcup_{i=1}^n U_A$. This establishes that $F_A$ is fuzzy soft compact.

**Theorem 3.13.** Every $L_{gfs}$-compact subset $F_A$ of an $L_{gfs}$-regular space $X$ is always an $L_{gfs}$-closed set.

**Proof:** Let $F_A$ be a $L_{gfs}$-compact set with $F_A \subseteq U_A$, where $U_A \in \tau_1(\tau_2)$. Let $e(G_A)$ be any fuzzy soft point in $X$ such that $e(G_A) \subseteq U_A$. Since $U_A$ is fuzzy soft open set, it is $L_{gfs}$-open set. By Theorem 3.3(c), $L_{gfs}$-regularity of $X$ implies that there exists fuzzy soft open set $V_A$ such that $e(G_A) \subseteq V_A \subseteq \text{cl}(V_A) \subseteq U_A$ and hence we have $F_A \subseteq \bigcup_{i \in A} V_A \subseteq \text{cl}(\bigcup_{i \in A} V_A) \subseteq U_A$, for $e(G_A) \in F_A$, and where each $V_A$ is $L_{gfs}$-open sets (as $V_A$ being fuzzy soft open set). Now, since $F_A$ is an $L_{gfs}$-compact set, exists a finite subset $\Lambda_0$ of $\Lambda$ such that $F_A \subseteq \bigcup_{i \in \Lambda_0} V_A \subseteq \text{cl}(\bigcup_{i \in \Lambda_0} V_A) \subseteq U_A$. Let $V_A = \bigcup_{i \in A} V_A$, then $V_A$ is fuzzy soft open set and $F_A \subseteq V_A \subseteq \text{cl}(V_A) \subseteq U_A$. Since $\text{cl}(F_A)$ is the smallest fuzzy soft closed set and $F_A \subseteq \text{cl}(F_A)$, we get $F_A \subseteq \text{cl}(F_A) \subseteq \text{cl}(V_A) \subseteq U_A$ and so $F_A \subseteq U_A$, this establishes that $F_A$ is $L_{gfs}$-closed set.

**Remark 3.14.** Every fuzzy soft compact subset of an $L_{gfs}$-regular space $X$ is always an $L_{gfs}$-closed set.

**Proof:** Using the above theorem 3.13, it can be proved the theorem easily and thus omitted.
4. Conclusion

In this article, a new type of generalized closed sets is introduced, whose characteristics are different up to some extent from the conventional ones. We have applied $L_{gfs}$-closed set to study few of the separation axioms and fuzzy soft compactness also. We have traditionally found that a generalized closed subset of a compact topological space is compact but in our context $L_{gfs}$-closed subset of a fuzzy soft compact space may not be a fuzzy soft compact in general. We emphasize that this traditional result will come true under certain conditions contained in this report.

Acknowledgments

The authors would like to thank the referees for their careful reading of the manuscript and valuable suggestions.

References


Md Mirazul Hoque,
Department of Mathematics,
NIT Agartala, Pin-799046,
India.
E-mail address: hoquemirazul.2012@gmail.com

and

Jayasree Chakraborty,
Department of Mathematics,
NIT Agartala, Pin-799046,
India.
E-mail address: chakrabortyjayasree1@gmail.com

and

Baby Bhattacharya,
Department of Mathematics,
NIT Agartala, Pin-799046,
India.
E-mail address: babybhatt75@gmail.com

and

Binod Chandra Tripathy,
Department of Mathematics,
Tripura University, Tripura, Pin-799022,
India.
E-mail address: tripathybc@yahoo.com