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Global Well-posedness and Exponential Stability Results of Bresse-Timoshenko System-type with Second Sound and Distributed Delay Term

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ABSTRACT: In this paper, we investigate a Bresse-Timoshenko-type system with a distributed delay term and second sound. Under suitable assumptions, we establish the global well-posedness of the initial and boundary value problem by using the Faedo-Galerkin approximations and some energy estimates.

Key Words: Bresse-Timoshenko-type systems, distributed delay term, second sound, well-posedness, exponential decay, Faedo-Galerkin approximations.

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1. Introduction

In engineering practice, questions about understanding the characteristics of natural vibrations of such coupled systems occur when tackling problems involving the dynamics of composite mechanical structures with various types of connections. When we talk about the combination of elements, the behavior of which is defined by equations of various types, we are talking about non-classical problems in mathematical physics. As a result of the difficulty in solving them, models of real structures are used in practice, which are simplified by incorporating extra hypotheses and assumptions.

Let us mention some references to the mechanics of technical structures and non-classical problems in mathematical physics [7], [8], [18], [19]. A new type of problem arises with the combination of the Timoshenko system [22] and Bresse system or the curved beam [10]. The coupled system from which we derive Bress-Timoshenko is derived from Elishakov [12] and combines D'Alembert's principle of dynamic equilibrium with Timoshenko's hypothesis to produce the following coupled system

$$\begin{cases} \rho_1 \partial_{tt}\varphi - \kappa\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2 \partial_{tt}\varphi_x - b\psi_{xx} + \kappa\left(\varphi_x + \psi\right) = 0. \end{cases} \tag{1.1}$$

The Cattaneo's law is one of the most well-known thermoelasticity laws, but it is unable to account for some physical properties and cannot answer all questions; therefore, its applications are limited. This leads us to consider coupling the fields of strain, temperature, and mass diffusion using the Gurtin-Pinkin model. Only a few authors have studied the stabilization of the Bress-Timoshenko model.

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Manevich and Kolakowski [21] obtained the first contribution in that direction. where They analyzed the dynamics of a Timoshenko model. in which,the damping mechanism is viscoelastic. More accurately, they considered the dissipative system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x+\psi)_x - \mu_1(\varphi_x+\psi)_{tx} = 0, \\ -\rho_2\psi_{tt} - b\psi_{xx} + \beta(\varphi_x+\psi) - \mu_2\psi_{ttx} + \mu_1(\varphi_x+\psi)_t = 0. \end{cases} \tag{1.2}$$

Second, based on Elishakoff's papers and collaborators and their studies on truncated versions for classical Timoshenko equations [1], Almeida Junior and Ramos [2] proved that the total energy for viscous damping acting on angle rotation of the simplified Timoshenko system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x+\psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x+\psi) + \mu_1\psi_t = 0, \end{cases} \tag{1.3}$$

There is a great difference in the model from a classical Timoshenko system, as it is consisted of three derivatives: two with respect to time and one with respect to space. This happened because the absence of the second spectrum, or nonphysical spectrum [1], [12], and its damage consequences for wave propagation speeds [2]. The historical and mathematical observations can be found in other works [1], [12]. The same results are accomplished for a dissipative truncated version, where the viscous damping acts on vertical displacement

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x+\psi)_x = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x+\psi) = 0. \end{cases} \tag{1.4}$$

We indicate some related work, [14] in this work, Guesmia and Soufyane studied the well posedness and proved the stability for the following system

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + \lambda_1\varphi_t + \mu_1\varphi_t(x,t-\tau_1) = 0, \\ \rho_2\varphi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) + \lambda_2\psi_t + \mu_2\psi_t(x,t-\tau_2) = 0. \end{cases} \tag{1.5}$$

In [6], the authors proved the well-posedness and establish uniform stability results for the following linear Timoshenko system with a linear frictional damping and an internal distributed delay acting on the transverse displacement

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \gamma_1\varphi_t + \int_{\tau_1}^{\tau_2}\gamma_2\varphi_t(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) = 0. \end{cases} \tag{1.6}$$

In [5],the authors proved the well-posedness and established exponential stability results regardless of the speeds of wave propagation for the following thermoelastic system of Timoshenko type with a linear frictional damping and an internal distributed delay acting on the displacement equation

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\varphi_t(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases} \tag{1.7}$$

In [15],they established the stability of the following Timoshenko-type system

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x+\psi)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\varphi_t(x,t-s)\,ds = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \int_0^t g(t-s)\,(a(x)\,\psi_x)_x\,ds \\ \quad + \mu_3(t)\,b(x)\,f(\psi_t) + \delta\theta_x = 0, \\ \rho_3\theta_t + kq_x + \delta\psi_{tx} = 0, \\ \tau q_t + \beta q + k\theta_x = 0. \end{cases} \tag{1.8}$$

In [3], Almeida Junior et al deemed two cases of dissipative systems for Bresse-Timoshenko-type systems with constant delay cases. For the first one, the authors established the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta(\varphi_x+\psi)_x + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau) = 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta(\varphi_x+\psi) = 0. \end{cases} \tag{1.9}$$

For the second one, the authors also studied the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x \ = \ 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) + \mu_1\varphi_t + \mu_2\psi_t\left(x, t-\tau\right) \ = \ 0, \end{cases} \tag{1.10}$$

The authors in [13] deemed two cases of dissipative systems for Bress-Timoshenko-type systems with time-varying delay cases. For the first one, the authors proved the exponential decay for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t-\tau\left(t\right)\right) \ = \ 0, \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) \ = \ 0, \end{cases} \tag{1.11}$$

for the second one, the authors also established the exponential decay result for the system given by

$$\begin{cases} \rho_1\varphi_{tt} - \beta\left(\varphi_x + \psi\right)_x \ = \ 0 \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + \beta\left(\varphi_x + \psi\right) + \mu_1\varphi_t + \mu_2\psi_t\left(x, t-\tau\left(t\right)\right) \ = \ 0 \end{cases} \tag{1.12}$$

In [11], They used the Faedo-Galerkin approximations and some energy estimates to establish the global well-posedness of the initial and boundary value problem, and they proved the exponential decay of dissipative systems for the following Bresse-Timoshenko-type systems with distributed delay, under appropriate assumptions

$$\begin{cases} \rho_1\varpi_{tt} - K\left(\varpi_x + v\right)_x + \mu_1\varpi_t + \int_{\tau_1}^{\tau_2}\mu_2\left(p\right)\varpi_t\left(x, t-p\right)dp \ = \ 0, \\ -\rho_2\varpi_{ttx} - bv_{xx} + K\left(\varpi_x + v\right) \ = \ 0. \end{cases} \tag{1.13}$$

See other works in [9], [23].

We escort the paper of [11] but in this present work we consider the following Bresse-Timoshenko system of second sound with distrubted delay term,

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x + \mu_1\varphi_t + \int_{\tau_1}^{\tau_2}\mu_2\left(s\right)\varphi_t\left(x, t-s\right)ds \ = \ 0, \text{ in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{ttx} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \gamma\theta_x \ = \ 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} \ = \ 0, \text{ in } (0,1)\times(0,\infty), \\ \tau_0 q_t + \delta q + \kappa\theta_x \ = \ 0, \text{in } (0,1)\times(0,\infty), \\ \varphi(x,0) \ = \ \varphi_0\left(x\right), \varphi_t(x,0) \ = \ \varphi_1\left(x\right), \psi(x,0) \ = \ \psi_0\left(x\right), \\ \psi_t(x,0) \ = \ \psi_1\left(x\right), \text{ in } (0,1), \\ \theta\left(x,0\right) = \theta_0\left(x\right), \ q\left(x,0\right) = q_0\left(x\right), \\ \psi_t(x,-t) = f_0\left(x,t\right), \text{ in } (0,1)\times(0,\tau_2), (0,1)\times(0,\infty), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = \theta(0,t) = \theta(1,t) = 0, \ \forall t \geq 0, \end{cases} \tag{1.14}$$

where $t \in (0,\infty)$ denotes the time variable and $x \in (0,1)$ is the space variable, the functions φ and ψ are respectively, the transverse displacement of the solid elastic material and the rotation angle, the function θ is the temperature difference, $q = q(t,x) \in \mathbb{R}$ is the heat flux, and ρ_1, ρ_2, ρ_3, γ, τ_0, δ, κ, μ_1, and K are positive constants, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds < \mu_1, \tag{1.15}$$

where τ_1 and τ_2 two real numbers satisfying $0 \leq \tau_1 \leq \tau_2$, and we study exponential stability results and the global well-posednes of a class of Bresse-Timoshenko system-type of second sound with distributed delay term.

This article is organized as follows: In section 1, we introduce some transformations needed in our work, In section 2, we use Faedo-Galerkin approximations to study the global well-posedness of the initial and boundary value problems, and in section 3, we show the exponential decay of dissipative systems for Bresse-Timoshenko-type second sound systems with distributed delay.

2. Preliminaries

Let us introduce a new dependent variable

$$y(x,\rho,s,t)=\varphi_t(x,t-\rho s) \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \tag{2.1}$$

then, we obtain

$$sy_t(x,\rho,s,t)+y_\rho(x,\rho,s,t)=0. \tag{2.2}$$

Consequently, the problem (1.14) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt}-K(\varphi_x+\psi)_x+\mu_1\varphi_t+\int_{\tau_1}^{\tau_2}\mu_2(s)\,y(x,1,s)\,ds=0, \text{ in } (0,1)\times(0,\infty), \\ -\rho_2\varphi_{ttx}-b\psi_{xx}+K(\varphi_x+\psi)+\gamma\theta_x=0, \text{ in } (0,1)\times(0,\infty), \\ \rho_3\theta_t+\kappa q_x+\gamma\psi_{tx}=0, \text{ in } (0,1)\times(0,\infty), \\ \tau_0 q_t+\delta q+\kappa\theta_x=0, \text{ in } (0,1)\times(0,\infty), \\ sy_t(x,\rho,s,t)+y_\rho(x,\rho,s,t)=0, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ sy_{tt}(x,\rho,s,t)+y_{\rho t}(x,\rho,s,t)=0, \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \\ \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x), \ \psi(x,0)=\psi_0(x) \\ , \psi_t(x,0)=\psi_1(x), \text{ in } (0,1), \\ \theta(x,0)=\theta_0(x), \ q(x,0)=q_0(x), \text{ in } (0,1), \\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=q(0,t)=q(1,t)=\theta(0,t)=\theta(1,t)=0, \ \forall t\geq 0, \\ y(x,\rho,s,0)=f_0(x,-\rho s), \text{ in } (0,1)\times(0,1)\times(\tau_1,\tau_2). \end{cases} \tag{2.3}$$

Let $\mathcal{V}(Q:=(0,1)\times(0,\infty))$ and $\mathcal{W}(Q)$ be the set spaces defined respectively by

$$\mathcal{V}(Q):=\left\{ \begin{array}{c} (\varphi,\psi,\theta,q,y):\varphi\in L^2\left(\mathbb{R}_+,H^2\cap H_0^1\right), \ \varphi_t\in L^2\left(\mathbb{R}_+,H^1\right), \ \varphi_{tt}\in L^2\left(\mathbb{R}_+,H^1\right), \\ \psi\in L^2\left(\mathbb{R}_+,H_0^1\right), \ \psi_{xt}\in L^2\left(\mathbb{R}_+,L^2\right), \ \theta, \ q\in L^2\left(\mathbb{R}_+,H_0^1\right), \\ \theta_t, \ q_t\in L^2\left(\mathbb{R}_+,L^2\right), \ y\in L^2\left(\mathbb{R}_+,H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)\right), \\ y_t\in L^2\left(\mathbb{R}_+,H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)\right), \end{array} \right\},$$

and

$$\mathcal{W}(Q):=\left\{(\varphi,\psi,\theta,q,y)\in\mathcal{V}(Q): \ \lim_{T\to\infty}w_l(T)=\lim_{T\to\infty}s_l(T)=\lim_{T\to\infty}v_l(T)=\lim_{T\to\infty}r_l(T)=\lim_{T\to\infty}p_l(T)=0\right\}.$$

Consider the system

$$\begin{cases} \rho_1(\varphi_{tt},u)+K((\varphi_x+\psi),u_x)+\mu_1(\psi_t,u) \\ +\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,y(x,1,s)\,ds,u\right)=0, \\ \rho_2(\varphi_{tt},v_x)+b(\psi_x,v_x)+K((\varphi_x+\psi),v) \\ +\gamma(\theta_x,v)=0, \\ \rho_3(\theta_t,w)+(q_x,w)+\gamma(\psi_{tx},w)=0, \\ \tau_0(q_t,z)+\delta(q,z)+\kappa(\theta_x,z)=0, \\ s(y_t(x,\rho,s,t),\phi)+(y_\rho(x,\rho,s,t),\phi)=0, \\ s(y_{tt}(x,\rho,s,t),\phi)+(y_{\rho t}(x,\rho,s,t),\phi)=0, \end{cases} \tag{2.4}$$

where $(.,.)_{L^2(Q)}$ stands for the inner product in $L^2(Q)$, $(\varphi,\psi,\theta,q,y)$ is supposed to be a solution of the system (2.3) and $(u,v,w,z,\phi)\in\mathcal{W}(Q)$. Evaluation of the inner product in (2.4) and use of the Dirichlet conditions $(2.3)_8$ leads to

$$\begin{aligned}
&-\rho_1(\varphi_t,u_t)_{L^2(Q)}-\rho_1(\varphi_t(x,0),u(x,0))_{L^2(0,1)}+K((\varphi_x+\psi),u_x)_{L^2(Q)}\\
&-\mu_1(\psi,u_t)_{L^2(Q)}-\mu_1(\psi(x,0),u(x,0))_{L^2(0,1)}+\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,y(x,1,s)\,ds,u\right)_{L^2(Q)} &=&\ 0,\\
&-\rho_2(\varphi_t,v_{xt})_{L^2(Q)}-\rho_2(\varphi_t(x,0),v_x(x,0))_{L^2(0,1)}+b(\psi_x,v_x)_{L^2(Q)}\\
&+K((\varphi_x+\psi),v)_{L^2(Q)}+\gamma(\theta_x,v)_{L^2(Q)} &=&\ 0,\\
&-\rho_3(\theta,w_t)_{L^2(Q)}-\rho_3(\theta(x,0),w(x,0))_{L^2(0,1)}+(q_x,w)_{L^2(Q)}-\gamma(\psi_x,w_t)_{L^2(Q)}\\
&-\gamma(\psi_x(x,0),w(x,0))_{L^2(0,1)} &=&\ 0,\\
&-\tau_0(q,z_t)_{L^2(Q)}-\tau_0(q,z)_{L^2(0,1)}+\delta(q,z)_{L^2(Q)}+\kappa(\theta_x,z)_{L^2(Q)} &=&\ 0,\\
&-s(y(x,\rho,s,t),\phi_t)_{L^2(Q)}-s(y(x,\rho,s,0),\phi(x,\rho,s,0))_{L^2(0,1)}+(y_\rho(x,\rho,s,t),\phi)_{L^2(Q)} &=&\ 0,\\
&-s(y_t(x,\rho,s,t),\phi_t)_{L^2(Q)}-s(y_t(x,\rho,s,0),\phi(x,\rho,s,0))_{L^2(0,1)}+(y_{\rho t}(x,\rho,s,t),\phi) &=&\ 0.
\end{aligned}\tag{2.5}$$

Definition 2.1. *functions $(\varphi,\psi,\theta,q,y)\in\mathcal{V}(Q)$ are called a generalized solution of system (2.3) if it satisfies (2.5) for each $(u,v,w,z,\phi)\in\mathcal{W}(Q)$.*

3. Well-posedness of problem

In this section, we will prove the global existence and the uniqueness of the generalized solution of system (2.3) by using the classical Faedo-Galerkin method

Theorem 3.1. *If $\varphi_0\in H_0^1(0,1)\cap H^2(0,1)$, $\varphi_1\in H^1(0,1)$, $\psi_0,\ q_0,\ \theta_0\in H_0^1(0,1)$, $\psi_1\in L^2(0,1)$, $f_0\in H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)$, and $f_1\in H^1\left((0,1)^2\times(\tau_1,\tau_2)\right)$, then there is at least one generalized solution in $\mathcal{V}(Q)$ to system.(2.3).*

By using Faedo-Galerkin approximations, we prove the global existence and the uniqueness of the generalized solution of system (2.3). for more detail, we refer to reader to see [4], [16], [13].

3.1. Approximate problem

let $\{u_j\},\{v_j\},\{w_j\},\{z_j\}$ be the Galerkin basis, For $m\geq 1$, let

$$\begin{aligned}
L_m &= span\{u_1,u_2,...,u_n\},\\
\Gamma_m &= span\{v_1,v_2,...,v_n\},\\
W_m &= span\{w_1,w_2,...,w_n\},\\
K_m &= span\{z_1,z_2,...,z_n\},
\end{aligned}\tag{3.1}$$

we define for $1\leq j\leq n$ the sequence $\phi_j(x,\rho,s)$ by

$$\phi_j(x,0,s)=u_j(x),\tag{3.2}$$

then, we can extend $\phi_j(x,0,s)$ by $\phi_j(x,\rho,s)$ over $L^2\left((0,1)^2\times(\tau_1,\tau_2)\right)$ and denote $Z^m=span\{\phi_1,...,\phi_n\}$. Given initial data $\varphi_0\in H_0^1(0,1)\cap H^2(0,1)$, $\varphi_1\in H^1(0,1)$, $\psi_0,\ q_0,\ \theta_0\in H_0^1(0,1)$, $\psi_1\in L^2(0,1)$,

$f_0,\ f_1 \in H^1\left((0,1)^2 \times (\tau_1,\tau_2)\right)$ define the approximations

$$\begin{aligned}
\varphi_m &= \sum_{j=1}^{n} \xi_{jm}(t)\, u_j(x), \\
\psi_m &= \sum_{j=1}^{n} k_{jm}(t)\, v_j(x), \\
\theta_m &= \sum_{j=1}^{n} l_{jm}(t)\, w_j(x), \\
q_m &= \sum_{j=1}^{n} f_{jm}(t)\, z_j(x), \\
y_m &= \sum_{j=1}^{n} h_{jm}(t)\, \phi_j(x,\rho,s),
\end{aligned} \tag{3.3}$$

where the constants $\xi_{jm}(t)$, $k_{jm}(t)$, $l_{jm}(t)$, $f_{jm}(t)$, and $h_{jm}(t)$ are defined by the conditions

$$\begin{aligned}
\xi_{jm}(t) &= (\varphi_m, u_j(x))_{L^2(0,1)}, \\
k_{jm}(t) &= (\psi_m, v_j(x))_{L^2(0,1)}, \\
l_{jm}(t) &= (\theta_m, w_j(x))_{L^2(0,1)}, \\
f_{jm}(t) &= (q_m, z_j(x))_{L^2(0,1)}, \\
h_{jm}(t) &= \left(y_m, \phi_j(x,\rho,s)\right)_{L^2(0,1)},
\end{aligned} \tag{3.4}$$

and can be determined from the relations

$$\left\{\begin{array}{l}
\rho_1(\varphi_{mtt}, u_l) + K((\varphi_{mx}+\psi_m), u_{lx}) + \mu_1(\psi_{mt}, u_l) \\
+\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_m(x,1,s)\,ds, u_l\right) = 0, \\
\rho_2(\varphi_{mtt}, v_{lx}) + b(\psi_{mx}, v_{lx}) + K((\varphi_{mx}+\psi_m), v_l) \\
+\gamma(\theta_{mx}, v_l) = 0, \\
\rho_3(\theta_{mt}, w_l) + (q_{mx}, w_l) + \gamma(\psi_{mtx}, w_l) = 0, \\
\tau_0(q_{mt}, z_l) + \delta(q_m, z_l) + \kappa(\theta_{mx}, z_l) = 0, \\
s(y_{mt}(x,\rho,s), \phi_l(x,\rho,s)) + (y_{m\rho}(x,\rho,s), \phi_l(x,\rho,s)) = 0, \\
s(y_{mtt}(x,\rho,s), \phi_l(x,\rho,s)) + (y_{m\rho t}(x,\rho,s), \phi_l(x,\rho,s)) = 0,
\end{array}\right. \tag{3.5}$$

substitution of (3.3) into (3.5) gives for $l = 1,..,n$

$$\left\{\begin{array}{l}
\int_0^1 \sum_{j=1}^n \left\{\rho_1 \xi_{jm}''(t)\, u_j(x)\, u_l(x) + K\xi_{jm}(t)\, u_{jx}(x)\, u_{lx}(x)\right. \\
+K k_{jm}(t)\, v_j(x)\, u_{lx}(x) + \mu_1 k_{jm}'(t)\, v_j(x)\, u_l(x) \\
\left.+u_l(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\, h_{jm}(t)\,\phi_j(x,\rho,s)\,ds\right\} dx = 0, \\
\int_0^1 \sum_{j=1}^n \left\{\rho_2 \xi_{jm}''(t)\, u_j(x)\, v_{lx}(x) + b k_{jm}(t)\, v_{jx}(x)\, v_{lx}(x)\right. \\
+K\xi_{jm}(t)\, u_{jx}(x)\, v_l(x) + K k_{jm}(t)\, v_j(x)\, v_l(x) \\
\left.+\gamma l_{jm}(t)\, w_{jx}(x)\, v_l(x)\right\} dx = 0, \\
\int_0^1 \sum_{j=1}^n \left\{\rho_3 l_{jm}'(t)\, w_j(x)\, w_l(x) + \kappa f_{jm}(t)\, z_{jx}(x)\, w_l(x)\right. \\
\left.+\gamma k_{jm}'(t)\, v_{jx}(x)\, w_l(x)\right\} dx = 0, \\
\int_0^1 \sum_{j=1}^n \left\{\tau_0 f_{jm}'(t)\, z_j(x)\, z_l(x) + \delta f_{jm}(t)\, z_j(x)\, z_l(x)\right. \\
\left.+\kappa l_{jm}(t)\, w_{jx}(x)\, z_l(x)\right\} dx = 0, \\
\int_0^1 \sum_{j=1}^n \left\{s h_{jm}'(t)\,\phi_j(x,\rho,s)\,\phi_l(x,\rho,s) + h_{jm\rho}(t)\,\phi_j(x,\rho,s)\,\phi_l(x,\rho,s)\right\} dx = 0, \\
\int_0^1 \sum_{j=1}^n \left\{s h_{jm}''(t)\,\phi_j(x,\rho,s)\,\phi_l(x,\rho,s) + h_{jm\rho}(t)\,\phi_j(x,\rho,s)\,\phi_l(x,\rho,s)\right\} dx = 0.
\end{array}\right. \tag{3.6}$$

From (3.6), it follows that

$$\left\{\begin{array}{l}\sum_{j=1}^{n}\left\{\rho_1\xi_{jm}^{''}(t)\left(u_j(x),u_l(x)\right)_{L^2(0,1)}+K\xi_{jm}(t)\left(u_{jx}(x),u_{lx}(x)\right)_{L^2(0,1)}\right.\\ +Kk_{jm}(t)\left(v_j(x),u_{lx}(x)\right)_{L^2(0,1)}+\mu_1k_{jm}^{'}(t)\left(v_j(x),u_l(x)\right)_{L^2(0,1)}\\ \left.+\int_{\tau_1}^{\tau_2}\mu_2(s)\,h_{jm}(t)\left(\phi_j(x,\rho,s),u_l(x)\right)_{L^2(0,1)}ds\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_2\xi_{jm}^{''}(t)\left(u_j(x),v_{lx}(x)\right)_{L^2(0,1)}+bk_{jm}(t)\left(v_{jx}(x),v_{lx}(x)\right)_{L^2(0,1)}\right.\\ +K\xi_{jm}(t)\left(u_{jx}(x),v_l(x)\right)_{L^2(0,1)}+Kk_{jm}(t)\left(v_j(x)\,v_l(x)\right)_{L^2(0,1)}\\ \left.+\gamma l_{jm}(t)\left(w_{jx}(x),v_l(x)\right)_{L^2(0,1)}\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_3l_{jm}^{'}(t)\left(w_j(x),w_l(x)\right)_{L^2(0,1)}+\kappa f_{jm}(t)\left(z_{jx}(x),w_l(x)\right)_{L^2(0,1)}\right.\\ \left.+\gamma k_{jm}^{'}(t)\left(v_{jx}(x),w_l(x)\right)_{L^2(0,1)}\right\}=0,\\ \sum_{j=1}^{n}\left\{\tau_0f_{jm}^{'}(t)\left(z_j(x),z_l(x)\right)_{L^2(0,1)}+\delta f_{jm}(t)\left(z_j(x),z_l(x)\right)_{L^2(0,1)}\right.\\ \left.+\kappa l_{jm}(t)\left(w_{jx}(x),z_l(x)\right)_{L^2(0,1)}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{'}(t)\left(\phi_j(x,\rho,s),\phi_l(x,\rho,s)\right)_{L^2(0,1)}+h_{jm\rho}(t)\left(\phi_j(x,\rho,s),\phi_l(x,\rho,s)\right)_{L^2(0,1)}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{''}(t)\left(\phi_j(x,\rho,s),\phi_l(x,\rho,s)\right)_{L^2(0,1)}+h_{jm\rho}^{'}(t)\left(\phi_j(x,\rho,s),\phi_l(x,\rho,s)\right)_{L^2(0,1)}\right\}=0.\end{array}\right. \tag{3.7}$$

Let

$$\begin{aligned}
\left(u_j(x),u_l(x)\right)_{L^2(0,1)} &= \delta_{jl} = \left\{\begin{array}{l}1,\ j=l\\ 0,\ j\neq l\end{array}\right., \\
\left(u_{jx}(x),u_{lx}(x)\right)_{L^2(0,1)} &= \gamma_{jl}, \\
\left(v_j(x),u_{lx}(x)\right)_{L^2(0,1)} &= \sigma_{jl}, \\
\left(u_j(x),v_{lx}(x)\right)_{L^2(0,1)} &= a_{jl} \\
\left(v_{jx}(x),v_{lx}(x)\right)_{L^2(0,1)} &= \vartheta_{jl}, \\
\left(u_{jx}(x),v_l(x)\right)_{L^2(0,1)} &= \nu_{jl}, \\
\left(v_j(x),u_l(x)\right)_{L^2(0,1)} &= \varsigma_{jl}, \\
\left(v_j(x),v_l(x)\right)_{L^2(0,1)} &= \delta_{jl} = \left\{\begin{array}{l}1,\ j=l\\ 0,\ j\neq l\end{array}\right., \\
\left(w_{jx}(x),v_l(x)\right)_{L^2(0,1)} &= \chi_{jl}, \\
\left(\phi_j(x,1,s),u_l(x)\right)_{L^2(0,1)} &= \omega_{jl}, \\
\left(w_j(x),w_l(x)\right)_{L^2(0,1)} &= \delta_{jl} = \left\{\begin{array}{l}1,\ j=l\\ 0,\ j\neq l\end{array}\right., \\
\left(z_{jx}(x),w_l(x)\right)_{L^2(0,1)} &= \varrho_{jl}, \\
\left(v_{jx}(x),w_l(x)\right)_{L^2(0,1)} &= \alpha_{jl}, \\
\left(z_j(x),z_l(x)\right)_{L^2(0,1)} &= \delta_{jl} = \left\{\begin{array}{l}1, j=l\\ 0, j\neq l\end{array}\right., \\
\left(w_{jx}(x),z_l(x)\right)_{L^2(0,1)} &= \beta_{jl}, \\
\left(\phi_j(x,\rho,s),\phi_l(x,\rho,s)\right)_{L^2(0,1)} &= \delta_{jl} = \left\{\begin{array}{l}1,\ j=l\\ 0,\ j\neq l\end{array}\right..
\end{aligned}$$

Then (3.7) can be written as

$$\begin{cases} \sum_{j=1}^{n}\left\{\rho_1\xi_{jm}^{''}(t)\,\delta_{jl}+K\xi_{jm}(t)\,\gamma_{jl,}+Kk_{jm}(t)\,\sigma_{jl}+\mu_1 k_{jm}^{'}(t)\,\varsigma_{jl}\right.\\ \left.+\int_{\tau_1}^{\tau_2}\mu_2(s)\,h_{jm}(t)\,\omega_{jl}ds\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_2\xi_{jm}^{''}(t)\,a_{jl}+bk_{jm}(t)\,\vartheta_{jl}+K\xi_{jm}(t)\,\nu_{jl}+Kk_{jm}(t)\,\delta_{jl}\right.\\ \left.+\gamma l_{jm}(t)\,\chi_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_3 l_{jm}^{'}(t)\,\delta_{jl}+\kappa f_{jm}(t)\,\varrho_{jl}+\gamma k_{jm}^{'}(t)\,\alpha_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{\tau_0 f_{jm}^{'}(t)\,\delta_{jl}+\delta f_{jm}(t)\,\delta_{jl}+\kappa l_{jm}(t)\,\beta_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{'}(t)\,\delta_{jl}+h_{jm\rho}(t)\,\delta_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{''}(t)\,\delta_{jl}+h_{jm\rho}^{'}(t)\,\delta_{jl}\right\}=0. \end{cases} \tag{3.8}$$

We put $\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds=c$, we obtain

$$\begin{cases} \sum_{j=1}^{n}\left\{\rho_1\xi_{jm}^{''}(t)\,\delta_{jl}+K\xi_{jm}(t)\,\gamma_{jl,}+Kk_{jm}(t)\,\sigma_{jl}+\mu_1 k_{jm}^{'}(t)\,\varsigma_{jl}\right.\\ \left.+ch_{jm}(t)\,\omega_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_2\xi_{jm}^{''}(t)\,a_{jl}+bk_{jm}(t)\,\vartheta_{jl}+K\xi_{jm}(t)\,\nu_{jl}+Kk_{jm}(t)\,\delta_{jl}\right.\\ \left.+\gamma l_{jm}(t)\,\chi_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{\rho_3 l_{jm}^{'}(t)\,\delta_{jl}+\kappa f_{jm}(t)\,\varrho_{jl}+\gamma k_{jm}^{'}(t)\,\alpha_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{\tau_0 f_{jm}^{'}(t)\,\delta_{jl}+\delta f_{jm}(t)\,\delta_{jl}+\kappa l_{jm}(t)\,\beta_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{'}(t)\,\delta_{jl}+h_{jm\rho}(t)\,\delta_{jl}\right\}=0,\\ \sum_{j=1}^{n}\left\{sh_{jm}^{''}(t)\,\delta_{jl}+h_{jm\rho}^{'}(t)\,\delta_{jl}\right\}=0, \end{cases} \tag{3.9}$$

with

$$\begin{aligned} \xi_{jm}(0) &= (\varphi_m(x,0)\,,u_j(x))_{L^2(0,1)}\,,\\ \xi_{jm}^{'}(0) &= (\varphi_{mt}(x,0)\,,u_j(x))_{L^2(0,1)}\,,\\ k_{jm}(0) &= (\psi_m(x,0)\,,v_j(x))_{L^2(0,1)}\,,\\ l_{jm}(0) &= (\theta_m(x,0)\,,w_j(x))_{L^2(0,1)}\,,\\ f_{jm}(0) &= (q_m(x,0)\,,z_j(x))_{L^2(0,1)}\,,\\ h_{jm}(0) &= \left(y_m(x,\rho,0)\,,\phi_j(x,\rho,s)\right)_{L^2(0,1)}\,,\\ h_{jm}^{'}(0) &= \left(y_{mt}(x,\rho,0)\,,\phi_j(x,\rho,s)\right)_{L^2(0,1)}\,. \end{aligned} \tag{3.10}$$

We obtain a system of differential equations of two orders with respect to the variable t with constant coefficients and the initial conditions (3.10), consequently, we get a Cauchy problem of linear differential equations with smooth coefficients that is uniquely solvable. Thus for every m there exists a function $(\varphi_m,\psi_m,\theta_m,q_m,y_m)$ satisfying (3.5).

3.2. A priori estimate I

Firstly, multiplying the first equation of (3.5) by $\acute{\xi}_{lm}$ and integrating over $(0,1)$, we get

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int_0^1\varphi_{mt}^2dx+K\int_0^1(\varphi_{mx}+\psi_m)\,\varphi_{mtx}dx\\ &+\mu_1\int_0^1\varphi_{mt}^2dx+\int_0^1\varphi_{mt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_m(x,1,s)\,dsdx \;=\; 0. \end{aligned} \tag{3.11}$$

Then, multiplying the second equation of (3.5) by k_{lm}' and integrating over $(0,1)$, we get

$$\rho_2 \int_0^1 \varphi_{mtt}\psi_{mxt}dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_{mx}^2 dx + K\int_0^1 (\varphi_{mx}+\psi_m)\psi_{mt}dx$$
$$+\gamma\int_0^1 \theta_{mx}\psi_{mt}dx = 0, \quad (3.12)$$

now, substituting: $\psi_{mxt} = \frac{\rho_1}{K}\varphi_{mttt} - \varphi_{mxxt} + \frac{\mu_1}{K}\varphi_{mtt} + \frac{1}{K}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mt}(x,1,s)\,ds$, obtained from the first equation of (3.5) ,we get

$$\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_0^1 \varphi_{mtt}^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \varphi_{mxt}^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_{mx}^2 dx$$
$$+K\int_0^1 (\varphi_{mx}+\psi_m)\psi_{mt}dx + \frac{\rho_2\mu_1}{K}\int_0^1 \varphi_{mtt}^2 dx$$
$$+\gamma\int_0^1 \theta_{mx}\psi_{mt}dx + \frac{\rho_2}{K}\int_0^1 \varphi_{mtt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mt}(x,1,s)\,dsdx = 0. \quad (3.13)$$

Next, multiplying the third equation of (3.5) by l_{lm} and integrating over $(0,1)$, we get

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta_m^2 dx + \kappa\int_0^1 q_{mx}\theta_m dx - \gamma\int_0^1 \psi_{mt}\theta_{mx}dx = 0. \quad (3.14)$$

Finally, multiplying the fourth equation of (3.5) by f_{lm} and integrating over $(0,1)$, we get

$$\frac{\tau_0}{2}\frac{d}{dt}\int_0^1 q_m^2 dx + \delta\int_0^1 q_m^2 dx - \kappa\int_0^1 \theta_m q_{mx}dx = 0. \quad (3.15)$$

By combining (3.11), (3.13), (3.14) and (3.15), we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_{mt}^2 + K(\varphi_{mx}+\psi_m)^2 + \frac{\rho_1\rho_2}{K}\varphi_{mtt}^2 + \rho_2\varphi_{mtx}^2 + b\psi_{mx}^2$$
$$+\rho_3\theta_m^2 + \tau_0 q_m^2\Big]dx + \mu_1\int_0^1 \varphi_{mt}^2 dx + \delta\int_0^1 q_m^2 dx + \frac{\rho_2\mu_1}{K}\int_0^1 \varphi_{mtt}^2 dx$$
$$+\int_0^1 \varphi_{mt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_m(x,1,s)\,dsdx$$
$$+\frac{\rho_2}{K}\int_0^1 \varphi_{mtt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mt}(x,1,s)\,dsdx = 0. \quad (3.16)$$

Now, multiplying the fifth equation of (3.5) by $\mu_2(s)\,h_{lm}$ and integrating over $(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\,y_m^2(x,\rho,s,t)\,dsd\rho dx$$
$$-\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\,y_m^2(x,\rho,s,0)\,dsd\rho dx$$
$$= -\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_m^2(x,1,s,\tau)\,dsdxd\tau$$
$$+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1 \psi_{mt}^2 dxd\tau. \quad (3.17)$$

Then, multiplying the last equation of (3.5) by $\mu_2(s)\, h'_{lm}$ and integrating over $(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mt}^2(x,\rho,s,t)\, dsd\rho dx \\
&-\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mt}^2(x,\rho,s,0)\, dsd\rho dx \\
= &-\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_{m\tau}^2(x,1,s,\tau)\, dsdxd\tau \\
&+\frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau}^2 dxd\tau. \qquad (3.18)
\end{aligned}$$

Next, integrating (3.16) over $(0,t)$ and using (3.17) and (3.18), we obtain

$$\begin{aligned}
&E_m(t)+\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1\varphi_{m\tau}^2 dxd\tau \\
&+\frac{\rho_2}{K}\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1\varphi_{m\tau\tau}^2 dxd\tau \\
&+\delta\int_0^t\int_0^1 q_m^2 dxd\tau+\int_0^t\int_0^1\varphi_{m\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_m(x,1,s)\, dsdxd\tau \\
&+\frac{\rho_2}{K}\int_0^t\int_0^1\varphi_{m\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_{m\tau}(x,1,s)\, dsdxd\tau \\
&+\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_m^2(x,1,s,\tau)\, dsdxd\tau - \\
&+\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_{m\tau}^2(x,1,s,\tau)\, dsdxd\tau \\
= &\ E_m(0), \qquad (3.19)
\end{aligned}$$

where

$$\begin{aligned}
E_m(t) = &\ \frac{1}{2}\int_0^1\Big[\rho_1\varphi_{mt}^2+K(\varphi_{mx}+\psi_m)^2+\frac{\rho_2\rho_1}{K}\varphi_{mtt}^2 \\
&+\rho_2\varphi_{mxt}^2+b\psi_{mx}^2+\rho_3\theta_m^2+\tau_0 q_m^2\Big]\, dx \\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_m^2(x,\rho,s)\, dsd\rho dx \\
&+\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_{mt}^2(x,\rho,s)\, dsd\rho dx,
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
&\int_0^t\int_0^1\varphi_{mt}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_m(x,1,s)\, dsdxd\tau \\
\geq &-\left(\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1\varphi_{mt}^2 dxd\tau \\
&-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_m^2(x,1,s,\tau)\, dsdxd\tau, \qquad (3.20)
\end{aligned}$$

and

$$\begin{aligned} & \frac{\rho_2}{K}\int_0^t\int_0^1 \varphi_{mtt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mt}(x,1,s)\,dsdxd\tau \\ \geq\; & -\frac{\rho_2}{2K}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\int_0^t\int_0^1\varphi_{mtt}^2dxd\tau \\ & -\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mt}^2(x,1,s,\tau)\,dsdxd\tau. \end{aligned} \tag{3.21}$$

Which, together with (3.19), yields

$$\begin{aligned} & E_m(t)+\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\varphi_{m\tau}^2dxd\tau \\ & +\frac{\rho_2}{K}\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\varphi_{m\tau\tau}^2dxd\tau \\ & +\delta\int_0^t\int_0^1 q_m^2dxd\tau \\ \leq\; & E_m(0), \end{aligned}$$

implies

$$\begin{aligned} & E_m(t)+\eta_0\int_0^t\int_0^1\varphi_{m\tau}^2dxd\tau \\ & +\delta\int_0^t\int_0^1 q_m^2dxd\tau+\frac{\rho_2}{K}\eta_0\int_0^t\int_0^1\varphi_{m\tau\tau}^2dxd\tau \\ \leq\; & E_m(0), \end{aligned} \tag{3.22}$$

where $\eta_0=\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds>0$.
So, we have

$$E_m(t)\leq E_m(0), \tag{3.23}$$

and make use of the following inequality

$$\begin{aligned} \rho_1\int_0^1\varphi_m^2dx \;\leq\; & \rho_1\int_0^t\int_0^1\varphi_m^2(x,\tau)\,dxd\tau \\ & +\rho_1\int_0^t\int_0^1\varphi_{m\tau}^2(x,\tau)\,dxd\tau+\rho_1\int_0^1\varphi_m^2(x,0)\,dx, \end{aligned} \tag{3.24}$$

combining inequalities (3.23) and (3.24), we get

$$\begin{aligned} E_m(t)+\rho_1\int_0^1\varphi_m^2dx \;\leq\; & E_m(0)+\rho_1\int_0^t\int_0^1\varphi_m^2(x,\tau)\,dxd\tau \\ & +\rho_1\int_0^t\int_0^1\varphi_{m\tau}^2(x,\tau)\,dxd\tau+\rho_1\int_0^1\varphi_m^2(x,0)\,dx, \end{aligned}$$

we put

$$\mathcal{P}_m(t)=E_m(t)+\rho_1\int_0^1\varphi_m^2dx, \tag{3.25}$$

we get

$$\mathcal{P}_m(t)\leq\mathcal{P}_m(0)+\int_0^t\mathcal{P}_m(\tau)\,d\tau. \tag{3.26}$$

Applying the Gronwall inequality to (3.26), we obtain

$$\mathcal{P}_m(t) \leq \mathcal{P}_m(0) \exp(T),$$

thus, there exist a positive constant C independent on m such that

$$\mathcal{P}_m(t) \leq C, \ t \geq 0, \tag{3.27}$$

it follows from (1.15) and (3.27) that

$$\begin{aligned} & \rho_1 \int_0^1 \varphi_m^2 dx + \rho_1 \int_0^1 \varphi_{mt}^2 dx + K \int_0^1 (\varphi_{mx} + \psi_m)^2 dx + \frac{\rho_1 \rho_2}{K} \int_0^1 \varphi_{mtt}^2 dx \\ & + \rho_2 \int_0^1 \varphi_{mxt}^2 dx + b \int_0^1 \psi_{mx}^2 dx + \rho_3 \int_0^1 \theta_m^2 dx + \tau_0 \int_0^1 q_m^2 dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_m^2(x,\rho,s) ds d\rho dx \\ & + \frac{\rho_2}{K} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_{mt}^2(x,\rho,s) ds d\rho dx \\ \leq \ & C. \end{aligned} \tag{3.28}$$

3.3. A priori estimate II

Firstly, differentiating the first equation of (3.5) and multiplying by $\acute{\xi}_{lm}$, and then integrating the result over $(0,1)$,we obtain

$$\begin{aligned} & \frac{\rho_1}{2} \frac{d}{dt} \int_0^1 \varphi_{mtt}^2 dx + K \int_0^1 (\varphi_{mxt} + \psi_{mt}) \varphi_{mttx} dx \\ & + \mu_1 \int_0^1 \varphi_{mtt}^2 dx + \int_0^1 \varphi_{mtt} \int_{\tau_1}^{\tau_2} \mu_2(s) y_{mt}(x,1,s) ds dx \ = \ . \end{aligned} \tag{3.29}$$

Next, differentiating the second equation of (3.5) and multiplying by $\acute{k}_{lm}$,and integrating over $(0,1)$, we obtain

$$\begin{aligned} & \rho_2 \int_0^1 \varphi_{mttt} \psi_{mxtt} dx + \frac{b}{2} \frac{d}{dt} \int_0^1 \psi_{mxt}^2 dx + K \int_0^1 (\varphi_{mxt} + \psi_{mt}) \psi_{mtt} dx \\ & + \gamma \int_0^1 \theta_{mxt} \psi_{mtt} dx \ = \ 0, \end{aligned}$$

now, substituting: $\psi_{mxtt} = \frac{\rho_1}{K} \varphi_{mtttt} - \varphi_{mxxtt} + \frac{\mu_1}{K} \varphi_{mttt} + \frac{1}{K} \int_{\tau_1}^{\tau_2} \mu_2(s) y_{mtt}(x,1,s) ds$, obtained from the first equation of (3.5) ,we obtain

$$\begin{aligned} & \frac{\rho_2 \rho_1}{2K} \frac{d}{dt} \int_0^1 \varphi_{mttt}^2 dx + \frac{\rho_2}{2} \frac{d}{dt} \int_0^1 \varphi_{mxtt}^2 dx + \frac{b}{2} \frac{d}{dt} \int_0^1 \psi_{mxt}^2 dx \\ & + K \int_0^1 (\varphi_{mxt} + \psi_{mt}) \psi_{mtt} dx + \frac{\rho_2 \mu_1}{K} \int_0^1 \varphi_{mttt}^2 dx \\ & + \gamma \int_0^1 \theta_{mxt} \psi_{mtt} dx + \frac{\rho_2}{K} \int_0^1 \varphi_{mttt} \int_{\tau_1}^{\tau_2} \mu_2(s) y_{mtt}(x,1,s) ds dx \ = \ 0. \end{aligned} \tag{3.30}$$

Then, differentiating the third equation of (3.5) and multiplying by $\acute{l}_{lm}$, and integrating over $(0,1)$, we obtain

$$\frac{\rho_3}{2} \frac{d}{dt} \int_0^1 \theta_{mt}^2 dx + \kappa \int_0^1 q_{mtx} \theta_{mt} dx - \gamma \int_0^1 \psi_{mtt} \theta_{mtx} dx \ = \ 0. \tag{3.31}$$

Finally, differentiating the fourth equation of (3.5) and multiplying by f'_{lm}, and integrating over $(0,1)$, we obtain

$$\frac{\tau_0}{2}\frac{d}{dt}\int_0^1 q_{mt}^2 dx + \delta\int_0^1 q_{mt}^2 dx - \kappa\int_0^1 \theta_{mt} q_{mxt} dx \;=\; 0. \tag{3.32}$$

By combining (3.29), (3.30), (3.31) and (3.32), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_{mtt}^2 + K\left(\varphi_{mxt}+\psi_{mt}\right)^2 + \frac{\rho_1\rho_2}{K}\varphi_{mttt}^2 + \rho_2\varphi_{mttx}^2 + b\psi_{mxt}^2 \\
&+\rho_3\theta_{mt}^2 + \tau_0 q_{mt}^2\Big] dx + \mu_1\int_0^1 \varphi_{mtt}^2 dx + \delta\int_0^1 q_{mt}^2 dx + \frac{\rho_2\mu_1}{K}\int_0^1 \varphi_{mttt}^2 dx \\
&+\int_0^1 \varphi_{mtt}\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_{mt}(x,1,s)\, ds dx \\
&+\frac{\rho_2}{K}\int_0^1 \varphi_{mttt}\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_{mtt}(x,1,s)\, ds dx \;=\; 0.
\end{aligned} \tag{3.33}$$

Now, differentiating the fifth equation of (3.5) and multiplying by $\mu_2(s)\, h'_{lm}$, and integrating over $(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mt}^2(x,\rho,s,t)\, ds d\rho dx \\
&-\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mt}^2(x,\rho,s,0)\, ds d\rho dx \\
=\;&-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_{m\tau}^2(x,1,s,\tau)\, ds dx d\tau \\
&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau}^2 dx d\tau,
\end{aligned} \tag{3.34}$$

then, differentiating the last equation of (3.5) and multiplying by $\mu_2(s)\, h'_{lm}$, and integrating over $(0,t)\times(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mtt}^2(x,\rho,s,t)\, ds d\rho dx \\
&-\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu_2(s)\, y_{mtt}^2(x,\rho,s,0)\, ds d\rho dx \\
=\;&-\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_{m\tau\tau}^2(x,1,s,\tau)\, ds dx d\tau \\
&+\frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\, ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau\tau}^2 dx d\tau.
\end{aligned} \tag{3.35}$$

next, integrating (3.33) over $(0,t)$ and using (3.34) and (3.35), we obtain

$$\begin{aligned}
&\mathcal{M}_m(t) + \left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} \mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau}^2 dxd\tau \\
&+\frac{\rho_2}{K}\left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} \mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau\tau}^2 dxd\tau \\
&+\delta\int_0^t\int_0^1 q_{m\tau}^2 dxd\tau + \int_0^t\int_0^1 \varphi_{m\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau}(x,1,s)\,dsdxd\tau \\
&+\frac{\rho_2}{K}\int_0^t\int_0^1 \varphi_{m\tau\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau\tau}(x,1,s)\,dsdxd\tau \\
&+\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau}^2(x,1,s,\tau)\,dsdxd\tau - \\
&+\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau\tau}^2(x,1,s,\tau)\,dsdxd\tau \\
=\;& \mathcal{M}_m(0), \qquad (3.36)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_m(t) =\;& \frac{1}{2}\int_0^1 \Big[\rho_1\varphi_{mtt}^2 + K(\varphi_{mxt}+\psi_{mt})^2 + \frac{\rho_2\rho_1}{K}\varphi_{mttt}^2 \\
&+\rho_2\varphi_{mxtt}^2 + b\psi_{mxt}^2 + \rho_3\theta_{mt}^2 + \tau_0 q_{mt}^2\Big]\,dx \\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,y_{mt}^2(x,\rho,s)\,dsd\rho dx \\
&+\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,y_{mtt}^2(x,\rho,s)\,dsd\rho dx,
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
&\int_0^t\int_0^1 \varphi_{m\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau}(x,1,s)\,dsdxd\tau \\
\geq\;& -\left(\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{m\tau\tau}^2 dxd\tau \\
&-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau}^2(x,1,s,\tau)\,dsdxd\tau, \qquad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\rho_2}{K}\int_0^t\int_0^1 \varphi_{m\tau\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau\tau}(x,1,s)\,dsdxd\tau \\
\geq\;& -\frac{\rho_2}{2K}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\int_0^t\int_0^1 \varphi_{m\tau\tau\tau}^2 dxd\tau \\
&-\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{m\tau\tau}^2(x,1,s,\tau)\,dsdxd\tau. \qquad (3.38)
\end{aligned}$$

Which, together with (3.36), yields

$$\begin{aligned}
&\mathcal{M}_m(t)+\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\varphi_{m\tau\tau}^2dxd\tau\\
&+\frac{\rho_2}{K}\left(\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\varphi_{m\tau\tau\tau}^2dxd\tau\\
&+\delta\int_0^t\int_0^1 q_{m\tau}^2dxd\tau\\
\leq\;&\mathcal{M}_m(0),
\end{aligned}\tag{3.39}$$

implies

$$\begin{aligned}
&\mathcal{M}_m(t)+\eta_0\int_0^t\int_0^1\varphi_{m\tau\tau}^2dxd\tau\\
&+\delta\int_0^t\int_0^1 q_{m\tau}^2dxd\tau+\frac{\rho_2}{K}\eta_0\int_0^t\int_0^1\varphi_{m\tau\tau\tau}^2dxd\tau\\
\leq\;&\mathcal{M}_m(0),
\end{aligned}\tag{3.40}$$

where $\eta_0=\mu_1-\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds>0$.

Then

$$\mathcal{M}_m(t)\leq\mathcal{M}_m(0),\tag{3.41}$$

thus, there exist a positive constant C independent on m such that

$$\mathcal{M}_m(t)\leq C,\;\; t\geq 0,\tag{3.42}$$

it follows from (1.15) and (3.42) that

$$\begin{aligned}
&\rho_1\int_0^1\varphi_{mtt}^2dx+K\int_0^1(\varphi_{mxt}+\psi_{mt})^2\,dx+\frac{\rho_1\rho_2}{K}\int_0^1\varphi_{mttt}^2dx\\
&+\rho_2\int_0^1\varphi_{mxtt}^2dx+b\int_0^1\psi_{mxt}^2dx+\rho_3\int_0^1\theta_{mt}^2dx+\tau_0\int_0^1 q_{mt}^2dx\\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,y_{mt}^2(x,\rho,s)\,dsd\rho dx\\
&+\frac{\rho_2}{K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\,|\mu_2(s)|\,y_{mtt}^2(x,\rho,s)\,dsd\rho dx\\
\leq\;&C.
\end{aligned}\tag{3.43}$$

3.4. A priori estimate III

Firstly, let $u_l=-\varphi_{mtxx}$ in the first equation of (3.5), we get

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int_0^1\varphi_{mtx}^2dx+K\int_0^1(\varphi_{mxx}+\psi_{mx})\,\varphi_{mtxx}dx\\
&+\mu_1\int_0^1\varphi_{mtx}^2dx+\int_0^1\varphi_{mtx}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}(x,1,s)\,dsdx\;=\;0.
\end{aligned}\tag{3.44}$$

Then, let $v_l=-\psi_{mtxx}$ in the second equation of (3.5), we get

$$\begin{aligned}
&\rho_2\int_0^1\varphi_{mxtt}\psi_{mxxt}dx+\frac{b}{2}\frac{d}{dt}\int_0^1\psi_{mxx}^2dx+K\int_0^1(\varphi_{mxx}+\psi_{mx})\,\psi_{mxt}dx\\
&+\gamma\int_0^1\theta_{mxx}\psi_{mxt}dx\;=\;0,
\end{aligned}\tag{3.45}$$

now, substituting: $\psi_{mxxt}=\frac{\rho_1}{K}\varphi_{mxttt}-\varphi_{mxxxt}+\frac{\mu_1}{K}\varphi_{mxtt}+\frac{1}{K}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mtx}(x,1,s)\,ds$, obtained from the first equation of (3.5) ,we get

$$\begin{aligned}&\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_0^1\varphi_{mxtt}^2dx+\frac{\rho_2}{2}\frac{d}{dt}\int_0^1\varphi_{mxxt}^2dx+\frac{b}{2}\frac{d}{dt}\int_0^1\psi_{mxx}^2dx\\&+K\int_0^1(\varphi_{mxx}+\psi_{mx})\,\psi_{mxt}dx+\frac{\rho_2\mu_1}{K}\int_0^1\varphi_{mxtt}^2dx\\&+\gamma\int_0^1\theta_{mxx}\psi_{mxt}dx+\frac{\rho_2}{K}\int_0^1\varphi_{mxtt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mxt}(x,1,s)\,dsdx\;=\;0.\end{aligned}\tag{3.46}$$

Next, let $w_l=-\theta_{mxx}$ in the third equation of (3.5) , we get

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1\theta_{mx}^2dx-\kappa\int_0^1q_{mx}\theta_{mxx}dx+\gamma\int_0^1\psi_{mtxx}\theta_{mx}dx\;=\;0.\tag{3.47}$$

Finally, let $z_l=-q_{mxx}$ in the fourth equation of (3.5) , we get

$$\frac{\tau_0}{2}\frac{d}{dt}\int_0^1q_{mx}^2dx+\delta\int_0^1q_{mx}^2dx+\kappa\int_0^1\theta_{mxx}q_{mx}dx\;=\;0.\tag{3.48}$$

By combining (3.44) , (3.46) , (3.47) and (3.48) , we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\Big[\rho_1\varphi_{mxt}^2+K(\varphi_{mxx}+\psi_{mx})^2+\frac{\rho_1\rho_2}{K}\varphi_{mxtt}^2+\rho_2\varphi_{mtxx}^2+b\psi_{mxx}^2\\&+\rho_3\theta_{mx}^2+\tau_0q_{mx}^2\Big]dx+\mu_1\int_0^1\varphi_{mxt}^2dx+\delta\int_0^1q_{mx}^2dx+\frac{\rho_2\mu_1}{K}\int_0^1\varphi_{mxtt}^2dx\\&+\int_0^1\varphi_{mxt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}(x,1,s)\,dsdx\\&+\frac{\rho_2}{K}\int_0^1\varphi_{mxtt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mxt}(x,1,s)\,dsdx\;=\;0.\end{aligned}\tag{3.49}$$

Now, let $\phi_l=-\mu_2(s)\,y_{mxx}$ in 5^{th} equation of (3.5) , and integrating over $(0,t)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}&\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,y_{mx}^2(x,\rho,s,t)\,dsd\rho dx\\&-\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,y_{mx}^2(x,\rho,s,0)\,dsd\rho dx\\=\;&-\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}^2(x,1,s,\tau)\,dsdxd\tau\\&+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\psi_{m\tau x}^2dxd\tau,\end{aligned}\tag{3.50}$$

and let $\phi_l=-\mu_2(s)\,y_{mxxt}$ in the last equation of (3.5) , and integrating over $(0,t)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}&\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,y_{mxt}^2(x,\rho,s,t)\,dsd\rho dx\\&-\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu_2(s)\,y_{mxt}^2(x,\rho,s,0)\,dsd\rho dx\\=\;&-\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx\tau}^2(x,1,s,\tau)\,dsdxd\tau\\&+\frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1\varphi_{mx\tau\tau}^2dxd\tau.\end{aligned}\tag{3.51}$$

Next, integrating (3.49) over $(0,t)$ and using (3.50) and (3.51), we obtain

$$\begin{aligned}
& \mathcal{K}_m(t) + \left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{mx\tau}^2 dxd\tau \\
& +\frac{\rho_2}{K}\left(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{mx\tau\tau}^2 dxd\tau \\
& +\delta\int_0^t\int_0^1 q_{mx}^2 dxd\tau + \int_0^t\int_0^1 \varphi_{mx\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}(x,1,s)\,dsdxd\tau \\
& +\frac{\rho_2}{K}\int_0^t\int_0^1 \varphi_{mx\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mxt}(x,1,s)\,dsdxd\tau \\
& +\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}^2(x,1,s,\tau)\,dsdxd\tau - \\
& +\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx\tau}^2(x,1,s,\tau)\,dsdxd\tau \\
= \; & \mathcal{K}_m(0), \qquad (3.52)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_m(t) = \; & \frac{1}{2}\int_0^1 \Big[\rho_1\varphi_{mxt}^2 + K(\varphi_{mxx}+\psi_{mx})^2 + \frac{\rho_2\rho_1}{K}\varphi_{mxtt}^2 \\
& +\rho_2\varphi_{mxxt}^2 + b\psi_{mxx}^2 + \rho_3\theta_{mx}^2 + \tau_0 q_{mx}^2\Big]\,dx \\
& +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,y_{mx}^2(x,\rho,s)\,dsd\rho dx \\
& +\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\,y_{mxt}^2(x,\rho,s)\,dsd\rho dx,
\end{aligned}$$

and using Young's inequality, we have

$$\begin{aligned}
& \int_0^t\int_0^1 \varphi_{mx\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}(x,1,s)\,dsdxd\tau \\
\geq \; & -\left(\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\right)\int_0^t\int_0^1 \varphi_{mx\tau}^2 dxd\tau \\
& -\frac{1}{2}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx}^2(x,1,s,\tau)\,dsdxd\tau, \qquad (3.53)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\rho_2}{K}\int_0^t\int_0^1 \varphi_{mx\tau\tau}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx\tau}(x,1,s)\,dsdxd\tau \\
\geq \; & -\frac{\rho_2}{2K}\int_{\tau_1}^{\tau_2}\mu_2(s)\,ds\int_0^t\int_0^1 \varphi_{mx\tau\tau}^2 dxd\tau \\
& -\frac{\rho_2}{2K}\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,y_{mx\tau}^2(x,1,s,\tau)\,dsdxd\tau. \qquad (3.54)
\end{aligned}$$

Which, together with (3.52), yields

$$\begin{aligned} & \mathcal{K}_m(t) + \left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds\right) \int_0^t \int_0^1 \varphi_{mx\tau}^2 dx d\tau \\ & + \frac{\rho_2}{K} \left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds\right) \int_0^t \int_0^1 \varphi_{mx\tau\tau}^2 dx d\tau \\ & + \delta \int_0^t \int_0^1 q_{mx}^2 dx d\tau \\ \leq\ & \mathcal{K}_m(0), \end{aligned}$$

implies

$$\begin{aligned} & \mathcal{K}_m(t) + \eta_0 \int_0^t \int_0^1 \varphi_{mx\tau}^2 dx d\tau \\ & + \delta \int_0^t \int_0^1 q_{mx}^2 dx d\tau + \frac{\rho_2}{K} \eta_0 \int_0^t \int_0^1 \varphi_{mx\tau\tau}^2 dx d\tau \\ \leq\ & \mathcal{K}_m(0), \end{aligned} \tag{3.55}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds > 0$.
Then

$$\mathcal{K}_m(t) \leq \mathcal{K}_m(0), \tag{3.56}$$

and make use of the following inequality

$$\begin{aligned} & \lambda_1 \int_0^1 \varphi_{mx}^2 dx + \lambda_2 \int_0^1 \varphi_{mxx}^2 dx \\ \leq\ & \lambda_1 \int_0^t \int_0^1 \varphi_{mx}^2(x,\tau)\, dx d\tau + \lambda_2 \int_0^t \int_0^1 \varphi_{mxx}^2(x,\tau)\, dx d\tau \\ & + \lambda_1 \int_0^t \int_0^1 \varphi_{mx\tau}^2(x,\tau)\, dx d\tau + \lambda_2 \int_0^t \int_0^1 \varphi_{mxx\tau}^2(x,\tau)\, dx d\tau \\ & + \lambda_1 \int_0^1 \varphi_{mx}^2(0)\, dx + \lambda_2 \int_0^1 \varphi_{mxx}^2(0)\, dx. \end{aligned} \tag{3.57}$$

Combining inequalities (3.56) and (3.57), we get

$$\begin{aligned} & \mathcal{K}_m(t) + \lambda_1 \int_0^1 \varphi_{mx}^2 dx + \lambda_2 \int_0^1 \varphi_{mxx}^2 dx \\ \leq\ & \mathcal{K}_m(0) + \lambda_1 \int_0^t \int_0^1 \varphi_{mx}^2(x,\tau)\, dx d\tau + \lambda_2 \int_0^t \int_0^1 \varphi_{mxx}^2(x,\tau)\, dx d\tau \\ & + \lambda_1 \int_0^t \int_0^1 \varphi_{mx\tau}^2(x,\tau)\, dx d\tau + \lambda_2 \int_0^t \int_0^1 \varphi_{mxx\tau}^2(x,\tau)\, dx d\tau \\ & + \lambda_1 \int_0^1 \varphi_{mx}^2(0)\, dx + \lambda_2 \int_0^1 \varphi_{mxx}^2(0)\, dx, \end{aligned}$$

we put

$$\mathcal{S}_m(t) = \mathcal{K}_m(t) + \lambda_1 \int_0^1 \varphi_{mx}^2 dx + \lambda_2 \int_0^1 \varphi_{mxx}^2 dx, \tag{3.58}$$

we get

$$\mathcal{S}_m(t) \leq \mathcal{S}_m(0) + \int_0^t \mathcal{S}_m(\tau)\, d\tau. \tag{3.59}$$

Applying the Gronwall inequality to (3.59), we obtain

$$\mathcal{S}_m(t) \leq \mathcal{S}_m(0)\exp(T), \tag{3.60}$$

thus, there exists a positive constant C independent on m such that

$$\mathcal{S}_m(t) \leq C, t \geq 0, \tag{3.61}$$

it follows from (1.15) and (3.61) that

$$\begin{aligned} & \rho_1 \int_0^1 \varphi_{mxt}^2 dx + K \int_0^1 (\varphi_{mxx} + \psi_{mx})^2 dx + \tau_0 \int_0^1 q_{mx}^2 dx \\ & + \lambda_1 \int_0^1 \varphi_{mx}^2 dx + \lambda_2 \int_0^1 \varphi_{mxx}^2 dx + \frac{\rho_2\rho_1}{K} \int_0^1 \varphi_{mxtt}^2 dx \\ & + \rho_2 \int_0^1 \varphi_{mxxt}^2 dx + b \int_0^1 \psi_{mxx}^2 dx + \rho_3 \int_0^1 \theta_{mx}^2 dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_{mx}^2(x,\rho,s)\, ds d\rho dx \\ & + \frac{\rho_2}{K} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_{mxt}^2(x,\rho,s)\, ds d\rho dx \\ \leq \; & C. \end{aligned} \tag{3.62}$$

Now, combining inequalities (3.62), (3.43), and (3.28) , we obtain

$$\begin{aligned} & \rho_1 \int_0^1 \varphi_m^2 dx + \lambda_1 \int_0^1 \varphi_{mx}^2 dx + \rho_1 \int_0^1 \varphi_{mt}^2 dx \\ & + (\rho_1 + \rho_2) \int_0^1 \varphi_{mxt}^2 dx + \left(\rho_1 + \frac{\rho_1\rho_2}{K}\right) \int_0^1 \varphi_{mtt}^2 dx \\ & + \lambda_2 \int_0^1 \varphi_{mxx}^2 dx + \left(\rho_2 + \frac{\rho_2\rho_1}{K}\right) \int_0^1 \varphi_{mxtt}^2 dx \\ & + \frac{\rho_1\rho_2}{K} \int_0^1 \varphi_{mttt}^2 dx + \rho_2 \int_0^1 \varphi_{mxxt}^2 dx \\ & + b \int_0^1 \psi_{mx}^2 dx + b \int_0^1 \psi_{mxt}^2 dx + b \int_0^1 \psi_{mxx}^2 dx \\ & + K \int_0^1 (\varphi_{mx} + \psi_m)^2 dx + K \int_0^1 (\varphi_{mxx} + \psi_{mx})^2 dx \\ & + K \int_0^1 (\varphi_{mxt} + \psi_{mt})^2 dx + \rho_3 \int_0^1 \theta_m^2 dx \\ & + \rho_3 \int_0^1 \theta_{mx}^2 dx + \rho_3 \int_0^1 \theta_{mt}^2 dx \\ & + \tau_0 \int_0^1 q_m^2 dx + \tau_0 \int_0^1 q_{mx}^2 dx + + \tau_0 \int_0^1 q_{mt}^2 dx \\ & + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_m^2(x,\rho,s)\, ds d\rho dx \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mx}^2(x,\rho,s)\,dsd\rho dx \\
&+\left(1+\frac{\rho_2}{K}\right)\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mt}^2(x,\rho,s)\,dsd\rho dx \\
&+\frac{\rho_2}{K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mxt}^2(x,\rho,s)\,dsd\rho dx \\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mtt}^2(x,\rho,s)\,dsd\rho dx \\
\leq& C, \qquad (3.63)
\end{aligned}
$$

using Young's inequality with ε, we have

$$
\begin{aligned}
&\rho_1\int_0^1 \varphi_m^2 dx+\lambda_1\int_0^1 \varphi_{mx}^2 dx+\rho_1\int_0^1 \varphi_{mt}^2 dx \\
&+(\rho_1+\rho_2)\int_0^1 \varphi_{mxt}^2 dx+\left(\rho_1+\frac{\rho_1\rho_2}{K}\right)\int_0^1 \varphi_{mtt}^2 dx \\
&+\lambda_2\int_0^1 \varphi_{mxx}^2 dx+\left(\rho_2+\frac{\rho_2\rho_1}{K}\right)\int_0^1 \varphi_{mxtt}^2 dx \\
&+\frac{\rho_1\rho_2}{K}\int_0^1 \varphi_{mttt}^2 dx+\rho_2\int_0^1 \varphi_{mxxt}^2 dx \\
&+b\int_0^1 \psi_{mx}^2 dx+b\int_0^1 \psi_{mxt}^2 dx+b\int_0^1 \psi_{mxx}^2 dx \\
&+K\left(1-\frac{1}{\varepsilon}\right)\int_0^1 \varphi_{mx}^2 dx+K(1-\varepsilon)\int_0^1 \psi_m^2 dx \\
&+K\left(1-\frac{1}{\varepsilon}\right)\int_0^1 \varphi_{mxt}^2 dx+K(1-\varepsilon)\int_0^1 \psi_{mt}^2 dx+\rho_3\int_0^1 \theta_m^2 dx \\
&+\rho_3\int_0^1 \theta_{mx}^2 dx+\rho_3\int_0^1 \theta_{mt}^2 dx \\
&+\tau_0\int_0^1 q_m^2 dx+\tau_0\int_0^1 q_{mx}^2 dx++\tau_0\int_0^1 q_{mt}^2 dx \\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_m^2(x,\rho,s)\,dsd\rho dx \\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mx}^2(x,\rho,s)\,dsd\rho dx \\
&+\left(1+\frac{\rho_2}{K}\right)\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mt}^2(x,\rho,s)\,dsd\rho dx \\
&+\frac{\rho_2}{K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mxt}^2(x,\rho,s)\,dsd\rho dx \\
&+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mtt}^2(x,\rho,s)\,dsd\rho dx \\
\leq\ & C, \qquad (3.64)
\end{aligned}
$$

we choose $\varepsilon = \frac{1}{2}$, and $\lambda = \lambda_1 - K > 0$,such that $\lambda_1 = \rho_1 + \rho_2 > K$, we obtain

$$\begin{aligned}
&\mathcal{N}_m(t) = \rho_1 \int_0^1 \varphi_m^2 dx + \lambda \int_0^1 \varphi_{mx}^2 dx + \rho_1 \int_0^1 \varphi_{mt}^2 dx \\
&+\lambda \int_0^1 \varphi_{mxt}^2 dx + \left(\rho_1 + \frac{\rho_1\rho_2}{K}\right) \int_0^1 \varphi_{mtt}^2 dx \\
&+\lambda_2 \int_0^1 \varphi_{mxx}^2 dx + \left(\rho_2 + \frac{\rho_2\rho_1}{K}\right) \int_0^1 \varphi_{mxtt}^2 dx \\
&+\frac{\rho_1\rho_2}{K} \int_0^1 \varphi_{mttt}^2 dx + \rho_2 \int_0^1 \varphi_{mxxt}^2 dx \\
&+\frac{K}{2} \int_0^1 \psi_m^2 dx + b \int_0^1 \psi_{mx}^2 dx \\
&+\frac{K}{2} \int_0^1 \psi_{mt}^2 dx + b \int_0^1 \psi_{mxt}^2 dx + b \int_0^1 \psi_{mxx}^2 dx \\
&+\rho_3 \int_0^1 \theta_m^2 dx + \rho_3 \int_0^1 \theta_{mx}^2 dx + \rho_3 \int_0^1 \theta_{mt}^2 dx \\
&+\tau_0 \int_0^1 q_m^2 dx + \tau_0 \int_0^1 q_{mx}^2 dx + +\tau_0 \int_0^1 q_{mt}^2 dx \\
&+\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_m^2(x,\rho,s)\, ds d\rho dx \\
&+\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mx}^2(x,\rho,s)\, ds d\rho dx \\
&+\left(1 + \frac{\rho_2}{K}\right) \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mt}^2(x,\rho,s)\, ds d\rho dx \\
&+\frac{\rho_2}{K} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mxt}^2(x,\rho,s)\, ds d\rho dx \\
&+\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_{mtt}^2(x,\rho,s)\, ds d\rho dx \\
\leq\ & C. \qquad (3.65)
\end{aligned}$$

3.5. Passage to limit

Now, to prove that (3.5) holds, we multiply each of the equation (3.5) by a functions $w_l(t)$, $s_l(t)$, $v_l(t)$, $r_l(t)$, and $p_l(t)$ respectivly, we obtain

$$\begin{cases}
\rho_1(\varphi_{mtt}, u_l) w_l(t) + K((\varphi_{mx} + \psi_m), u_{lx}) w_l(t) + \mu_1(\psi_{mt}, u_l) w_l(t) \\
+ \left(\int_{\tau_1}^{\tau_2} \mu_2(s) y_m(x,1,s) ds, u_l\right) w_l(t) = 0, \\
\rho_2(\varphi_{mtt}, v_{lx}) s_l(t) + b(\psi_{mx}, v_{lx}) s_l(t) + K((\varphi_{mx} + \psi_m), v_l) s_l(t) \\
+\gamma(\theta_{mx}, v_l) s_l(t) = 0, \\
\rho_3(\theta_{mt}, w_l) v_l(t) + (q_{mx}, w_l) v_l(t) + \gamma(\psi_{mtx}, w_l) v_l(t) = 0, \\
\tau_0(q_{mt}, z_l) r_l(t) + \delta(q_m, z_l) r_l(t) + \kappa(\theta_{mx}, z_l) r_l(t) = 0, \\
s(y_{mt}(x,\rho,s), \phi_l) p_l(t) + (y_{m\rho}(x,\rho,s), \phi_l) p_l(t) = 0, \\
s(y_{mtt}(x,\rho,s), \phi_l) p_l(t) + (y_{m\rho t}(x,\rho,s), \phi_l) p_l(t) = 0.
\end{cases} \qquad (3.66)$$

Then, summing over l from 1 to m and if we let

$$\begin{aligned} \lambda_m &= \sum_{l=1}^{l=m} u_l(x)\, w_l(t),\ \gamma_m = \sum_{l=1}^{l=m} v_l(x)\, s_l(t), \\ \mu_m &= \sum_{l=1}^{l=m} w_l(x)\, v_l(t),\ \eta_m = \sum_{l=1}^{l=m} z_l(x)\, r_l(t), \\ \sigma_m &= \sum_{l=1}^{l=m} \phi_l(x,\rho,s)\, p_l(t), \end{aligned} \tag{3.67}$$

then, we have

$$\begin{cases} \rho_1(\varphi_{mtt}, \lambda_m) + K((\varphi_{mx} + \psi_m), \lambda_{mx}) + \mu_1(\psi_{mt}, \lambda_m) \\ + \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_m(x,1,s)\, ds, \lambda_m \right) = 0, \\ \rho_2(\varphi_{mtt}, \gamma_{mx}) + b(\psi_{mx}, \gamma_{mx}) + K((\varphi_{mx} + \psi_m), \gamma_m) \\ + \gamma(\theta_{mx}, \gamma_m) = 0, \\ \rho_3(\theta_{mt}, \mu_m) + (q_{mx}, \mu_m) + \gamma(\psi_{mtx}, \mu_m) = 0, \\ \tau_0(q_{mt}, \eta_m) + \delta(q_m, \eta_m) + \kappa(\theta_{mx}, \eta_m) = 0, \\ s(y_{mt}(x,\rho,s,t), \sigma_m(x,\rho,s)) + (y_{m\rho}(x,\rho,s,t), \sigma_m(x,\rho,s)) = 0, \\ s(y_{mtt}(x,\rho,s,t), \sigma_m(x,\rho,s,t)) + (y_{m\rho t}(x,\rho,s,t), \sigma_m(x,\rho,s,t)) = 0. \end{cases} \tag{3.68}$$

Now, we integrate over t on $(0,\infty)$, we obtain

$$\begin{aligned} -\rho_1(\varphi_{mt}, \lambda_{mt})_{L^2(Q)} - \rho_1(\varphi_{mt}(x,0), \lambda_m(x,0))_{L^2(0,1)} & \\ +K((\varphi_{mx} + \psi_m), \lambda_{mx})_{L^2(Q)} - \mu_1(\psi_m, \lambda_{mt})_{L^2(Q)} & \\ -\mu_1(\psi_m(x,0), \lambda_m(x,0))_{L^2(0,1)} + \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, y_m(x,1,s)\, ds, \lambda_m \right)_{L^2(Q)} &= 0, \\ -\rho_2(\varphi_{mt}, \gamma_{mxt})_{L^2(Q)} - \rho_2(\varphi_{mt}(x,0), \gamma_{mx}(x,0))_{L^2(0,1)} & \\ +b(\psi_{mx}, \gamma_{mx})_{L^2(Q)} + K((\varphi_{mx} + \psi_m), \gamma_m)_{L^2(Q)} + \gamma(\theta_{mx}, \gamma_m)_{L^2(Q)} &= 0, \\ -\rho_3(\theta_m, \mu_{mt})_{L^2(Q)} - \rho_3(\theta_m(x,0), \mu_m(x,0))_{L^2(0,1)} + (q_{mx}, \mu_m)_{L^2(Q)} & \\ -\gamma(\psi_{mx}, \mu_{mt})_{L^2(Q)} - \gamma(\psi_{mx}(x,0), \mu_m(x,0))_{L^2(0,1)} &= 0, \\ -\tau_0(q_m, \eta_{mt})_{L^2(Q)} - \tau_0(q_m(x,0), \eta_m(x,0))_{L^2(0,1)} & \\ +\delta(q_m, \eta_m)_{L^2(Q)} + \kappa(\theta_{mx}, \eta_m)_{L^2(Q)} &= 0, \\ -s(y_m(x,\rho,s,t), \sigma_{mt}(x,\rho,s,t))_{L^2(Q)} - s(y_m(x,\rho,s,0), \sigma_m(x,\rho,s,0))_{L^2(0,1)} & \\ +(y_{m\rho}(x,\rho,s,t), \sigma_m(x,\rho,s,t))_{L^2(Q)} &= 0, \\ -s(y_{mt}(x,\rho,s,t), \sigma_{mt}(x,\rho,s,t))_{L^2(Q)} - s(y_{mt}(x,\rho,s,0), \sigma_m(x,\rho,s,0))_{L^2(0,1)} & \\ +(y_{m\rho}(x,\rho,s,t), \sigma_m(x,\rho,s,t))_{L^2(Q)} &= 0. \end{aligned} \tag{3.69}$$

From (3.27), (3.42) and (3.61), we conclude that for any $m \in \mathbb{N}$,

$$\begin{aligned}
&\varphi_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^2 \cap H_0^1\right),\\
&\varphi_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\right),\\
&\varphi_{mtt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\right),\\
&\psi_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H_0^1\right),\\
&\psi_{mxt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&\theta_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H_0^1\right),\\
&\theta_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&q_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H_0^1\right),\\
&q_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, L^2\right),\\
&y_m \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right),\\
&y_{mt} \text{ is bounded in } L^\infty\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right).
\end{aligned} \tag{3.70}$$

Thus, we get

$$\begin{aligned}
&\varphi_m \text{ weakly in } L^2\left(\mathbb{R}_+, H^2 \cap H_0^1\right),\\
&\varphi_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\right),\\
&\varphi_{mtt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\right),\\
&\psi_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&\psi_{mxt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&\theta_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&\theta_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&q_m \text{ weakly in } L^2\left(\mathbb{R}_+, H_0^1\right),\\
&q_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, L^2\right),\\
&y_m \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right),\\
&y_{mt} \text{ weakly in } L^2\left(\mathbb{R}_+, H^1\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\right).
\end{aligned} \tag{3.71}$$

Thus, the limit function $(\varphi, \psi, \theta, q, y)$ satisfies (2.5) for every (3.67). We denote by Q_m the totality of all functions of the forme (3.67) with $\lim\limits_{T\to\infty} w_l(T) = \lim\limits_{T\to\infty} s_l(T) = \lim\limits_{T\to\infty} v_l(T) = \lim\limits_{T\to\infty} r_l(T) = \lim\limits_{T\to\infty} p_l(T) = 0$. But $\cup_{m=1}^{\infty} Q_m$ is dense in $\mathcal{W}(Q)$, then the relation (2.5) holds for all $(\varphi, \psi, \theta, q, y) \in \mathcal{W}(Q)$. Thus, we have shown that the limit function $(\varphi, \psi, \theta, q, y)$ is a generalized solution of system (2.3) in $\mathcal{V}(Q)$.

3.6. Continuous dependence and uniqueness

First, we prove the continuous dependence and uniqueness for weak solutions of system (2.3).Let $(\varphi, \varphi_t, \varphi_{tt}, \psi, \theta, q, y, y_t)$ and $(\Gamma, \Gamma_t, \Gamma_{tt}, \Xi, \Omega, \xi, \Pi, \Pi_t)$ be two global solutions of system (2.3) with respect to initial data $(\varphi_0, \varphi_1, \varphi_2, \psi_0, \theta_0, q_0, \Theta_0, \Theta_1)$ and $(\Gamma_0, \Gamma_1, \Gamma_2, \Xi_0, \Omega_0, \xi_0, \Phi_0, \Phi_1)$, respectively. Let

$$\begin{aligned}
\Lambda(t) &= \varphi - \Gamma,\\
\Sigma(t) &= \psi - \Xi,\\
\mathcal{M}(t) &= \theta - \Omega,\\
\mathcal{R}(t) &= q - \xi,\\
\chi(t) &= \Pi - \Phi.
\end{aligned} \tag{3.72}$$

Then, $(\Lambda, \lambda, \mathcal{M}, \mathcal{R}, \chi)$ verifies (2.3), and we have

$$\begin{aligned}
&\rho_1 \Lambda_{tt} - K(\Lambda_x + \Sigma)_x + \mu_1 \Lambda_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\chi(x,1,s)\,ds = 0,\\
&-\rho_2 \Lambda_{ttx} - b\lambda_{xx} + K(\Lambda_x + \lambda) + \gamma \mathcal{M}_x = 0,\\
&\rho_3 \mathcal{M}_t + \kappa \mathcal{R}_x + \gamma \lambda_{tx} = 0,\\
&\tau_0 \mathcal{R}_t + \delta \mathcal{R} + \kappa \mathcal{M}_x = 0,\\
&s\chi_t + \chi_\rho = 0,\\
&s\chi_{tt} + \chi_{\rho t} = 0. \qquad (3.73)
\end{aligned}$$

Now, multiplying $(3.73)_1$, $(3.73)_2$, $(3.73)_3$ and $(3.73)_4$ by Λ_t, λ_t, $\mathcal{M}_t, \mathcal{R}_t$, respectively, and integrating over $(0,1)$ (the same arguments as in energy method), we get

$$\begin{aligned}
&\frac{\rho_1}{2}\frac{d}{dt}\int_0^1 \Lambda_t^2 dx + K\int_0^1 (\Lambda_x + \Sigma)\Lambda_{tx} dx + \mu_1 \int_0^1 \Lambda_t^2 dx\\
&+\int_0^1 \Lambda_t \int_{\tau_1}^{\tau_2} \mu_2(s)\chi(x,1,s)\,ds dx = 0, \qquad (3.74)
\end{aligned}$$

then,

$$\begin{aligned}
&\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_0^1 \Lambda_{tt}^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \Lambda_{xt}^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \Sigma_x^2 dx\\
&+K\int_0^1 \Sigma_t(\Lambda_x + \Sigma)\,dx + \gamma\int_0^1 \Sigma_t \mathcal{M}_x dx\\
&\frac{\rho_2\mu_1}{K}\int_0^1 \Lambda_{tt}^2 dx + \frac{\rho_2}{K}\int_0^1 \Lambda_{tt}\int_{\tau_1}^{\tau_2} \mu_2(s)\chi_t(x,1,s)\,ds dx = 0, \qquad (3.75)
\end{aligned}$$

after that,

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \mathcal{M}^2 dx + \kappa\int_0^1 \mathcal{M}\mathcal{R}_x dx - \gamma\int_0^1 \Sigma_t \mathcal{M}_x dx = 0, \qquad (3.76)$$

finally,

$$\frac{\tau_0}{2}\frac{d}{dt}\int_0^1 \mathcal{R}^2 dx + \delta\int_0^1 \mathcal{R}^2 dx - \kappa\int_0^1 \mathcal{M}\mathcal{R}_x dx = 0. \qquad (3.77)$$

By combining (3.74), (3.75), (3.76) and (3.77), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\Lambda_t^2 + K(\Lambda_x + \Sigma)^2 + \frac{\rho_2\rho_1}{K}\Lambda_{tt}^2 + b\Sigma_x^2 + \rho_2\Lambda_{xt}^2 \qquad (3.78)\\
&\rho_3\mathcal{M}^2 + \tau_0\mathcal{R}^2\Big] dx + \mu_1\int_0^1 \Lambda_t^2 dx + \delta\int_0^1 \mathcal{R}^2 dx\\
&+\int_0^1 \Lambda_t\int_{\tau_1}^{\tau_2} \mu_2(s)\chi(x,1,s)\,ds dx + \frac{\rho_2\mu_1}{K}\int_0^1 \Lambda_{tt}^2 dx\\
&+\frac{\rho_2}{K}\int_0^1 \Lambda_{tt}\int_{\tau_1}^{\tau_2} \mu_2(s)\chi_t(x,1,s)\,ds dx = 0.
\end{aligned}$$

Now, multiplying $(3.73)_5$ by $|\mu_2(s)|\chi(x,\rho,s)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\chi^2(x,\rho,s)\,ds d\rho dx\\
&-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1 \Lambda_t^2 dx\\
&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\chi^2(x,1,s)\,ds dx = 0. \qquad (3.79)
\end{aligned}$$

Then, multiplying (3.73)$_6$ by $|\mu_2(s)|\chi_t(x,\rho,s)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\begin{aligned}&\frac{\rho_2}{2K}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\chi_t^2(x,\rho,s)\,dsd\rho dx\\&-\frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1\Lambda_{tt}^2dx\\&+\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\chi_t^2(x,1,s)\,dsdx = 0.\end{aligned}\tag{3.80}$$

By combining (3.78), (3.79) and (3.80), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= -\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1\Lambda_t^2dx\\&\quad-\frac{\rho_2}{K}\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\int_0^1\Lambda_{tt}^2dx\\&\quad-\int_0^1\Lambda_t\int_{\tau_1}^{\tau_2}\mu_2(s)\chi(x,1,s)\,dsdx-\delta\int_0^1\mathcal{R}^2dx\\&\quad-\frac{\rho_2}{K}\int_0^1\Lambda_{tt}\int_{\tau_1}^{\tau_2}\mu_2(s)\chi_t(x,1,s)\,dsdx\\&\quad-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\chi^2(x,1,s)\,dsdx\\&\quad-\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\chi_t^2(x,1,s)\,dsdx\\&\le -\eta_0\int_0^1\Lambda_t^2dx-\eta_0\frac{\rho_2}{K}\int_0^1\Lambda_{tt}^2dx-\delta\int_0^1\mathcal{R}^2dx\le 0\\&\le c\left(\int_0^1\left[\Lambda_t^2+(\Lambda_x+\Sigma)^2+\Lambda_{tt}^2+\Lambda_{xt}^2+\Sigma_x^2+\mathcal{M}^2+\mathcal{R}^2\right]dx\right.\\&\quad+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\chi^2(x,\rho,s)\,dsdxd\rho\\&\quad\left.+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\chi_t^2(x,\rho,s)\,dsdxd\rho\right).\end{aligned}\tag{3.81}$$

By integrating (3.81) over $(0,t)$, we obtain

$$\begin{aligned}E(t)-E(0) &\le c\left(\int_0^t\left[\|\Lambda_t\|^2+\|(\Lambda_x+\Sigma)\|^2+\|\Lambda_{tt}\|^2+\|\Lambda_{xt}\|^2+\|\Sigma_x\|^2+\|\mathcal{M}\|^2+\|\mathcal{R}\|^2\right]d\tau\right.\\&\quad\left.+\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho d\tau+\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho d\tau\right),\end{aligned}$$

implies,

$$\begin{aligned}E(t) &\le E(0)+c\int_0^t\left[\|\Lambda_t\|^2+\|(\Lambda_x+\Sigma)\|^2+\|\Lambda_{tt}\|^2\right.\\&\quad\left.+\|\Lambda_{xt}\|^2+\|\mathcal{M}\|^2+\|\Sigma_x\|^2+\|\mathcal{R}\|^2\right]d\tau\\&\quad+c\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho d\tau\\&\quad+c\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho d\tau.\end{aligned}\tag{3.82}$$

On the other hand, we have

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\Lambda_t^2 + K(\Lambda_x+\Sigma)^2 + \frac{\rho_2\rho_1}{K}\Lambda_{tt}^2 + \rho_2\Lambda_{xt}^2 + b\Sigma_x^2 + \rho_3\mathcal{M}^2 + \tau_0\mathcal{R}^2\right]dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\chi^2(x,\rho,s)\,dsd\rho dx \\ &+\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\chi_t^2(x,\rho,s)\,dsd\rho dx \\ &\geq \frac{1}{2}\min\left(\rho_1, K, \frac{\rho_2\rho_1}{K}, \rho_2, b, \rho_3, \tau_0, \frac{\rho_2}{K}, 1\right) \\ &\times\Big(\|\Lambda_t\|^2 + \|(\Lambda_x+\lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{xt}\|^2 + \|\lambda_x\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho\Big), \end{aligned}$$

implies,

$$\begin{aligned} E(t) &\geq m_0\Big(\|\Lambda_t\|^2 + \|(\Lambda_x+\lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{xt}\|^2 + \|\lambda_x\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho\Big). \end{aligned} \tag{3.83}$$

So, we have

$$\begin{aligned} & m_0\Big(\|\Lambda_t\|^2 + \|(\Lambda_x+\lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{xt}\|^2 \\ &+\|\lambda_x\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 + \int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho\Big) \\ \leq\ & E(0) + c\int_0^t \Big[\|\Lambda_t\|^2 + \|(\Lambda_x+\lambda)\|^2 + \|\Lambda_{tt}\|^2 \\ &+\|\Lambda_{xt}\|^2 + \|\mathcal{M}\|^2 + \|\lambda_x\|^2 + \|\mathcal{R}\|^2\,d\tau\Big] \\ &+c\int_0^t\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho d\tau \\ &+c\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho. \end{aligned} \tag{3.84}$$

Applying Gronwall's inequality to (3.84), we get

$$\begin{aligned} & \|\Lambda_t\|^2 + \|(\Lambda_x+\lambda)\|^2 + \|\Lambda_{tt}\|^2 + \|\Lambda_{xt}\|^2 + \|\lambda_x\|^2 + \|\mathcal{M}\|^2 + \|\mathcal{R}\|^2 \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi(x,\rho,s)\|^2\,dsd\rho \\ &+\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\,\|\chi_t(x,\rho,s)\|^2\,dsd\rho \\ \leq\ & \frac{1}{m_0}E(0)\exp(Mt), \end{aligned} \tag{3.85}$$

where $M = \frac{c}{m_0}$.

This shows that solution of system (2.3) , depends continuously on the initial data.

4. Stability Results

In this section, we state and prove our stability results for the energy of the solution of system (2.3), using the multiplier technique. To achieve our goal, we need the following lemmas.

Lemma 4.1. *Define the energy of solution as*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2 + K(\varphi_x+\psi)^2 + \frac{\rho_2\rho_1}{K}\varphi_{tt}^2 + \rho_2\varphi_{xt}^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\right] dx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y^2(x,\rho,s)\, ds d\rho dx \\ &+\frac{\rho_2}{2K}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right| y_t^2(x,\rho,s)\, ds d\rho dx, \end{aligned}$$

satisfies

$$\begin{aligned} E'(t) &\le -\left(\mu_1 - \int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\int_0^1 \varphi_t^2 dx - \delta\int_0^1 q^2 dx \\ &\quad -\frac{\rho_2}{K}\left(\mu_1 - \int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\int_0^1 \varphi_{tt}^2 dx \\ &\le -\eta_0\int_0^1 \varphi_t^2 dx - \delta\int_0^1 q^2 dx \\ &\quad -\frac{\rho_2}{K}\eta_0\int_0^1 \varphi_{tt}^2 dx \\ &\le 0, \end{aligned}$$

where $\eta_0 = \left(\mu_1 - \int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right) > 0.$

Proof. First, multiplying $(2.3)_1$, $(2.3)_2$, $(2.3)_3$ and $(2.3)_4$ by φ_t, ψ_t, θ and q,respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain

$$\begin{aligned} &\frac{\rho_1}{2}\frac{d}{dt}\int \varphi_t^2 dx + K\int_0^1 \varphi_{tx}(\varphi_x+\psi)\, dx + \mu_1\int_0^1 \varphi_t^2 dx \\ &+\int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)\, y(x,1,s)\, ds dx \;=\; 0, \end{aligned} \tag{4.1}$$

then,

$$\begin{aligned} &\rho_2\int_0^1 \psi_{tx}\varphi_{tt} dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx + K\int_0^1 \psi_t(\varphi_x+\psi)\, dx \\ &+\gamma\int_0^1 \psi_t\theta_x dx \;=\; 0, \end{aligned} \tag{4.2}$$

next,

$$\frac{\rho_3}{2}\frac{d}{dt}\int_0^1 \theta^2 dx - \kappa\int_0^1 \theta_x q dx - \gamma\int_0^1 \theta_x\psi_t dx \;=\; 0, \tag{4.3}$$

finally,

$$\frac{\tau_0}{2}\frac{d}{dt}\int_0^1 q^2 dx + \delta\int_0^1 q^2 dx + \kappa\int_0^1 q\theta_x dx \;=\; 0, \tag{4.4}$$

now, substituting $\psi_{tx} = \frac{\rho_1}{K}\varphi_{ttt} - \varphi_{xxt} + \frac{\mu_1}{K}\varphi_{tt} + \frac{1}{K}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_t(x,1,s)\, dx$ into first integral of (4.2) and using the integral by parts, we get

$$\begin{aligned}
&\frac{\rho_2\rho_1}{2K}\frac{d}{dt}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2}{2}\frac{d}{dt}\int_0^1 \varphi_{xt}^2 dx + \frac{b}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx \\
&+K\int_0^1 \psi_t(\varphi_x + \psi)\, dx + \gamma\int_0^1 \psi_t\theta_x dx + \frac{\mu_1\rho_2}{K}\int_0^1 \varphi_{tt}^2 dx \\
&+\frac{\rho_2}{K}\int_0^1 \varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_t(x,1,s)\, dsdx \;=\; 0,
\end{aligned} \tag{4.5}$$

summing $(4.1), (4.3), (4.4)$ and (4.5), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_t^2 + K(\varphi_x+\psi)^2 + \frac{\rho_2\rho_1}{K}\varphi_{tt}^2 \\
&+\rho_2\varphi_{xt}^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\Big]\, dx \\
&+\mu_1\int_0^1 \varphi_t^2 dx + \delta\int_0^1 q^2 dx + \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}\mu_2(s)\, y(x,1,s)\, dsdx \\
&+\mu_1\frac{\rho_2}{K}\int_0^1 \varphi_{tt}^2 dx + \frac{\rho_2}{K}\int_0^1 \varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2(s)\, y_t(x,1,s)\, dsdx \;=\; 0.
\end{aligned} \tag{4.6}$$

Second, multiplying $(2.3)_5$ by $(y\,|\mu_2(s)|)$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and recall that $y(x,0,s) = \varphi_t$, yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, dsd\rho dx \\
=\;& -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left(y^2(x,1,s) - y^2(x,0,s)\right) dsdx \\
=\;& \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\, ds\right)\int_0^1 \varphi_t^2 dx \\
&-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\, y^2(x,1,s)\, dsdx \;=\; 0,
\end{aligned} \tag{4.7}$$

now, differentiating and multiplying $(2.3)_5$ by $(y_t\,|\mu_2(s)|)$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we get

$$\begin{aligned}
&\frac{\rho_2}{2K}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx \\
=\;& -\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left(y_t^2(x,1,s) - y_t^2(x,0,s)\right) dsdx \\
&\frac{\rho_2}{2K}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx \\
=\;& \frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\, ds\right)\int_0^1 \varphi_{tt}^2 dx \\
&-\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\, y_t^2(x,1,s)\, dsdx \;=\; 0.
\end{aligned} \tag{4.8}$$

A combination of (4.6), (4.7) and (4.8), gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho_1\varphi_t^2+K\left(\varphi_x+\psi\right)^2+\frac{\rho_2\rho_1}{K}\varphi_{tt}^2+\rho_2\varphi_{xt}^2+b\psi_x^2+\rho_3\theta^2+\tau_0q^2\right]dx\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|y^2\left(x,\rho,s\right)dsd\rho dx\\
&+\frac{\rho_2}{2K}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|y_t^2\left(x,\rho,s\right)dsd\rho dx\\
&=\ -\left(\mu_1-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\right)\int_0^1\varphi_t^2dx-\delta\int_0^1q^2dx\\
&-\frac{\rho_2}{K}\left(\mu_1-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\right)\int_0^1\varphi_{tt}^2dx\\
&-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y\left(x,1,s\right)dsdx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y^2\left(x,1,s\right)dsdx\\
&-\frac{\rho_2}{K}\int_0^1\varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y_t\left(x,1,s\right)dsdx-\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y_t^2\left(x,1,s\right)dsdx,
\end{aligned}$$

where

$$\begin{aligned}
E\left(t\right)\ =\ &\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho_1\varphi_t^2+K\left(\varphi_x+\psi\right)^2+\frac{\rho_2\rho_1}{K}\varphi_{tt}^2\right.\\
&\left.+\rho_2\varphi_{xt}^2+b\psi_x^2+\rho_3\theta^2+\tau_0q^2\right]dx\\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|y^2\left(x,\rho,s\right)dsd\rho dx\\
&+\frac{\rho_2}{2K}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|y_t^2\left(x,\rho,s\right)dsd\rho dx,
\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}
&\frac{d}{dt}E\left(t\right)=-\left(\mu_1-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\right)\int_0^1\varphi_t^2dx-\delta\int_0^1q^2dx\\
&-\frac{\rho_2}{K}\left(\mu_1-\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\right)\int_0^1\varphi_{tt}^2dx\\
&-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y\left(x,1,s\right)dsdx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y^2\left(x,1,s\right)dsdx\\
&-\frac{\rho_2}{K}\int_0^1\varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y_t\left(x,1,s\right)dsdx-\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y_t^2\left(x,1,s\right)dsdx.
\end{aligned}\tag{4.10}$$

Meanwhile, using Young's and Cauchy Shwarz's inequalities, we have

$$\begin{aligned}
&-\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y\left(x,1,s\right)dsdx\\
\leq\ &\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y^2\left(x,1,s\right)dsdx+\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\int_0^1\varphi_t^2dx,
\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}
&-\frac{\rho_2}{K}\int_0^1\varphi_t\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)y_t\left(x,1,s\right)dsdx\\
\leq\ &\frac{\rho_2}{2K}\int_0^1\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|y_t^2\left(x,1,s\right)dsdx+\frac{\rho_2}{2K}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds\right)\int_0^1\varphi_{tt}^2dx.
\end{aligned}\tag{4.12}$$

Now, substituting (4.11) and (4.12) into (4.10), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &\leq -\left(\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\int_0^1\varphi_t^2dx-\delta\int_0^1 q^2dx\\ &-\frac{\rho_2}{K}\left(\mu_1-\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\int_0^1\varphi_{tt}^2dx\\ &\leq -\eta_0\int_0^1\varphi_t^2dx-\delta\int_0^1 q^2dx-\frac{\rho_2}{K}\eta_0\int_0^1\varphi_{tt}^2dx, \qquad (4.13)\end{aligned}$$

where $\eta_0=\left(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0$.then we obtain that E is decreasing. □

Lemma 4.1. *The functional*

$$F_1(t)=-\frac{\mu_1}{2}\int_0^1\varphi_t^2dx-K\int_0^1\varphi_{tx}\varphi_x dx, \qquad (4.14)$$

satisfies

$$\begin{aligned}F_1'(t) &\leq -K\int_0^1\varphi_{tx}^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\varphi_{tt}^2dx+\varepsilon_1\int_0^1\psi_x^2dx\\ &+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,y^2(x,1,s)\,dsdx. \qquad (4.15)\end{aligned}$$

Proof. A simple differentiation of $F_1(t)$, using parametric integral, $(2.3)_1$, integration by parts, Young's and Poincaré inequalities, we get

$$\begin{aligned}F_1'(t) &= \rho_1\int_0^1\varphi_{tt}^2dx-K\int_0^1\varphi_{tt}\psi_x dx-K\int_0^1\varphi_{tx}^2dx\\ &+\int_0^1\varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y(x,1,s)\,dsdx\\ &\leq -K\int_0^1\varphi_{tx}^2dx+\rho_1\int_0^1\varphi_{tt}^2dx+\frac{K^2}{\varepsilon_1}\int_0^1\varphi_{tt}^2dx+\varepsilon_1\int_0^1\psi_x^2dx\\ &+\rho_1\int_0^1\varphi_{tt}^2dx+\frac{\mu_1}{\rho_1}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,y^2(x,1,s)\,dsdx\\ &\leq -K\int_0^1\varphi_{tx}^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\varphi_{tt}^2dx+\varepsilon_1\int_0^1\psi_x^2dx\\ &+\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,y^2(x,1,s)\,dsdx,\end{aligned}$$

where $c=\max\left(\frac{\mu_1}{\rho_1},2\rho_1\right)$ and

$$\begin{aligned}&\int_0^1\varphi_{tt}\int_{\tau_1}^{\tau_2}\mu_2(s)\,y(x,1,s)\,dsdx\\ &\leq \rho_1\int_0^1\varphi_{tt}^2dx+\frac{\mu_1}{\rho_1}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,y^2(x,1,s)\,dsdx.\end{aligned}$$

□

Lemma 4.2. *The functional*

$$F_2(t) = \rho_1 \int_0^1 \varphi\varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx + \frac{\mu_1\rho_2}{2K} \int_0^1 \varphi_t^2 dx + \rho_2 \int_0^1 \varphi_{tx}\varphi_x dx, \tag{4.16}$$

satisfies

$$\begin{aligned} F_2'(t) \leq & -\frac{b}{2}\int_0^1 \psi_x^2 dx - \frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx - \frac{\rho_1\rho_2}{2K}\int_0^1 \varphi_{tt}^2 dx \\ & +\rho_2 \int_0^1 \varphi_{tx}^2 dx + \frac{\rho_3\kappa}{4}\int_0^1 \theta^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx \\ & +c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx. \end{aligned} \tag{4.17}$$

Proof. A simple differentiation of $F_2(t)$, using parametric integral, $(2.3)_1$, $(2.3)_2$ integration by parts, Young's, Cauchy Schwarz and Poincaré inequalities, we get

$$\begin{aligned} F_2'(t) = & \rho_1 \int_0^1 \varphi_t^2 dx - K\int_0^1 (\varphi_x+\psi)^2 dx - \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)\, y(x,1,s)\, ds dx \\ & -\frac{\rho_1\rho_2}{K}\int_0^1 \varphi_{tt}^2 dx - b\int_0^1 \psi_x^2 dx + \gamma \int_0^1 \theta\psi_x dx + \rho_2 \int_0^1 \varphi_{tx}^2 dx \\ & -\frac{\rho_2}{K}\int_0^1 \varphi_{tt}\int_{\tau_1}^{\tau_2} \mu_2(s)\, y(x,1,s)\, ds dx \\ \leq & -\frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx - \frac{b}{2}\int_0^1 \psi_x^2 dx - \frac{\rho_1\rho_2}{2K}\int_0^1 \varphi_{tt}^2 dx \\ & +\rho_2 \int_0^1 \varphi_{tx}^2 dx + \rho_1 \int_0^1 \varphi_t^2 dx + \frac{\rho_3\kappa}{4}\int_0^1 \theta^2 dx \\ & +c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx, \end{aligned}$$

where

$$\begin{aligned} & -\int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu_2(s)\, y(x,1,s)\, ds dx \\ \leq & \frac{1}{2}\int_0^1 \varphi^2 dx + \frac{1}{2}\int_0^1 \left[\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds\right] dx \\ \leq & \frac{K}{2}\int_0^1 \varphi_x^2 dx + \frac{\mu_1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx \\ \leq & \frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx + \frac{\mu_1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx, \end{aligned}$$

and

$$\begin{aligned} & -\int_0^1 \varphi_{tt} \int_{\tau_1}^{\tau_2} \mu_2(s)\, y(x,1,s)\, ds dx \\ \leq & \frac{\rho_1}{2}\int_0^1 \varphi_{tt}^2 dx + \frac{\mu_1}{2\rho_1}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx. \end{aligned}$$

□

Lemma 4.3. *The functional*

$$\begin{aligned} F_3(t) &= -\rho_3\tau_0 \int_0^1 q \int_0^x \theta(y,t)\, dydx \\ &\quad -\tau_0\gamma \int_0^1 q\psi dx, \end{aligned} \tag{4.18}$$

satisfies

$$\begin{aligned} F_3'(t) &\le -\frac{\kappa\rho_3}{2}\int_0^1 \theta^2 dx + \frac{b}{4}\int_0^1 \psi_x^2 dx \\ &\quad +c\int_0^1 q^2 dx. \end{aligned} \tag{4.19}$$

Proof. A simple differentiation of $F_3(t)$, using parametric integral, $(2.3)_3$, $(2.3)_4$, integration by parts, Young's, Poincaré and Cauchy Schwarz inequalities, we get

$$\begin{aligned} F_3'(t) &= \rho_3\delta \int_0^1 q \int_0^x \theta(y,t)\, dydx - \rho_3\kappa \int_0^1 \theta^2 dx + \tau_0\kappa \int_0^1 q^2 dx \\ &\quad +\gamma\delta \int_0^1 \psi q dx - \gamma\kappa \int_0^1 \psi_x \theta dx \\ &\le -\frac{\rho_3\kappa}{2}\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx + \frac{b}{4}\int_0^1 \psi_x^2 dx, \end{aligned}$$

□

Lemma 4.4. *The functional*

$$F_4(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\exp(-s\rho)\,|\mu_2(s)|\, y^2(x,\rho,s)\, dsd\rho dx, \tag{4.20}$$

satisfies

$$\begin{aligned} F_4'(t) &\le -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_t^2(t)\, dx \\ &\quad -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, dsd\rho dx. \end{aligned} \tag{4.21}$$

Proof. A simple differentiation of $F_4(t)$, using parametric integral and $(2.3)_5$, we get

$$\begin{aligned} F_4'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\exp(-s\rho)\,|\mu_2(s)|\, y(x,\rho,s)\, y_\rho(x,\rho,s)\, dsd\rho dx \\ &= -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[\exp(-s)\, y^2(x,1,s) - y^2(x,0,s)\right] dsdx \\ &\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\exp(-s\rho)\,|\mu_2(s)|\, y^2(x,\rho,s)\, dsd\rho dx. \end{aligned}$$

Using the equality $y(x,0,s) = \varphi_t(t)$ and $-\exp(-s\rho) \le -\exp(-s) \le -\exp(-\tau_2)$ for all $0 \le \rho \le 1$

and $\tau_1 \leq s \leq \tau_2$, we get

$$\begin{aligned}
F_4'(t) &\leq -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \exp(-s) y^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_t^2(t)\, dx \\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s)|\mu_2(s)| y^2(x,\rho,s)\, dsd\rho dx \\
&\leq -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \exp(-\tau_2) y^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_t^2(t)\, dx \\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-\tau_2)|\mu_2(s)| y^2(x,\rho,s)\, dsd\rho dx \\
&\leq -m_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_t^2(t)\, dx \\
&\quad -m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| y^2(x,\rho,s)\, dsd\rho dx,
\end{aligned}$$

where $m_1 = \exp(-\tau_2)$. □

Lemma 4.5. *The functional*

$$F_5(t) = \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s\rho)|\mu_2(s)| y_t^2(x,\rho,s)\, dsd\rho dx, \tag{4.22}$$

satisfies

$$\begin{aligned}
F_5'(t) &\leq -m_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| y_t^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_{tt}^2(t)\, dx \\
&\quad -m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| y_t^2(x,\rho,s)\, dsd\rho dx.
\end{aligned} \tag{4.23}$$

Proof. A simple differentiation of $F_5(t)$, using parametric integral and $(2.3)_6$, we get

$$\begin{aligned}
F_5'(t) &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s\rho)|\mu_2(s)| y_t(x,\rho,s) y_{t\rho}(x,\rho,s)\, dsd\rho dx \\
&= -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{\partial}{\partial\rho}\left[\exp(-s\rho) y_t^2(x,\rho,s)\right] d\rho dsdx \\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s\rho)|\mu_2(s)| y_t^2(x,\rho,s)\, dsd\rho dx \\
&= -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[\exp(-s) y_t^2(x,1,s) - y_t^2(x,0,s)\right] dsdx \\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-s\rho)|\mu_2(s)| y_t^2(x,\rho,s)\, dsd\rho dx.
\end{aligned}$$

Using the equality $y_t(x,0,s) = \varphi_{tt}(t)$ and $-\exp(-s\rho) \leq -\exp(-s) \leq -\exp(-\tau_2)$ for all $0 \leq \rho \leq 1$

and $\tau_1 \leq s \leq \tau_2$, we get

$$\begin{aligned}
F_5'(t) &\leq -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \exp(-s)\, y_t^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_{tt}^2(t)\, dx \\
&\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \exp(-s) |\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx \\
&\leq -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \exp(-\tau_2)\, y_t^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_{tt}^2(t)\, dx \\
&\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s \exp(-\tau_2) |\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx \\
&\leq -m_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y_t^2(x,1,s)\, dsdx + \mu_1 \int_0^1 \varphi_{tt}^2(t)\, dx \\
&\quad -m_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx,
\end{aligned}$$

where $m_1 = \exp(-\tau_2)$. □

Next, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional E.

Theorem 4.2. *Assume that (1.15) holds, then there exist positive constants ℓ_1 and ℓ_2 such that the energy functional (4.9) satisfies*

$$E(t) \leq \ell_2 \exp(-\ell_1 t), \ \forall t \geq 0. \tag{4.24}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2 (F_2(t) + F_3(t)) + N_4 (F_4(t) + F_5(t)), \tag{4.25}$$

where $N,\ N_1,\ N_2,\ N_4 > 0$, by differentiating (4.25) and using (4.13), (4.15), (4.17), (4.19), (4.21) and

(4.23), we have

$$\begin{aligned}
\mathcal{L}'(t) &= NE'(t) + N_1F_1'(t) + N_2(F_2'(t) + F_3'(t)) + N_4(F_4'(t) + F_5'(t)) \\
&\leq -[N\eta_0 - N_2\rho_1 - N_4\mu_1]\int_0^1 \varphi_t^2 dx \\
&\quad -\left[N\frac{\rho_2}{K}\eta_0 + N_2\frac{\rho_1\rho_2}{2K} - N_1c\left(1+\frac{1}{\varepsilon_1}\right) - N_4\mu_1\right]\int_0^1 \varphi_{tt}^2 dx \\
&\quad -[N\delta - cN_2]\int_0^1 q^2 dx \\
&\quad -[N_1K - N_2\rho_2]\int_0^1 \varphi_{xt}^2 dx \\
&\quad -\left[N_2\frac{b}{2} - N_1\varepsilon_1 - N_2\frac{b}{4}\right]\int_0^1 \psi_x^2 dx \\
&\quad -N_2\frac{K}{2}\int_0^1 (\varphi_x+\psi)^2 dx \\
&\quad -N_2\frac{\rho_3\kappa}{4}\int_0^1 \theta^2 dx \\
&\quad -[N_4m_1 - N_1c - N_2c]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y^2(x,1,s)\, dsdx \\
&\quad -N_4m_1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y_t^2(x,1,s)\, dsdx \\
&\quad -N_4m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, dsd\rho dx \\
&\quad -N_4m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, dsd\rho dx,
\end{aligned}$$

By setting

$$\varepsilon_1 = \frac{bN_2}{8N_1}.$$

Once N_2 is fixed, we then choose N_1 large enough such that

$$\gamma_4 = N_1K - N_2\rho_2 > 0.$$

Then we choose N_4 large enough so that

$$\begin{aligned}
\gamma_5 &= m_1N_4 > 0, \\
\gamma_6 &= N_4m_1 - N_1c - N_2c > 0.
\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) &\leq -(N\eta_0 - c)\int_0^1 \varphi_t^2 dx - \left(N\frac{\rho_2}{K}\eta_0 + \gamma_0 - c\right)\int_0^1 \varphi_{tt}^2 dx \\
&\quad -\gamma_1 \int_0^1 \psi_x^2 dx - \gamma_2 \int_0^1 (\varphi_x + \psi)^2 dx - \gamma_3 \int_0^1 \theta^2 dx \\
&\quad -(N\eta_0 - c)\int_0^1 q^2 dx \\
&\quad -\gamma_4 \int_0^1 \varphi_{xt}^2 dx - \gamma_6 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y^2(x,1,s)\, ds dx \\
&\quad -\gamma_5 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y_t^2(x,1,s)\, ds dx \\
&\quad -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, ds d\rho dx \\
&\quad -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, ds d\rho dx, \qquad (4.26)
\end{aligned}$$

where $\gamma_0 = N_2\frac{\rho_1\rho_2}{2K}, \gamma_1 = N_2\frac{b}{8}, \gamma_2 = N_2\frac{K}{2}$ and $\gamma_3 = N_2\frac{\rho_3\kappa}{4}$ .On the other hand, if we let

$$\mathcal{Z}(t) = N_1 F_1(t) + N_2(F_2(t) + F_3(t)) + N_4(F_4(t) + F_5(t)),$$

then

$$\begin{aligned}
|\mathcal{Z}(t)| &= |N_1 F_1(t) + N_2(F_2(t) + F_3(t)) + N_4(F_4(t) + F_5(t))| \\
&\leq N_1|F_1(t)| + N_2|F_2(t)| + N_2|F_3(t)| + N_4|F_4(t)| + N_4|F_5(t)| \\
&\leq N_1\frac{\mu_1}{2}\int_0^1 \varphi_t^2 dx + N_1 K \int_0^1 |\varphi_{tx}\varphi_x|\, dx \\
&\quad +N_2\rho_1 \int_0^1 |\varphi\varphi_t|\, dx + N_2\frac{\mu_1}{2}\int_0^1 \varphi^2 dx \\
&\quad +N_2\frac{\mu_1\rho_2}{2K}\int_0^1 \varphi_t^2 dx + N_2\rho_2 \int_0^1 |\varphi_x\varphi_{tx}|\, dx \\
&\quad +N_2\rho_3\tau_0 \int_0^1 \left| q\int_0^x \theta(y,t)\, dy \right| dx + N_2 \int_0^1 |q\psi|\, dx \\
&\quad +N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\exp(-s\rho)|\,|\mu_2(s)|\, y^2(x,\rho,s)\, ds d\rho dx \\
&\quad +N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\exp(-s\rho)|\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, ds d\rho dx.
\end{aligned}$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, we get

$$\begin{aligned}
|\mathcal{Z}(t)| \leq\ & \left[N_1\frac{\mu_1}{2}+\frac{N_2\rho_1}{2}+N_2\frac{\mu_1\rho_2}{2K}\right]\int_0^1 \varphi_t^2 dx \\
&+\left[\frac{N_1K}{2}+\frac{N_2\rho_2}{2}\right]\int_0^1 \varphi_{tx}^2 dx \\
&+\frac{N_2c}{2}\int_0^1 \psi_x^2 dx+\int_0^1 \varphi_{tt}^2 dx \\
&+\left[\frac{N_1K}{2}+\frac{N_2\rho_1 c}{2}+N_2\frac{\mu_{1C}}{2}+\frac{N_2\rho_2}{2}\right]\int_0^1 (\varphi_x+\psi)^2 dx \\
&+\left(\frac{N_2\rho_3\tau_0}{2}+\frac{N_2}{2}\right)\int_0^1 q^2 dx+\frac{N_2\rho_3\tau_0 C}{2}\int_0^1 \theta^2 dx \\
&+N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, ds d\rho dx \\
&+N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\ & c\int_0^1\left[\varphi_t^2+\varphi_{tt}^2+\varphi_{tx}^2+\psi_x^2+(\varphi_x+\psi)^2+q^2+\theta^2\right] dx \\
&+c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y^2(x,\rho,s)\, ds d\rho dx \\
&+c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu_2(s)|\, y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\ & cE(t).
\end{aligned}$$

Consequently

$$|\mathcal{Z}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t),$$

implies

$$-cE(t) \leq \mathcal{L}(t) - NE(t) \leq cE(t),$$

which yields

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (c+N)E(t).$$

Now, we choose N large enough so that

$$N\frac{\rho_2}{K}\eta_0 - c > 0, N\eta_0 - c > 0, N\delta - c > 0, N - c > 0,$$

we obtain

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \forall t \geq 0, \tag{4.27}$$

where β_1 and β_2 are positive constants.

So

$$\begin{aligned}
\mathcal{L}'(t) \leq\; & -\gamma_7 \int_0^1 \varphi_t^2 dx - \gamma_8 \int_0^1 \varphi_{tt}^2 dx - \gamma_1 \int_0^1 \psi_x^2 dx \\
& -\gamma_2 \int_0^1 (\varphi_x + \psi)^2 dx - \gamma_4 \int_0^1 \varphi_{xt}^2 dx \\
& -\gamma_9 \int_0^1 q^2 dx - \gamma_3 \int_0^1 \theta^2 dx \\
& -\gamma_6 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y^2(x,1,s)\, ds dx \\
& -\gamma_5 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| y_t^2(x,1,s)\, ds dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y^2(x,\rho,s)\, ds d\rho dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\; & -\gamma_7 \int_0^1 \varphi_t^2 dx - \gamma_8 \int_0^1 \varphi_{tt}^2 dx - \gamma_1 \int_0^1 \psi_x^2 dx \\
& -\gamma_2 \int_0^1 (\varphi_x + \psi)^2 dx - \gamma_4 \int_0^1 \varphi_{xt}^2 dx \\
& -\gamma_9 \int_0^1 q^2 dx - \gamma_3 \int_0^1 \theta^2 dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y^2(x,\rho,s)\, ds d\rho dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\; & -\gamma_7 \int_0^1 \varphi_t^2 dx - \gamma_8 \int_0^1 \varphi_{tt}^2 dx - \gamma_1 \int_0^1 \psi_x^2 dx \\
& -\gamma_2 \int_0^1 (\varphi_x + \psi)^2 dx - \gamma_4 \int_0^1 \varphi_{xt}^2 dx \\
& -\gamma_9 \int_0^1 q^2 dx - \gamma_3 \int_0^1 \theta^2 dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y^2(x,\rho,s)\, ds d\rho dx \\
& -\gamma_5 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\; & -\eta \int_0^1 \left[\varphi_t^2 + \varphi_{tt}^2 + \varphi_{tx}^2 + \psi_x^2 + (\varphi_x + \psi)^2 + q^2 + \theta^2\right] dx \\
& -\eta \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y^2(x,\rho,s)\, ds d\rho dx \\
& -\eta \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| y_t^2(x,\rho,s)\, ds d\rho dx \\
\leq\; & -\eta E(t), \ \forall t \geq 0, \qquad (4.28)
\end{aligned}$$

where $\eta = \min(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9) > 0$.

A combination (4.27) with (4.28) gives,

$$\mathcal{L}'(t) \leq -\eta E(t) \leq -\frac{\eta}{\beta_2}\mathcal{L}(t).$$

We choose $h_1 = \frac{m}{\alpha_2}$, we get

$$\mathcal{L}'(t) \leq -\ell_1 \mathcal{L}(t), \ \forall t \geq 0. \tag{4.29}$$

A simple integration of (4.29) over $(0, t)$ and using (4.28), we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0) \exp(-\ell_1 t), \ \forall t \geq 0,$$

implies

$$E(t) \leq \ell_2 \exp(-\ell_1 t), \ \forall t \geq 0,$$

because

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \mathcal{L}(0) \exp(-\ell_1 t), \ \forall t \geq 0,$$

where $\ell_2 = \frac{\mathcal{L}(0)}{\beta_1} \leq \frac{\beta_2}{\beta_1} E(0)$. □

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