Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2026 (44)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.63155

Quasi Bi-Slant Pseudo-Riemannian Submersions in Para-Complex Geometry

Esra Başarir Noyan and Yılmaz Gündüzalp

ABSTRACT: We presented quasi-bi-slant pseudo-Riemannian submersions in para-complex geometry from para-Kaehler manifolds onto pseudo-Riemannian manifolds in our study. We get these submersions which is the expansion of hemi-slant submersions and semi slant submersions. We give non-trivial examples of such submersions. Further, some geometric properties with two types of quasi-bi-slant submersions were investigated.

Keywords: Quasi bi-slant submersion, bi-slant submersion, pseudo-Riemannian submersion, para-Kaehler manifold.

Contents

1. Introduction

A $C^{\infty}-$submersion ψ can be defined according to the following conditions. A pseudo-Riemannian submersion ([4],[15],[20],[21],[32]), an almost Hermitian submersion ([2],[18],[33],[35]), an anti-invariant submersion ([10],[12],[30],[7]), a slant submersion ([19],[8],[14],[1],[23],[29]), a para quaternionic submersion ([16]), a Clairaut submersion ([11]), anti-invariant Riemnnian submersion from cosymplectic manifolds ([13]), bi-slant submanifold ([3],[5]), a hemi-slant submersion ([34],[28]), bi-slant submersion([27],[24]), a quasi-bi-slant submersion ([26]), a semi-invariant submersion ([22],[31]), a semi-slant $\xi^{\perp}$-Riemannian submersions ([25]), etc. As we know, Riemannian submersions were severally introduced by B. O'Neill ([21]) and A. Gray ([15]) in 1960s. In particular, by using the concept of almost Hermitian submersions, B. Watson ([36]) gave some differential geometric properties among fibers, base manifolds, and total manifolds. Some interesting results concerning para-Kaehler-like statistical submersions were obtained by G.E. Vîlcu ([35]).

We organized our work in three sections. In section 2, we gather basic concepts and definitions needed in the following parts. In section 3, we examined quasi-bi-slant pseudo-Riemannian submersions in para-complex geometry that satisfies certain conditions. We give some non-trivial examples of these submersions which satisfy the conditions of two types, while in we study the decomposition theorem of two types of the distributions.

2. Preliminaries

By a para-Hermitian manifold we mean a triple $(\mathcal{B},\mathcal{P},g_{\mathcal{B}})$, where $\mathcal{B}$ is connected differentiable manifold of $2n$- dimensional , $\mathcal{P}$ is a tensor field of type (1,1) and a pseudo-Riemannian metric $g_{\mathcal{B}}$ on $\mathcal{B}$, satisfying

$$\mathcal{P}^2E_1 = E_1, \quad g_{\mathcal{B}}(\mathcal{P}E_1,\mathcal{P}E_2) = -g_{\mathcal{B}}(E_1,E_2) \tag{2.1}$$

where E_1, E_2 are vector fields on $\mathcal{B}$. Then we can say that $\mathcal{B}$ is a para-Kaehler manifold such that

$$\nabla\mathcal{P} = 0; \tag{2.2}$$

where ∇ denotes the Levi-Civita connection on $\mathcal{B}$ ([18]).

2020 *Mathematics Subject Classification:* 53C43, 53B20, 53C40.
Submitted April 05, 2022. Published March 19, 2026

Let $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be two pseudo-Riemannian manifolds. Being a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ provides the following two properties;
(i) the fibres $\psi^{-1}(q)$, $q \in \tilde{\mathcal{B}}$, are $r-$ dimensional pseudo-Riemannian submanifolds of $\mathcal{B}$, where $r = dim(\mathcal{B}) - dim(\tilde{\mathcal{B}})$.
(ii) ψ_* preserves scalar products of vectors normal to fibres.
The vectors tangent to the fibres are called vertical and those normal to the fibres are called horizontal. A vector field U on $\mathcal{B}$ is called basic if U is horizontal and ψ- related to a vector field U_* on $\tilde{\mathcal{B}}$, i.e., $\psi_* U_p = U_{*\psi_p}$ for all $p \in \mathcal{B}$. We indicate by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. We know that $(\mathcal{B}, g_{\mathcal{B}})$ is called total manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is called base manifold of the submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$.

Now, let's denote O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$:

$$\mathcal{T}_U \mathcal{W} = h\nabla_{vU} v\mathcal{W} + v\nabla_{vU} h\mathcal{W} \tag{2.3}$$

and

$$\mathcal{A}_U \mathcal{W} = v\nabla_{hU} h\mathcal{W} + h\nabla_{hU} v\mathcal{W} \tag{2.4}$$

for every $U, \mathcal{W} \in \chi(\mathcal{B})$, on $\mathcal{B}$ where ∇ is the Levi-Civita connection of $g_{\mathcal{B}}$.
Further, a pseudo-Riemannian submersion $\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ has totally geodesic fibers if and only if $\mathcal{T} \equiv 0$. Also, if $\mathcal{A}$ vanishes then the horizontal distribution is integrable(see [4], [6]). Using (2.3) and (2.4), we get

$$\nabla_U W = \mathcal{T}_U W + \hat{\nabla}_U W; \tag{2.5}$$

$$\nabla_U \zeta = \mathcal{T}_U \zeta + h\nabla_U \zeta; \tag{2.6}$$

$$\nabla_\zeta U = \mathcal{A}_\zeta U + v\nabla_\zeta U; \tag{2.7}$$

$$\nabla_\zeta \eta = \mathcal{A}_\zeta \eta + h\nabla_\zeta \eta, \tag{2.8}$$

for any $\zeta, \eta \in \Gamma((ker\psi_*)^\perp)$, $U, W \in \Gamma(ker\psi_*)$. Also, if ζ is basic then $h\nabla_U \zeta = h\nabla_\zeta U = \mathcal{A}_\zeta U$.

We can easily see that $\mathcal{T}$ is symmetric on the vertical distribution and $\mathcal{A}$ is alternating on the horizontal distribution such that

$$\mathcal{T}_{\mathcal{W}} U = \mathcal{T}_U \mathcal{W}, \quad \mathcal{W}, U \in \Gamma(ker\psi_*); \tag{2.9}$$

$$\mathcal{A}_Y V = -\mathcal{A}_V Y = \frac{1}{2} v[Y, V], \quad Y, V \in \Gamma((ker\psi_*)^\perp). \tag{2.10}$$

Also, ıt is easily seen that for any $\wp \in \Gamma(T\mathcal{B})$, $\mathcal{T}_\wp$ and $\mathcal{A}_\wp$ are skew-symmetric operators on $\Gamma(T\mathcal{B})$, such that

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}} \mathcal{X}, U) \tag{2.11}$$

$$g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} U, \mathcal{X}) = -g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} \mathcal{X}, U) \tag{2.12}$$

Remark 2.1 *Our work, all horizontal vector fields are accepted as basic vector fields.*

Definition 2.2. ([12]) Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let's assume that the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ is an invariant pseudo-Riemannian submersion if the vertical distribution is invariant with respect to $\mathcal{P}$, i.e., $\mathcal{P}(ker\psi_*) = (ker\psi_*)$.

Definition 2.3. ([10]) Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ such that $ker\psi_*$ is anti-invariant with respect to $\mathcal{P}$, i.e., $\mathcal{P}(ker\psi_*) \subseteq (ker\psi_*)^\perp$. So, we can say ψ is an anti-invariant pseudo-Riemannian submersion.

Definition 2.4. ([31]) Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, there exists a pseudo-Riemannian submersion ψ is a semi-invariant pseudo-Riemannian submersion if there is a distribution $\mathtt{D_1} \subseteq ker\psi_*$, such that

$$ker\psi_* = \mathtt{D_1} \oplus \mathtt{D_2},$$

and

$$\mathcal{P}\mathtt{D_1} = \mathtt{D_1}, \mathcal{P}\mathtt{D_2} \subseteq (ker\psi_*)^{\perp}$$

where $\mathtt{D_2}$ is orthogonal complementary to $\mathtt{D_1}$ in $ker\psi_*$.
We know that μ is the complementary orthogonal subbundle to $\mathcal{P}\ ker\psi_*$ in $(ker\psi_*)^{\perp}$.
Also we have;

$$(ker\psi_*)^{\perp} = \mathcal{P}\mathtt{D_2} \oplus \mu.$$

From here we can say that μ is an invariant subbundle of $(ker\psi_*)^{\perp}$ with respect to the para-complex structure $\mathcal{P}$.

For any non-null vector field $U_2 \in (ker\psi_*)$, we get

$$\mathcal{P}U_2 = qU_2 + rU_2,$$

where qU_2 is vertical part and rU_2 is horizontal part.

If for non-null vector field $U_2 \in ker\psi_*$, the quotient $\frac{g_{\mathcal{B}}(qU_2,qU_2)}{g_{\mathcal{B}}(\mathcal{P}U_2,\mathcal{P}U_2)}$ is constant, i.e. it is independent of the choice of the point $\bar{q} \in \mathcal{B}$ and choice of the non-null vector field $U_2 \in \Gamma(ker\psi_*)$, we can say that ψ is a slant submersion. So, the angle is called the slant angle of the slant submersion ([12]). A submersion is called invariant([9]) if it is slant with slant angle zero, that is if $g_{\mathcal{B}}(qU_2, qU_2)/g_{\mathcal{B}}(\mathcal{P}U_2, \mathcal{P}U_2) = 1$ for all non-null $U_2 \in \Gamma(ker\psi_*)$. It is called anti-invariant([12]) if $qU_2 = 0$ for all non-null $U_2 \in \Gamma(ker\psi_*)$. In other cases, it is called a proper slant submersions.

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. We have;
type $\sim$1 if for every space-like (time-like) vector field $U_2 \in \Gamma(ker\psi_*)$, qU_2 is time-like (space-like), and $\frac{\|qU_2\|}{\|\mathcal{P}U_2\|} > 1$,
type $\sim$ 2 if for every space-like (time-like) vector field $U_2 \in \Gamma(ker\psi_*)$, qU_2 is time-like (space-like), and $\frac{\|qU_2\|}{\|\mathcal{P}U_2\|} < 1$([11]).

Theorem 2.5. ([11]) Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper slant submersion. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then,
(a) ψ is slant submersion of type-1 if and only if for any space-like (time-like) vector field $U_1 \in ker\psi_*$, qU_1 is time-like (space-like) and there exists a constant $\mu \in (1, +\infty)$ such that

$$q^2 = \mu Id.$$

If ψ is a proper slant submersion of type-1, then $\mu = \cosh^2\varphi$, with $\varphi > 0$.
(b) ψ is slant submersion of type-1 if and only if for any space-like (time-like) vector field $U_1 \in ker\psi_*$, qU_1 is time-like (space-like) and there exists a constant $\mu \in (0, 1)$ such that

$$q^2 = \mu Id.$$

If ψ is a proper slant submersion of type-1, then $\mu = \cos^2\varphi$, with $0 < \varphi < \frac{\pi}{2}$.

Definition 2.6. Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a semi-slant submersion if there is a distribution $\mathtt{D}_1 \in ker\psi_*$ such that

$$ker\psi_* = \mathtt{D}_1 \oplus \mathtt{D}_2, \quad \mathcal{P}(\mathtt{D}_1) = \mathtt{D}_1$$

and the angle φ is known the semi-slant angle of the submersion where $\mathtt{D}_2$ is the orthogonal complement of $\mathtt{D}_1$ in $ker\psi_*$.

Definition 2.7. Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a hemi-slant submersion if the vertical distribution $ker\psi_*$ of ψ accepts two orthogonal complementary distribution $\mathtt{D}^{\varphi}$ and $\mathtt{D}^{\perp}$, such that $\mathtt{D}^{\varphi}$ is slant and $\mathtt{D}^{\perp}$ is anti-invariant, i.e., we can show

$$ker\psi_* = \mathtt{D}^{\varphi} \oplus \mathtt{D}^{\perp}$$

Therefore, the angle φ is known the hemi-slant angle of the submersion.

Definition 2.8. Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a bi-slant submersion if there are two slant distribution $\mathtt{D}^{\varphi_1} \in ker\psi_*$ and $\mathtt{D}^{\varphi_2} \in ker\psi_*$ such that

$$ker\psi_* = \mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2}$$

where $\mathtt{D}^{\varphi_1}$ and $\mathtt{D}^{\varphi_2}$ have slant angles φ_1 and φ_2, respectively.

$\psi : \mathcal{B} \to \tilde{\mathcal{B}}$ is a differentiable map. So, $(\mathcal{B}, g_{\mathcal{B}})$ and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be pseudo-Riemannian manifolds. Then, the second fundamental form of ψ is described by

$$(\nabla\psi_*)(\zeta, V) = \nabla^{\psi}_{\zeta}\psi_* V - \psi_*(\nabla_{\zeta} V) \tag{2.13}$$

for $\zeta, V \in \Gamma(\mathcal{B})$. When $trace(\nabla\psi_*) = 0$, we can say that ψ is *harmonic* and ψ is a *totally geodesic* map when $(\nabla\psi_*)(\zeta, V) = 0$ for $\zeta, V \in \Gamma(T\mathcal{B})$ ([17]). Recall that ∇^{ψ} is the pullback connection.

Lemma 2.9. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian submersion from a pseudo-Riemannian manifold $(\mathcal{B}, g_{\mathcal{B}})$ onto an other pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$, we have;*

(i) $(\nabla\psi_*)(\zeta, V) = 0$;

(ii) $(\nabla\psi_*)(U, W) = -\psi_*(\mathcal{T}_U W) = -\psi_*(\nabla_U W)$;

(iii) $(\nabla\psi_*)(\zeta, U) = -\psi_*(\nabla_{\zeta} U) = -\psi_*(\mathcal{A}_{\zeta} U)$.

where ζ and V are horizontal vectors and U and W are vertical vectors.

3. Quasi bi-slant submersions

Definition 3.1. *Let $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ be an almost para-Hermitian manifold and $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ is known a quasi-bi-slant submersion if there are three orthogonal distributions $\mathtt{D}$, $\mathtt{D}^{\varphi_1}$ and $\mathtt{D}^{\varphi_2}$, such that*

- $ker\psi_* = \mathtt{D} \oplus \mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2}$,
- $\mathcal{P}(\mathtt{D}) = \mathtt{D}$ *i.e.,* $\mathtt{D}$ *is invariant,*
- $\mathcal{P}(\mathtt{D}^{\varphi_1}) \perp \mathtt{D}^{\varphi_2}$,
- *the angle φ_1 between $\mathcal{P}U$ and $(\mathtt{D}^{\varphi_1})_q$ is constant and independent of the choice of point q and U in $(\mathtt{D}^{\varphi_1})_q$, for any non-null vector field $U \in (\mathtt{D}^{\varphi_1})_q$,*

- *the angle* φ_2 *between* $\mathcal{P}U$ *and* $(\mathtt{D}^{\varphi_2})_q$ *is constant and independent of the choice of point* q *and* U *in* $(\mathtt{D}^{\varphi_2})_q$*, for any non-null vector field* $U \in (\mathtt{D}^{\varphi_2})_q$*,*

where D^{φ_1} *and* D^{φ_2} *have slant angles* φ_1 *and* φ_2*, respectively.*

Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a quasi-bi-slant submersion with type-1 or 2. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, we obtain;*

$$TB = ker\psi_* \oplus (ker\psi_*)^{\perp} \tag{3.1}$$

For any non-null vector field $U \in (ker\psi_*)$*, we get*

$$U = KU + LU + RU + \eta(U)\xi, \tag{3.2}$$

where KU*,* LU *and* RU *are projection morphisms of* $ker\psi_*$ *onto* $\mathtt{D}, \mathtt{D}^{\varphi_1}$ *and* $\mathtt{D}^{\varphi_2}$*, respectively. For non-null vector field* $U \in (ker\psi_*)$*, we have*

$$\mathcal{P}U = \phi U + fU, \tag{3.3}$$

where $\phi U \in ker\psi_*$ *and* $fU \in (ker\psi_*)^{\perp}$ *.*

From (3.2) and (3.3) we get:

$$\begin{aligned} \mathcal{P}U &= \mathcal{P}(KU) + \mathcal{P}(LU) + \mathcal{P}(RU), \\ &= \phi(KU) + f(KU) + \phi(LU) + f(LU) + \phi(RU) + f(RU). \end{aligned}$$

Since $\mathcal{P}(\mathtt{D}) = (\mathtt{D})$*, we obtain* $f(KU) = 0$*. Now, let's arrange the above equation*

$$\mathcal{P}U = \phi(KU) + \phi(LU) + f(LU) + \phi(RU) + f(RU). \tag{3.4}$$

So, we have the following decomposition:

$$\mathcal{P}(ker\psi_*) = \mathtt{D} \oplus (\phi\mathtt{D}^{\varphi_1} \oplus \phi\mathtt{D}^{\varphi_2}) \oplus (f\mathtt{D}^{\varphi_1} \oplus f\mathtt{D}^{\varphi_2}). \tag{3.5}$$

Also, for $U \in \Gamma(\mathtt{D}^{\varphi_1})$ *and* $W \in \Gamma(\mathtt{D}^{\varphi_2})$ *we have;*

$$g_{\mathcal{B}}(U, W) = 0.$$

From Definition 3.1, we obtain;

$$g_{\mathcal{B}}(\mathcal{P}U, W) = g_{\mathcal{B}}(U, \mathcal{P}W) = 0.$$

If we rearrange the equation, we get;

$$\begin{aligned} g_{\mathcal{B}}(\phi U, W) &= g_{\mathcal{B}}(\mathcal{P}U - fU, W), \\ &= g_{\mathcal{B}}(\mathcal{P}U, W), \\ &= 0. \end{aligned}$$

In a similar way, we get;

$$g_{\mathcal{B}}(U, \phi W) = 0.$$

for non-null vector fields $\mathcal{Z} \in \Gamma(\mathtt{D})$ *and* $U \in \Gamma(\mathtt{D}^{\varphi_1})$ *we have;*

$$\begin{aligned} g_{\mathcal{B}}(\phi U, \mathcal{Z}) &= g_{\mathcal{B}}(\mathcal{P}U - fU, \mathcal{Z}), \\ &= g_{\mathcal{B}}(\mathcal{P}U, \mathcal{Z}), \\ &= -g_{\mathcal{B}}(U, \mathcal{P}\mathcal{Z}), \\ &= 0. \end{aligned}$$

Since $\mathtt{D}$ *is invariant, we know that* $\mathcal{P}\mathcal{Z} \in \Gamma(\mathtt{D})$.

Then, for non-null vector field $\mathcal{Z} \in \Gamma(\mathtt{D})$ *and* $W \in \Gamma(\mathtt{D}^{\varphi_2})$ *we have;*

$$g_{\mathcal{B}}(\phi W, \mathcal{Z}) = 0.$$

From above equations, we obtain;

$$g_{\mathcal{B}}(\phi U, \phi W) = 0,$$

and

$$g_{\mathcal{B}}(fU, fW) = 0,$$

for all non-null vector field $U \in \Gamma(\mathtt{D}^{\varphi_1})$ *and* $W \in \Gamma(\mathtt{D}^{\varphi_2})$.
After, we obtain;

$$\phi\mathtt{D}^{\varphi_1} \cap \phi\mathtt{D}^{\varphi_2} = 0, \;\; f\mathtt{D}^{\varphi_1} \cap f\mathtt{D}^{\varphi_2} = 0.$$

If $\phi R = 0$, *then* $\mathtt{D}^{\varphi_2}$ *is anti-invariant, i.e.,* $\mathcal{P}(\mathtt{D}^{\varphi_2}) \subseteq (ker\psi_*)^{\perp}$.
So, we define $\mathtt{D}^{\varphi_2}$ *by* $\mathtt{D}^{\perp}$. *Such that;*

$$\mathcal{P}(ker\psi_*) = \mathtt{D} \oplus \phi\mathtt{D}^{\varphi_1} \oplus f\mathtt{D}^{\varphi_1} \oplus \mathcal{P}\mathtt{D}^{\perp}. \tag{3.6}$$

Since, $f\mathtt{D}^{\varphi_1} \subseteq (ker\psi_*)^{\perp}, f\mathtt{D}^{\varphi_2} \subseteq (ker\psi_*)^{\perp}$. *We have;*

$$(ker\psi_*)^{\perp} = f\mathtt{D}^{\varphi_1} \oplus f\mathtt{D}^{\varphi_2} \oplus \mu$$

where μ *is the orthogonal complementary distribution of* $f\mathtt{D}^{\varphi_1} \oplus f\mathtt{D}^{\varphi_2}$ *in* $(ker\psi_*)^{\perp}$.

In adittion, for any non-null vector field $W \in (ker\psi_*)^{\perp}$ *is decomposed as*

$$\mathcal{P}W = BW + CW \tag{3.7}$$

where $BW \in \Gamma(ker\psi_*)$ *and* $CW \in \Gamma(ker\psi_*)^{\perp}$.
Then, we can easily see that $\mathcal{P}^2 = I$ *and from (3.3) and (3.7) we get:*

Lemma 3.2. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *is a quasi-bi-slant submersion with type ∼1 or 2. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. Then, we obtain the following equations:*
(a) $\phi^2\mathcal{Z} + Bf\mathcal{Z} = \mathcal{Z}$ **(b)** $C^2U + fBU = U$
(c) $\phi BU + BCU = \{0\}$ **(d)** $f\phi\mathcal{Z} + Cf\mathcal{Z} = \{0\}$
for all non-null vectors $\mathcal{Z} \in \Gamma(ker\psi_*)$ *and* $U \in \Gamma(ker\psi_*)^{\perp}$.

The proofs of the following theorems are similar to those given in ([12]). Therefore we skip its proof.

Theorem 3.3. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a quasi-bi-slant submersion with type∼1. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. In this case,* ψ *is a quasi-bi-slant submersion such that:*
(a) $\phi^2\mathcal{Z} = \cosh^2\varphi_1\mathcal{Z}$ **(b)** $\phi^2U = \cosh^2\varphi_2U$
(c) $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -\cosh^2\varphi_1 g_{\mathcal{B}}(\mathcal{Z}, Y)$ **(d)** $g_{\mathcal{B}}(\phi U, \phi W) = -\cosh^2\varphi_2 g_{\mathcal{B}}(U, W)$
(e) $g_{\mathcal{B}}(f\mathcal{Z}, fY) = \sinh^2\varphi_1 g_{\mathcal{B}}(\mathcal{Z}, Y)$ **(f)** $g_{\mathcal{B}}(fU, fW) = \sinh^2\varphi_2 g_{\mathcal{B}}(U, W)$
for any space-like(time-like) vector field $\mathcal{Z}, Y \in \mathtt{D}^{\varphi_1}$ *and* $U, W \in \mathtt{D}^{\varphi_2}$.

Theorem 3.4. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a quasi-bi-slant submersion with type∼2. Let's suppose the total manifold as an almost para-Hermitian manifold and base manifold as a pseudo-Riemannian manifold. In this case,* ψ *is quasi-bi-slant submersion such that:*
(a) $\phi^2\mathcal{Z} = \cos^2\varphi_1\mathcal{Z}$ **(b)** $\phi^2U = \cos^2\varphi_2U$

(c) $g_{\mathcal{B}}(\phi\mathcal{Z}, \phi Y) = -\cos^2\varphi_1 g_{\mathcal{B}}(\mathcal{Z}, Y)$ **(d)** $g_{\mathcal{B}}(\phi U, \phi W) = -\cos^2\varphi_2 g_{\mathcal{B}}(U, W)$
(e) $g_{\mathcal{B}}(f\mathcal{Z}, fY) = -\sin^2\varphi_1 g_{\mathcal{B}}(\mathcal{Z}, Y)$ **(f)** $g_{\mathcal{B}}(fU, fW) = -\sin^2\varphi_2 g_{\mathcal{B}}(U, W)$.
for any space-like(time-like) vector field $\mathcal{Z}, Y \in \mathtt{D}^{\varphi_1}$ *and* $U, W \in \mathtt{D}^{\varphi_2}$.

Now, we can easily notice the following situations:

(1) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) = 0$ *and* $dim(\mathtt{D}^{\varphi_2}) = 0$, *then* ψ *is an invariant submersion of types∼1 and 2.*

(2) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1$ *and* $dim(\mathtt{D}^{\varphi_2}) = 0$, *then* ψ *is proper semi-slant submersion of type∼1.*

(3) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1$ *and* $dim(\mathtt{D}^{\varphi_2}) = 0$, *then* ψ *is proper slant submersion of type∼1 with slant angle* φ_1.

(4) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) = 0$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $0 < \varphi_2$, *then* ψ *is proper slant submersion of type∼1 with slant angle* φ_2.

(5) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $\varphi_1 = \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) = 0$, *then* ψ *is anti-invariant submersion of type∼2.*

(6) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $\varphi_1 = \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) = 0$, *then* ψ *is semi-invariant submersion of type∼2.*

(7) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1 < \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $\varphi_2 = \frac{\pi}{2}$ *then* ψ *is a hemi-slant submersion of type∼2.*

(8) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $0 < \varphi_2$ *then* ψ *is a bi-slant submersion of type∼1.*

(9) If $dim(\mathtt{D}) = 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1 < \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $0 < \varphi_2 < \frac{\pi}{2}$ *then* ψ *is a bi-slant submersion of type∼2.*

(10) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1 < \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $\varphi_2 = \frac{\pi}{2}$ *then* ψ *is a quasi-hemi-slant submersion of type∼2.*

(11) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $0 < \varphi_2$ *then* ψ *is a proper quasi-bi-slant submersion of type∼1.*

(12) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $0 < \varphi_1 < \frac{\pi}{2}$ *and* $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $0 < \varphi_2 < \frac{\pi}{2}$ *then* ψ *is a proper quasi-bi-slant submersion of type∼2.*

(13) If $dim(\mathtt{D}) \neq 0$, $dim(\mathtt{D}^{\varphi_1}) \neq 0$, $dim(\mathtt{D}^{\varphi_2}) \neq 0$, $\varphi_1 = \varphi_2 = \varphi$ *then* ψ *is semi-slant submersion of types∼1 and 2 with semi-slant angle* φ.

Let's consider para-Kaehler structure on R_n^{2n} :

$$P(\frac{\partial}{\partial y_{2i}}) = \frac{\partial}{\partial y_{2i-1}}, \ P(\frac{\partial}{\partial y_{2i-1}}) = \frac{\partial}{\partial y_{2i}}, \ g = (dy^1)^2 - (dy^2)^2 + (dy^3)^2 - ... - (dy^{2n})^2$$

here $i \in \{1, ..., n\}$. *Also,* $(y_1, y_2, ..., y_{2n})$ *denotes the cartesian coordinates over* R_n^{2n}.

We can easily present non-trivial examples of proper quasi-bi-slant pseudo-Riemannian submersions of type∼1 and 2.

Example 3.5. *Let's determine map* $\psi : R_6^{12} \to R_3^6$

$$\psi(y_1, ..., y_{12}) = (y_2 \sinh\beta_1 + y_3 \cosh\beta_1, y_7, y_4 \cosh\beta_2 + y_5 \sinh\beta_2, y_8, y_{11}, y_{12}),$$

So, ψ is a proper quasi-bi-slant pseudo-Riemannian submersion with type ~ 1. By direct calculations, we have

$\mathtt{D} =< \frac{\partial}{\partial y_9}, \frac{\partial}{\partial y_{10}} >$

$\mathtt{D}^{\varphi_1} =< -\cosh\beta_1 \frac{\partial}{\partial y_2} + \sinh\beta_1 \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_1} >$

$\mathtt{D}^{\varphi_2} =< -\sinh\beta_2 \frac{\partial}{\partial y_4} + \cosh\beta_2 \frac{\partial}{\partial y_5}, \frac{\partial}{\partial y_6} >$

with bi-slant angles β_1 and β_2.

Example 3.6. *Let's determine map $\psi : R^{12}_6 \to R^6_3$*

$$\psi(y_1, ..., y_{12}) = (\frac{y_1 - y_3}{\sqrt{2}}, y_4, \frac{\sqrt{3}y_5 - y_7}{2}, y_8, y_{11}, y_{12})$$

So, ψ is a proper quasi-bi-slant pseudo-Riemannian submersion with type ~ 2. By direct calculations, we get

$\mathtt{D} =< \frac{\partial}{\partial y_9}, \frac{\partial}{\partial y_{10}} >$

$\mathtt{D}^{\varphi_1} =< \frac{1}{\sqrt{2}}(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}), \frac{\partial}{\partial y_2} >$

$\mathtt{D}^{\varphi_2} =< \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial y_5} + \frac{\partial}{\partial y_7}), \frac{\partial}{\partial y_6} >$

with bi-slant angles $\varphi_1 = \frac{\pi}{4}$ and $\varphi_2 = \frac{\pi}{3}$.

Using equations (2.1), (2.5)~(2.6)~(2.7)~(2.8) and (3.3)~(3.7), we get:

Lemma 3.7. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi-bi-slant pseudo-Riemannian submersion with type ~1 or 2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. So, we obtain the following equations.*

$$\hat{\nabla}_U \phi W + \mathcal{T}_U f W = \phi\hat{\nabla}_U W + \mathcal{B}\mathcal{T}_U W \tag{3.8}$$

$$\mathcal{T}_U \phi W + \mathcal{H}\nabla_U f W = f\hat{\nabla}_U W + \mathcal{C}\mathcal{T}_U W \tag{3.9}$$

$$\mathcal{V}\nabla_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{A}_{\mathcal{X}}\mathcal{C}\mathcal{Y} = \phi\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{B}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{3.10}$$

$$\mathcal{A}_{\mathcal{X}}\mathcal{B}\mathcal{Y} + \mathcal{H}\nabla_{\mathcal{X}}\mathcal{C}\mathcal{Y} = f\mathcal{A}_{\mathcal{X}}\mathcal{Y} + \mathcal{C}\mathcal{H}\nabla_{\mathcal{X}}\mathcal{Y} \tag{3.11}$$

$$\hat{\nabla}_U \mathcal{B}\mathcal{X} + \mathcal{T}_U\mathcal{C}\mathcal{X} = \phi\mathcal{T}_U\mathcal{X} + \mathcal{B}\mathcal{H}\nabla_U\mathcal{X} \tag{3.12}$$

$$\mathcal{T}_U\mathcal{B}\mathcal{X} + \mathcal{H}\nabla_U\mathcal{C}\mathcal{X} = f\mathcal{T}_U\mathcal{X} + \mathcal{C}\mathcal{H}\nabla_U\mathcal{X}, \tag{3.13}$$

for any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $\mathcal{X}, \mathcal{Y} \in \Gamma(ker\psi_*)^\perp$.

Now we can show

$$(\nabla_U \phi)W = \hat{\nabla}_U \phi W - \phi\hat{\nabla}_U W$$

$$(\nabla_U f)W = \mathcal{H}\nabla_U f W - f\hat{\nabla}_U W,$$

$$(\nabla_X B)\zeta = \hat{\nabla}_X B\zeta - B\mathcal{H}\nabla_X\zeta$$

$$(\nabla_X C)\zeta = \mathcal{H}\nabla_X C\zeta - C\mathcal{H}\nabla_X\zeta$$

for any non-null vector fields $U, W \in ker\psi_*$ and $X, \zeta \in (ker\psi_*)^{\perp}$.

Lemma 3.8. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a quasi-bi-slant pseudo-Riemannian submersion with type $\sim$1 and type $\sim$ 2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. So, we obtain the following equations.*

$$(\nabla_U \phi)W = \mathcal{B}\mathcal{T}_U W - \mathcal{T}_U fW$$

$$(\nabla_U f)W = \mathcal{C}\mathcal{T}_U W - \mathcal{T}_U \phi W$$

$$(\nabla_X B)\zeta = \phi\mathcal{A}_X\zeta - \mathcal{A}_X\mathcal{B}\zeta$$

$$(\nabla_X C)\zeta = f\mathcal{A}_X\zeta - \mathcal{A}_X\mathcal{C}\zeta$$

for any non-null vector fields $U, W \in ker\psi_*$ and $X, \zeta \in (ker\psi_*)^{\perp}$.

Proof. *The proof is simple.*

If ϕ and f are parallel with respect to ∇ on $\mathcal{B}$ respectively, we have

$\mathcal{B}\mathcal{T}_U W = \mathcal{T}_U fW$ *and* $\mathcal{C}\mathcal{T}_U W = \mathcal{T}_U \phi W$

for any $U, W \in \Gamma(TB)$.

Theorem 3.9. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with type$\sim$1 or 2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. The invariant distribution $\mathtt{D}$ is integrable if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_W\phi U - \mathcal{T}_U\phi W, fL\zeta + fR\zeta) = -g_{\mathcal{B}}(\mathcal{V}\nabla_W\phi U - \mathcal{V}\nabla_U\phi W, \phi L\zeta + \phi R\zeta) \tag{3.14}$$

for any non-null vector fields $U, W \in \Gamma(\mathtt{D})$ and $\zeta \in \Gamma(\mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2})$.

Proof. *For any non-null vector fields $U, W \in \Gamma(\mathtt{D})$ and $\zeta \in \Gamma(\mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2})$. Then using (2.1),(2.2),(2.5) and (3.3) obtained:*

$$\begin{aligned}
g_{\mathcal{B}}([U, W], \zeta) &= -g_{\mathcal{B}}(\nabla_U\mathcal{P}W, \mathcal{P}\zeta) + g_{\mathcal{B}}(\nabla_W\mathcal{P}U, \mathcal{P}\zeta) \\
&= -g_{\mathcal{B}}(\nabla_U\phi W, \mathcal{P}\zeta) + g_{\mathcal{B}}(\nabla_W\phi U, \mathcal{P}\zeta) \\
&= g_{\mathcal{B}}(\mathcal{T}_W\phi U - \mathcal{T}_U\phi W, fL\zeta + fR\zeta) \\
&+ g_{\mathcal{B}}(\mathcal{V}\nabla_W\phi U - \mathcal{V}\nabla_U\phi W, \phi L\zeta + \phi R\zeta).
\end{aligned}$$

So, the proof is complete.

Theorem 3.10. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi- bi-slant pseudo-Riemannian submersion with types:1,2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. The slant distribution $\mathtt{D}^{\varphi_1}$ is integrable if and only if*

$$\begin{aligned}
g_{\mathcal{B}}(\mathcal{T}_W fU - \mathcal{T}_U fW, \phi K\mathcal{X} + \phi R\mathcal{X}) &= g_{\mathcal{B}}(\mathcal{H}\nabla_U fW - \mathcal{H}\nabla_W fU, fR\mathcal{X}) \\
&- g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X})
\end{aligned} \tag{3.15}$$

for any non-null vector fields $U, W \in \Gamma(\mathtt{D}^{\varphi_1})$ and $\mathcal{X} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_2})$.

Proof. *We only give its proof ψ is type∼1. For any non-null vector fields $U, W \in \Gamma(\mathtt{D}^{\varphi_1})$ and $\mathcal{X} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_2})$. Then using (2.1),(2.2),(2.6), (3.3) and Lemma 3.3(a), we get:*

$$\begin{aligned} g_{\mathcal{B}}([U,W],\mathcal{X}) &= -g_{\mathcal{B}}(\nabla_U \mathcal{P}W, \mathcal{P}\mathcal{X}) + g_{\mathcal{B}}(\nabla_W \mathcal{P}U, \mathcal{P}\mathcal{X}) \\ &= -g_{\mathcal{B}}(\nabla_U \phi W, \mathcal{P}\mathcal{X}) - g_{\mathcal{B}}(\nabla_U fW, \mathcal{P}\mathcal{X}) \\ &+ g_{\mathcal{B}}(\nabla_W \phi U, \mathcal{P}\mathcal{X}) + g_{\mathcal{B}}(\nabla_W fU, \mathcal{P}\mathcal{X}) \\ &= \cosh^2\varphi_1 g_{\mathcal{B}}(\nabla_U W, \mathcal{X}) - \cosh^2\varphi_1 g_{\mathcal{B}}(\nabla_W U, \mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X}) \\ &- g_{\mathcal{B}}(\mathcal{T}_U fW + \mathcal{H}\nabla_U fW, \phi K\mathcal{X} + \phi R\mathcal{X} + fR\mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{T}_W fU + \mathcal{H}\nabla_W fU, \phi K\mathcal{X} + \phi R\mathcal{X} + fR\mathcal{X}). \end{aligned}$$

Then, we have;

$$\begin{aligned} -\sinh^2\varphi_1 g_{\mathcal{B}}([U,W],\mathcal{X}) &= g_{\mathcal{B}}(\mathcal{T}_W fU - \mathcal{T}_U fW, \phi K\mathcal{X} + \phi R\mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{H}\nabla_W fU - \mathcal{H}\nabla_U fW, fR\mathcal{X}) \\ &+ g_{\mathcal{B}}(\mathcal{T}_U f\phi W - \mathcal{T}_W f\phi U, \mathcal{X}) \end{aligned}$$

which completes proof.

Similarly, the following conclusion is obtained.

Theorem 3.11. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with types:1,2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. The slant distribution $\mathtt{D}^{\varphi_2}$ is integrable if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_V f\zeta - \mathcal{T}_\zeta fV, \phi\mathcal{W}L) &= g_{\mathcal{B}}(\mathcal{H}\nabla_\zeta fV - \mathcal{H}\nabla_V f\zeta, fL\mathcal{W}) \\ &- g_{\mathcal{B}}(\mathcal{T}_\zeta f\phi V - \mathcal{T}_V f\phi\zeta, \mathcal{W}) \end{aligned} \tag{3.16}$$

for all non-null vectors $\zeta, V \in \Gamma(\mathtt{D}^{\varphi_2})$ and $\mathcal{W} \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_1})$.

Theorem 3.12. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with type-1. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the horizontal distribution $(ker\psi_*)^\perp$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$\begin{aligned} & g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z}, K\zeta + \cosh^2\varphi_1 L\zeta + \cosh^2\varphi_2 R\zeta) \\ & \quad = g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z}, f\phi K\zeta + f\phi L\zeta + f\phi R\zeta) \\ & \qquad + g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} B\mathcal{Z} + \mathcal{H}\nabla_{\mathcal{W}} C\mathcal{Z}, f\zeta). \end{aligned} \tag{3.17}$$

for any non-null vector fields $\mathcal{W}, \mathcal{Z} \in (ker\psi_*)^\perp$ and $\zeta \in (ker\psi_*)$.

Proof. *For any non-null vectors $\mathcal{W}, \mathcal{Z} \in (ker\psi_*)^\perp$ and $\zeta \in (ker\psi_*)$, we get:*

$$g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, \zeta) = g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, K\zeta + L\zeta + R\zeta)$$

Then using (2.1),(2.2),(2.7), (2.8), (3.3), (3.4) and Lemma 3.3(a),(b) we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, \zeta) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}K\zeta) - g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}L\zeta) \\ &- g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}R\zeta) \\ &= g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z}, K\zeta + \cosh^2\varphi_1 L\zeta + \cosh^2\varphi_2 R\zeta) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z}, f\phi K\zeta + f\phi L\zeta + f\phi R\zeta) \\ &+ g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}} B\mathcal{Z} + \mathcal{H}\nabla_{\mathcal{W}} C\mathcal{Z}, fK\zeta + fL\zeta + fR\zeta). \end{aligned}$$

Since $fK\zeta = 0$ and $fK\zeta + fL\zeta + fR\zeta = f\zeta$, we obtain;

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, \zeta) &= g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z}, K\zeta + \cosh^2\varphi_1 L\zeta + \cosh^2\varphi_2 R\zeta) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z}, f\phi K\zeta + f\phi L\zeta + f\phi R\zeta) \\ &+ g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}B\mathcal{Z} + \mathcal{H}\nabla_{\mathcal{W}}C\mathcal{Z}, f\zeta) \end{aligned}$$

which gives proof.

Similarly, the following conclusion is obtained.

Theorem 3.13. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with type∼1. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the vertical distribution $(ker\psi_*)$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_U\zeta, \mathcal{W}) + \cosh^2\varphi_1 g_{\mathcal{B}}(\mathcal{T}_U L\zeta, \mathcal{W}) + \cosh^2\varphi_2 g_{\mathcal{B}}(\mathcal{T}_U R\zeta, \mathcal{W}) \\ = g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi K\zeta + \mathcal{H}\nabla_U f\phi L\zeta + \mathcal{H}\nabla_U f\phi R\zeta, \mathcal{W}) \\ + g_{\mathcal{B}}(\mathcal{T}_U f\zeta, B\mathcal{W}) + g_{\mathcal{B}}(\mathcal{H}\nabla_U f\zeta, C\mathcal{W}). \end{aligned} \tag{3.18}$$

for any non-null vector fields $U, \zeta \in \Gamma(ker\psi_*)$ and $\mathcal{W} \in \Gamma(ker\psi_*)^\perp$.

Using Theorem 3.12 and Theorem 3.13, we get the Theorem 3.14.

Theorem 3.14. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with type∼1. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the total space is a locally product $\mathcal{B}_{ker\psi_*} \times \mathcal{B}_{ker\psi_*^\perp}$ where $\mathcal{B}_{ker\psi_*}$ and $\mathcal{B}_{ker\psi_*^\perp}$ are leaves of $(ker\psi_*)$ and $(ker\psi_*)^\perp$, respectively, if and only if;*

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}\mathcal{Z}, K\zeta + \cosh^2\varphi_1 L\zeta + \cosh^2\varphi_2 R\zeta) \\ = g_{\mathcal{B}}(\mathcal{H}\nabla_{\mathcal{W}}\mathcal{Z}, f\phi K\zeta + f\phi L\zeta + f\phi R\zeta) \\ + g_{\mathcal{B}}(\mathcal{A}_{\mathcal{W}}B\mathcal{Z} + \mathcal{H}\nabla_{\mathcal{W}}C\mathcal{Z}, f\zeta). \end{aligned}$$

and

$$\begin{aligned} g_{\mathcal{B}}(\mathcal{T}_U\zeta, \mathcal{W}) + \cosh^2\varphi_1 g_{\mathcal{B}}(\mathcal{T}_U L\zeta, \mathcal{W}) + \cosh^2\varphi_2 g_{\mathcal{B}}(\mathcal{T}_U R\zeta, \mathcal{W}) \\ = g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi K\zeta + \mathcal{H}\nabla_U f\phi L\zeta + \mathcal{H}\nabla_U f\phi R\zeta, \mathcal{W}) \\ + g_{\mathcal{B}}(\mathcal{T}_U f\zeta, B\mathcal{W}) + g_{\mathcal{B}}(\mathcal{H}\nabla_U f\zeta, C\mathcal{W}). \end{aligned}$$

for any non-null vector fields $U, \zeta \in \Gamma(ker\psi_*)$ and $\mathcal{W}, \mathcal{Z} \in \Gamma(ker\psi_*)^\perp$.

Theorem 3.15. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with types:1,2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the invariant distribution $\mathtt{D}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi K\mathcal{Z}, fLY + fRY) = -g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi K\mathcal{Z}, \phi LY + \phi RY) \tag{3.19}$$

and

$$g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi K\mathcal{Z}, C\xi) = -g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi K\mathcal{Z}, B\xi) \tag{3.20}$$

Proof. *For all non-null vectors* $\mathcal{W}, \mathcal{Z} \in \Gamma(\mathtt{D})$ *and* $Y \in \Gamma(\mathtt{D}^{\varphi_1} \oplus \mathtt{D}^{\varphi_2})$ *and* $\xi \in \Gamma(ker\psi_*)^{\perp}$*. Then using (2.1),(2.2),(2.5) and (3.3), we get:*

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, Y) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}Y) \\ &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}K\mathcal{Z}, \mathcal{P}LY + \mathcal{P}RY) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi K\mathcal{Z}, fLY + fRY) - g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi K\mathcal{Z}, \phi LY + \phi RY) \end{aligned}$$

Then, again using (2.1),(2.2),(2.5),(3.3) and (3.7), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{Z}, \xi) &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\mathcal{P}\mathcal{Z}, \mathcal{P}\xi) \\ &= -g_{\mathcal{B}}(\nabla_{\mathcal{W}}\phi K\mathcal{Z}, B\xi + C\xi) \\ &= -g_{\mathcal{B}}(\mathcal{T}_{\mathcal{W}}\phi K\mathcal{Z}, C\xi) - g_{\mathcal{B}}(\mathcal{V}\nabla_{\mathcal{W}}\phi K\mathcal{Z}, B\xi) \end{aligned}$$

so, the proof is complete.

Theorem 3.16. *Let* $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ *be a proper quasi-bi-slant pseudo-Riemannian submersion with types:1,2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the slant distribution* $\mathtt{D}^{\varphi_1}$ *describes a totally geodesic foliation on* $\mathcal{B}$ *if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) = g_{\mathcal{B}}(\mathcal{T}_U fLV, \mathcal{P}KY + \phi RY) + g_{\mathcal{B}}(\mathcal{H}\nabla_U fLV, fRY) \tag{3.21}$$

and

$$g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) = g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) + g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi) \tag{3.22}$$

for any non-null vector fields $U, V \in \Gamma(\mathtt{D}^{\varphi_1})$ *and* $Y \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_2})$ *and* $\xi \in \Gamma(ker\psi_*)^{\perp}$*.*

Proof. *We will show it when* ψ *is type∼1. For all non-null vectors* $U, V \in \Gamma(\mathtt{D}^{\varphi_1})$ *and* $Y \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_2})$ *and* $\xi \in \Gamma(ker\psi_*)^{\perp}$*. Then using (2.1),(2.2),(2.5),(3.3) and Lemma 3.3(a),(b), we get:*

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U V, Y) &= -g_{\mathcal{B}}(\nabla_U \phi V, \mathcal{P}Y) - g_{\mathcal{B}}(\nabla_U fV, \mathcal{P}Y) \\ &= \cosh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, Y) + g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) \\ &- g_{\mathcal{B}}(\mathcal{T}_U fLV, \mathcal{P}KY + \phi RY) - g_{\mathcal{B}}(\mathcal{H}\nabla_U fLV, fRY). \end{aligned}$$

Hence we obtain;

$$\begin{aligned} -\sinh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, Y) &= g_{\mathcal{B}}(\mathcal{T}_U f\phi V, Y) - g_{\mathcal{B}}(\mathcal{T}_U fLV, \mathcal{P}KY + \phi RY) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_U fLV, fRY). \end{aligned}$$

Similarly, using (2.1),(2.2),(2.7),(3.3),(3.7) and Lemma 3.3(a),(b), we get:

$$\begin{aligned} g_{\mathcal{B}}(\nabla_U V, \xi) &= -g_{\mathcal{B}}(\nabla_U \phi V, \mathcal{P}\xi) - g_{\mathcal{B}}(\nabla_U fV, \mathcal{P}\xi) \\ &= \cosh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, \xi) + g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) \\ &- g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) - g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi). \end{aligned}$$

Hence, arrive at

$$\begin{aligned} -\sinh^2\varphi_1 g_{\mathcal{B}}(\nabla_U V, \xi) &= g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi V, \xi) - g_{\mathcal{B}}(\mathcal{H}\nabla_U fV, C\xi) \\ &- g_{\mathcal{B}}(\mathcal{T}_U fV, B\xi) \end{aligned}$$

which gives proof.

Similarly, the following conclusion is obtained.

Theorem 3.17. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with types:1,2. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the slant distribution $\mathtt{D}^{\varphi_2}$ describes a totally geodesic foliation on $\mathcal{B}$ if and only if*

$$g_{\mathcal{B}}(\mathcal{T}_U f\phi\zeta, V) = g_{\mathcal{B}}(\mathcal{T}_U fL\zeta, \mathcal{P}KV + \phi RV) + g_{\mathcal{B}}(\mathcal{H}\nabla_U fL\zeta, fRV) \tag{3.23}$$

and

$$g_{\mathcal{B}}(\mathcal{H}\nabla_U f\phi\zeta, \xi) = g_{\mathcal{B}}(\mathcal{H}\nabla_U f\zeta, C\xi) + g_{\mathcal{B}}(\mathcal{T}_U f\zeta, B\xi) \tag{3.24}$$

for any non-null vector fields $U, \zeta \in \Gamma(\mathtt{D}^{\varphi_2})$ and $V \in \Gamma(\mathtt{D} \oplus \mathtt{D}^{\varphi_1})$ and $\xi \in \Gamma(ker\psi_*)^{\perp}$.

Now, from Theorem 3.15, Theorem 3.16 and Theorem 3.17 we arrive at the Theorem 3.18. This is decomposition theorem for the fiber:

Theorem 3.18. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with types:1,2 from a para-Kaehler manifold $(\mathcal{B}, g_{\mathcal{B}}, \mathcal{P})$ onto a pseudo-Riemannian manifold $(\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, the fibers of ψ are locally product $\mathcal{B}_{\mathtt{D}} \times \mathcal{B}_{\mathtt{D}^{\varphi_1}} \times \mathcal{B}_{\mathtt{D}^{\varphi_2}}$ are leaves of $\mathtt{D}$, $\mathtt{D}^{\varphi_1}$ and $\mathtt{D}^{\varphi_2}$,respectively, if and only if the conditions (3.19),(3.20),(3.21),(3.22)(3.23) and (3.24) hold.*

Theorem 3.19. *Let $\psi : (\mathcal{B}, g_{\mathcal{B}}, \mathcal{P}) \to (\tilde{\mathcal{B}}, g_{\tilde{\mathcal{B}}})$ be a proper quasi-bi-slant pseudo-Riemannian submersion with type∼1. Let's suppose the total manifold as a para-Kaehler manifold and base manifold as a pseudo-Riemannian manifold. In this case, ψ is a totally geodesic map on $\mathcal{B}$ if and only if*

$$\begin{aligned} g_{\mathcal{B}}(\cosh^2\varphi_1 \nabla_U LW + \cosh^2\varphi_2 \nabla_U RW + \mathcal{H}\nabla_U f\phi LW + \mathcal{H}\nabla_U f\phi RW, Y) \\ = g_{\mathcal{B}}(\mathcal{V}\nabla_U \mathcal{P}KW + \mathcal{T}_U fLW + \mathcal{T}_U fRW, BY) \\ + g_{\mathcal{B}}(\mathcal{T}_U \mathcal{P}KW + \mathcal{H}\nabla_U fLW + \mathcal{H}\nabla_U fRW, CY) \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} g_{\mathcal{B}}(\cosh^2\varphi_1 \nabla_Y LU + \cosh^2\varphi_2 \nabla_Y RU + \mathcal{H}\nabla_Y f\phi LU + \mathcal{H}\nabla_Y f\phi RU, Z) \\ = g_{\mathcal{B}}(\mathcal{V}\nabla_Y \mathcal{P}KU + \mathcal{T}_Y fLU + \mathcal{T}_{UY} fRU, BZ) \\ g_{\mathcal{B}}(\mathcal{T}_Y \mathcal{P}KU + \mathcal{H}\nabla_Y fLU + \mathcal{H}\nabla_Y fRU, CZ) \end{aligned} \tag{3.26}$$

For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $Y, Z \in \Gamma(ker\psi_*)^{\perp}$.

Proof. *For any non-null vector fields $U, W \in \Gamma(ker\psi_*)$ and $Y, Z \in \Gamma(ker\psi_*)^{\perp}$. Then, using Lemma 2.9, (2.1),(2.2),(2.5),(3.3),(3.7) and Lemma 3.3(a),(b), we get:*

$$
\begin{aligned}
g_{\tilde{\mathcal{B}}}((\nabla\psi_*)(U,W),\psi_* Y) &= -g_{\mathcal{B}}(\nabla_U W, Y) \\
&= g_{\mathcal{B}}(\nabla_U \mathcal{P}W, \mathcal{P}Y) \\
&= g_{\mathcal{B}}(\nabla_U \mathcal{P}KW, \mathcal{P}Y) + g_{\mathcal{B}}(\nabla_U \mathcal{P}LW, \mathcal{P}Y) \\
&+ g_{\mathcal{B}}(\nabla_U \mathcal{P}RW, \mathcal{P}Y) \\
&= g_{\mathcal{B}}(\nabla_U \mathcal{P}KW, \mathcal{P}Y) + g_{\mathcal{B}}(\nabla_U \phi LW, \mathcal{P}Y) \\
&+ g_{\mathcal{B}}(\nabla_U \phi RW, \mathcal{P}Y) + g_{\mathcal{B}}(\nabla_U fLW, \mathcal{P}Y) \\
&+ g_{\mathcal{B}}(\nabla_U fRW, \mathcal{P}Y) \\
g_{\tilde{\mathcal{B}}}((\nabla\psi_*)(U,W),\psi_* X) &= g_{\mathcal{B}}(\mathcal{V}\nabla_U \mathcal{P}KW + \mathcal{T}_U fLW + \mathcal{T}_U fRW, \mathcal{P}Y) \\
&+ g_{\mathcal{B}}(\mathcal{T}_U \mathcal{P}KW + \mathcal{H}\nabla_U fLW + \mathcal{H}\nabla_U fRW, CY) \\
&- \{g_{\mathcal{B}}(\cosh^2\varphi_1 \nabla_U LW + \cosh^2\varphi_2 \nabla_U RW \\
&+ \mathcal{H}\nabla_U f\phi LW + \mathcal{H}\nabla_U f\phi RW, Y)\}
\end{aligned}
$$

Then, again using (2.1),(2.2),(2.7),(3.3),(3.7) and Lemma 3.3(a),(b), we get:

$$
\begin{aligned}
g_{\tilde{\mathcal{B}}}((\nabla\psi_*)(Y,U),\psi_* Z) &= -g_{\mathcal{B}}(\nabla_Y U, Z) \\
&= g_{\mathcal{B}}(\nabla_Y \mathcal{P}U, \mathcal{P}Z) \\
&= g_{\mathcal{B}}(\nabla_Y \mathcal{P}KU, \mathcal{P}Z) + g_{\mathcal{B}}(\nabla_Y \mathcal{P}LU, \mathcal{P}Z) \\
&+ g_{\mathcal{B}}(\nabla_Y \mathcal{P}RU, \mathcal{P}Z) \\
&= g_{\mathcal{B}}(\nabla_Y \mathcal{P}KU, \mathcal{P}Z) + g_{\mathcal{B}}(\nabla_Y \phi LU, \mathcal{P}Z) \\
&+ g_{\mathcal{B}}(\nabla_Y \phi RU, \mathcal{P}Z) + g_{\mathcal{B}}(\nabla_Y fLU, \mathcal{P}Z) \\
&+ g_{\mathcal{B}}(\nabla_Y fRU, \mathcal{P}Z) \\
g_{\tilde{\mathcal{B}}}((\nabla\psi_*)(Y,U),\psi_* Z) &= g_{\mathcal{B}}(\mathcal{V}\nabla_Y \mathcal{P}KU + \mathcal{T}_Y fLU + \mathcal{T}_{UY} fRU, BZ) \\
&+ g_{\mathcal{B}}(\mathcal{T}_Y \mathcal{P}KU + \mathcal{H}\nabla_Y fLU + \mathcal{H}\nabla_Y fRU, CZ) \\
&- \{g_{\mathcal{B}}(\cosh^2\varphi_1 \nabla_Y LU + \cosh^2\varphi_2 \nabla_Y RU) \\
&+ \mathcal{H}\nabla_Y f\phi LU + \mathcal{H}\nabla_Y f\phi RU, Z)\}.
\end{aligned}
$$

Acknowledgments

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

References

1. *M. A. Akyol , B. Şahin Conformal slant submersions. Hacettepe Journal of Mathematics and Statistics, 48(1), 28-44,(2019).*
2. *P. Alegre , A. Carriazo Slant submanifolds of para-Hermitian manifolds. Mediterranean Journal of Mathematics, 14(5), 1-14, (2017).*
3. *P. Alegre , A. Carriazo Bi-slant submanifolds of para-Hermitian manifolds. Mathematics, 7(7), 618, (2019).*
4. *G. Baditoiu , S. Ianus Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces. Differential Geometry and its Applications, 16(1), 79-94 (2002).*
5. *A. Carriazo Bi-slant immersions. Proc. ICRAMS 2000, 88-97, (2000).*
6. *M. Falcitelli , S. Ianus , A.M. Pastore Riemannian Submersions and Related Topics. World Scientific, 2004.*
7. *P. Gilkey , M. Itoh , J.H. Park Anti-invariant Riemannian submersions: A Lietheoretical approach. Taiwanese Journal of Mathematics , 20(4), 787-800, (2016).*
8. *Y. Gündüzalp Slant submersions in paracontact geometry. Hacettepe Journal of Mathematics and Statistics , 49(2), 822-834, (2019).*
9. *Y. Gündüzalp Almost para-Hermitian submersions. Matematicki Vesnik, 68(4), 241-253, (2016).*

10. Y. Gündüzalp Anti-invariant semi-Riemannian submersions from almost para-Hermitian manifolds. Journal of Function Spaces and Applications, ID 720623, (2013).

11. Y. Gündüzalp Anti-invariant pseudo-Riemannian submersions and Clairaut submersions from paracosymplectic manifolds. Mediterranean Journal of Mathematics, 16(4), 1-18, (2019).

12. Y. Gündüzalp Neutral slant submersions in paracomplex geometry. Afrika Matematika, 32(5), 1095-1110, (2021).

13. Y. Gündüzalp, M. A. Akyol Conformal slant submersions from cosymplectic manifolds. Turkish Journal of Mathematics, 42(5), 2672-2689, (2018).

14. Y. Gündüzalp Slant submersions from almost product Riemannian manifolds. Turkish Journal of Mathematics, 37(5), 863-873, (2013).

15. A. Gray Pseudo-Riemannian almost product manifolds and submersions. Journal of Mathematics and Mechanics, 16(7), 715-737, (1967).

16. S. Ianus , R. Mazzocco , G.E. Vîlcu Riemannian submersions from quaternionic manifolds. Acta Applicandae Mathematicae, 104(1), 83 , (2008).

17. S. Ianus , G.E. Vîlcu, R.C. Voicu Harmonic maps and Riemannian submersions between manifolds endowed with special structures. Banach Center Publications, 93, 277-288, (2011).

18. S. Ivanov , S. Zamkovoy Parahermitian and paraquaternionic manifolds. Differential Geometry and its Applications, 23(2), 205-234 (2005).

19. I. Kupeli Erken, C. Murathan On slant Riemannian submersions for cosymplectic manifolds. arXiv preprint arXiv:1311.3657, (2013).

20. C.W. Lee , J.W. Lee , B. Şahin , G.E. Vîlcu Optimal inequalities for Riemannian maps and Riemannian submersions involving Casorati curvatures. Annali di Matematica,200 ,1277–1295, (2021).

21. B. O'Neill The fundamental equations of a submersion. Michigan Mathematical Journal, 13(4), 459-469, (1966).

22. F. Özdemir, C. Sayar , H.M. Taştan Semi-invariant submersions whose total manifolds are locally product Riemannian. Quaestiones Mathematicae, 40(7), 909-926, (2017) .

23. S.A. Sepet , M. Ergüt Pointwise slant submersions from cosymplectic manifolds. Turkish Journal of Mathematics, 40(3), 582-593, (2016).

24. S.A. Sepet Bi-slant $\xi^{\perp}-$Riemannian submersions. Hacettepe Journal of Mathematics and Statistics, (2020).

25. K.S. Park , B. Şahin Semi-slant Riemannian maps into almost Hermitian manifolds. Czechoslovak Mathematical Journal, 64(4), 1045-1061, (2014).

26. R. Prasad , S.S. Shukla , S. Kumar On Quasi-bi-slant Submersions. Mediterranean Journal of Mathematics, 16(6), 1-18, (2019).

27. C. Sayar , M.A. Akyol , R. Prasad Bi-slant submersions in complex geometry. International Journal of Geometric Methods in Modern Physics, 17(04), 2050055, (2020).

28. R. Sarı , M.A. Akyol Hemi-slant $\xi^{\perp}$ -Riemannian submersions in contact geometry. Filomat, 34(11), 3747–3758, (2020).

29. B. Şahin Slant submersions from almost Hermitian manifolds. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, 93-105, (2011).

30. B. Şahin Anti-invariant Riemannian submersions from almost Hermitian manifolds. Open Mathematics, 8(3), 437-447, (2010).

31. B. Şahin Semi-invariant Submersions from Almost Hermitian Manifold. Canadian Mathematical Bulletin-Bulletin Canadien de Mathematiques, 56(1), 173-183 , (2013).

32. B. Şahin Riemannian submersions, Riemannian maps in Hermitian geometry and their applications. Academic Press, (2017).

33. B. Şahin Riemannian submersions from almost Hermitian manifolds. Taiwanese Journal of Mathematics, 17(2), 629-659, (2013).

34. H.M. Tastan, B. Şahin, Ş. Yanan Hemi-slant submersions. Mediterranean Journal of Mathematics,13(4), 2171-2184, (2016).

35. G.E. Vîlcu Almost product structures on statistical manifolds and para-Kähler-like statistical submersions. Bulletin des Sciences Mathématiques, 171, 103018, (2021).

36. B. Watson Almost Hermitian submersions. Journal of Differential Geometry, 11(1), 147-165, (1976).

Esra BAŞARIR NOYAN,
Department of Mathematics,
Dicle University,
21280, Sur, Diyarbakır, Turkey,
E-mail address: bsrrnoyan@gmail.com

and

Yılmaz GÜNDÜZALP,
Department of Mathematics,
Dicle University,
21280, Sur, Diyarbakır, Turkey,
E-mail address: ygunduzalp@dicle.edu.tr