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Existence of two solutions for $(p(x), q(x))$-Laplacian problems with Steklov boundary conditions

M. Rahmani, B. Karim*, A. Lakhdi, and A. Zerouali

ABSTRACT: The present paper discusses an elliptic equation with Steklov boundary conditions and $(p(x), q(x))$-Laplacian. Using mountain pass theorem together with Ekeland's variational principle, we prove, under appropriate conditions on the functions involved, that the problem admits at least two solutions.

Key Words: Ekeland's variational principle, Elliptic equations, $(p(x), q(x))$-Laplacian, Mountain pass theorem, Steklov eigenvalue problem.

Contents

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the following elliptic problem with variable exponents

$$\begin{cases} -\triangle_{p(x)}u - \triangle_{q(x)}u = \lambda(x)f(x,u) & \text{in } \Omega, \\ \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right)\frac{\partial u}{\partial \nu} = \mu(x)g(x,u) - |u|^{p(x)-2}u - |u|^{q(x)-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative of the outer unit on $\partial\Omega$ and $p, q \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : 1 < h^- < h^+ < \infty\}$ such that $q(.) \leq p(.)$. λ and μ are two functions, which will be described later. $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are two Carathéodory functions fulfilling appropriate conditions with potentials $F(x,u) = \int_0^u f(x,s)ds$ and $G(x,u) = \int_0^u g(x,s)ds$, respectively.

For every $m \in C^+(\overline{\Omega})$, we define

$$m^- = \min_{x\in\overline{\Omega}} m(x); \quad m^+ = \max_{x\in\overline{\Omega}} m(x);$$

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

In recent years, $p(x)$-Laplacian equations have attracted a lot of interest and importance because they can model a wide range of phenomena that appear throughout the study of electrorheological and thermorheological fluids [3,22], elastic mechanics [24], image restoration [5] and mathematical biology [14]. For more results and details on variable exponent problems, we refer the interested reader to [10,11,18,19]. The case when $p(.) = q(.)$ subjected to different types of boundary conditions is widely investigated (see, for example, [4,6,7,16]) and references therein. Particularly, in [23], A. Zerouali et al. studied the case where λ and μ are positive constants and established, under suitable hypotheses on f and

* Corresponding author
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g, that the $p(x)$-Laplacian problem admits at least three solutions, their approach is based on Ricceri's theorem [20,21]. The purpose of this study, is to generalize the results found in [23] by considering a more generalized form problem (1.1).

The Lebesgue space with variable exponent $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \subset \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} := \inf \left\{ \alpha > 0 \; ; \; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\},$$

and the generalized Sobolev space $W^{1,p(x)}(\Omega)$ as follows

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) \; / \; |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

endowed with the norm

$$\|u\|_\Omega := \inf \left\{ \alpha > 0 \; ; \; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx + \int_\Omega \left| \frac{\nabla u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

We shall first state the hypotheses on the functions involved in the current problem setting. Assume that f and g satisfy the following assumptions:

(F_1) There exist a positive constant c and two functions β, $m \in C_+(\overline{\Omega})$ with $\beta^+ < p^- < p^+ \leq m(x) < p^*(x)$, $\forall x \in \Omega$, such that $|f(x,s)| \leq c|s|^{\beta(x)-1}$, $(x,s) \in \Omega \times \mathbb{R}$ and $\lambda \in L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)$, with $\beta(x) + 1 \leq m(x)$, $\forall x \in \Omega$;

(F_2) There exist $d > 0$ and two functions $k_0 \in C(\Omega, \mathbb{R})$ and $\beta_0 \in C_+(\overline{\Omega})$ with $k_0(x) \geq 0$, $\forall x \in \Omega$, $k_0 \not\equiv 0$ and $\beta_0^+ < q^-$, such that $F(x,s) \geq k_0(x) s^{\beta_0(x)}$ for a.e. $x \in \Omega$ and all $s \in]0, d]$;

(G_1) There exist a positive constant C and two functions δ, $t \in C_+(\overline{\Omega})$ with $p^+ < \delta^- < \delta^+ < p^\partial(x)$, $\forall x \in \partial\Omega$, such that $|g(x,s)| \leq C|s|^{\delta(x)-1}$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ and $\mu \in L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)$, with $\delta(x) + 1 \leq t(x)$, $\forall x \in \partial\Omega$;

(G_2) There exist $\theta > p^+$ and $l > 0$ such that for a.e $x \in \partial\Omega$ and all $|t| > l$,

$$0 < \theta G(x,t) \leq g(x,t)t.$$

This article consists of three sections. In the second section, we state some necessary preliminary knowledge and known results. The following theorem, which is the main result of this work, is proved in Section 3.

Theorem 1.1 *Assume that conditions $(F_1) - (F_2)$ and $(G_1) - (G_2)$ are satisfied. If $\lambda(x) > 0$ a.e $x \in \Omega$ and $\mu(x) > 0$ a.e $x \in \partial\Omega$, then there exists a positive constant λ^* such that the problem (1.1) possesses at least two nontrivial weak solutions, for every $|\lambda|_{L^{\frac{s(x)}{s(x)-\alpha(x)}}(\Omega)} \in (0, \lambda^*)$, .*

2. Preliminaries

We give some fundamental facts concerning the generalized Lebesgue and Sobolev spaces, which we refer to as $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively. See [8,13,17] for more details on these spaces. Here also, we collect the ingredients of our proofs.

Proposition 2.1 *[8,12,13]*

1. *The space* $\left(L^{p(\cdot)}(\Omega), |u|_{L^{p(\cdot)}(\Omega)}\right)$ *is a separable and uniformly convex Banach space. The conjugate space of* $L^{p(\cdot)}(\Omega)$ *is* $L^{p'(\cdot)}(\Omega)$*, where* $p'(.) := \frac{p(.)}{p(.)-1}$ *and for every* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$*, the following Hölder inequality*

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{L^{p(x)}(\Omega)} |u|_{L^{p'(x)}(\Omega)} \leq 2|u|_{L^{p(x)}(\Omega)} |u|_{L^{p'(x)}(\Omega)}.$$

   holds true

2. *If* $\frac{1}{p_1(.)} + \frac{1}{p_2(.)} + \frac{1}{p_3(.)} = 1$*, then*

$$\begin{aligned}\left|\int_\Omega uvw dx\right| &\leq \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right) |u|_{L^{p_1(\cdot)}(\Omega)} |v|_{L^{p_2(\cdot)}(\Omega)} |w|_{L^{p_3(\cdot)}(\Omega)} \\ &\leq 3|u|_{L^{p_1(\cdot)}(\Omega)} |v|_{L^{p_2(\cdot)}(\Omega)} |w|_{L^{p_3(\cdot)}(\Omega)}.\end{aligned} \tag{2.1}$$

   for each $u \in L^{p_1(\cdot)}(\Omega)$*,* $v \in L^{p_2(\cdot)}(\Omega)$ *and* $w \in L^{p_3(\cdot)}(\Omega)$*.*

Now we will define the norm that will be applied afterward.

Proposition 2.2 *[7] The norm* $\|u\| := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\partial\Omega)}$ *is an equivalent norm to* $\|.\|_\Omega$ *on* $W^{1,p(x)}(\Omega)$*.*

The following proposition describes both the properties of $W^{1,p(x)}(\Omega)$ and the embedding results.

Proposition 2.3 *[10,17]*

(1) The space $\left(W^{1,p(x)}(\Omega), \|.\|\right)$ *is a separable and reflexive Banach space;*

(2) If $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$*,* $\forall x \in \overline{\Omega}$*, then the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous;*

(3) If $q \in C_+(\bar{\Omega})$ *and* $q(x) < p^\partial(x)$*,* $\forall x \in \bar{\Omega}$*, then the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ *is compact and continuous.*

When dealing with variable exponent Lebesgue-Sobolev spaces, the map $\rho_p : W^{1,p(x)}(\Omega) \to \mathbb{R}$ provided by

$$\rho_p(u) := \int_\Omega |\nabla u|^{p(x)} dx + \int_{\partial\Omega} |u|^{p(x)} d\sigma,$$

is particularly useful.

The following proposition clarifies the relationship between the norm $\|.\|$ and the map ρ_p.

Proposition 2.4 *[13] Let* $u, u_n \in W^{1,p(x)}(\Omega)$*;* $\quad n = 1, 2, \ldots,$ *we have*

1. *If* $||u|| \geq 1$ *then* $||u||^{p^-} \leq \rho_p(u) \leq ||u||^{p^+}$*;*
2. *If* $||u|| \leq 1$ *then* $||u||^{p^+} \leq \rho_p(u) \leq ||u||^{p^-}$*;*
3. $||u_n|| \to 0 (\to \infty)$ *if and only if* $\rho_p(u_n) \to 0 (\to \infty)$*.*

Lemma 2.1 *[2] If* r*,* $m \in C_+(\overline{\Omega})$ *with* $r(.) \leq m(.)$ *on* $\overline{\Omega}$*, then for all* $u \in L^{m(x)}(\Omega)$*, we have* $|u|^{r(x)} \in L^{\frac{m(x)}{r(x)}}(\Omega)$ *and*

$$\left||u|^{r(x)}\right|_{L^{\frac{m(x)}{r(x)}}(\Omega)} \leq |u|^{r^-}_{L^{m(x)}(\Omega)} + |u|^{r^+}_{L^{m(x)}(\Omega)},$$

or

$$\left||u|^{r(x)}\right|_{L^{\frac{m(x)}{r(x)}}(\Omega)} = |u|^{\tilde{q}}_{L^{m(x)}(\Omega)},$$

with $\tilde{r} \in [r^-, r^+]$*.*

We will use the following form of the mountain pass theorem (see [15]) to establish the existence result of Theorem 1.1.

Theorem 2.1 *Let $(E, \|.\|_E)$ be a Banach space and let $\Phi \in C^1(E;\mathbb{R})$ be a functional, which satisfies the Palais-Smale condition and*

(i) there exist $R > 0$ and $\rho \in \mathbb{R}$ such that $\Phi(u) \geq \rho$ if $\|u\|_E = R$;

(ii) $\Phi(0) < \rho$ and there exists $e \in E$ such that $\|e\|_E > R$ and $\Phi(e) < \rho$.

Then, Φ has a critical point $u_0 \in E$ such that $u_0 \neq 0$ and $u_0 \neq e$ with critical value

$$\Phi(u_0) = \inf_{\gamma \in P} \sup_{u \in \gamma} \Phi(u) \geq \rho > 0,$$

where P denotes the class of the paths $\gamma \in C([0,1]; E)$ joining 0 to e.

Finally, we will apply the Ekeland's variational principle to establish the existence of a second solution.

Theorem 2.2 *[9] Let (E, d) be a complete metric space and $\Phi : E \mapsto \mathbb{R} \cup \{+\infty\}$ be a l.s.c. function, not identically equal to $+\infty$ and bounded from below. Then, for all $\delta > 0$ and all $u \in E$ such that*

$$\Phi(u) \leq \inf_E \Phi + \varepsilon, \quad \forall \varepsilon > 0,$$

and all $\delta > 0$, there exists $v \in E$ such that

i) $\Phi(v) \leq \Phi(u)$;

ii) $d(u, v) \leq \delta$;

iii) for all $w \neq v$, $\Phi(v) < \Phi(w) + \frac{\varepsilon}{\delta} d(v, w)$.

3. Proof of main results

We first make some notations before proving our main result. The Sobolev space $W^{1,p(x)}(\Omega)$ will be denoted, from now on, by E. The strong and the weak convergence of u_n to u are denoted by $u_n \rightharpoonup u$ and $u_n \to u$, respectively. We use the notations c_i and C_i, with $i = 0, 1, 2, \ldots$ to indicate positive constants in inequalities which we derive in this section.

We start with proving the lemma below, which is crucial in the definition of a weak solution of problem (1.1).

Lemma 3.1 *Assume that (F_1) and (G_1) hold, then the functional J defined by $J(u) = \int_\Omega \lambda(x) F(x, u) dx + \int_{\partial\Omega} \mu(x) G(x, u) d\sigma$ satisfies*

(a) J is well defined and $J \in C^1(E, \mathbb{R})$;

(b) J, J' are completely continuous.

Proof:

(a) We use the same technic as in [1] with slight changes. By Proposition 2.1,Lemma 2.1, (F_1) and (G_1), we have

$$\begin{aligned} |J(u)| &\leq \int_\Omega |\lambda(x) F(x, u)| dx + \int_{\partial\Omega} |\mu(x) G(x, u)| d\sigma \\ &\leq 2\frac{c}{\beta^-} |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} |u^{\beta(x)}|_{L^{\frac{m(x)}{\beta(x)}}(\Omega)} + 2\frac{C}{\delta^-} |\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)} |u^{\delta(x)}|_{L^{\frac{t(x)}{\delta(x)}}(\partial\Omega)}. \end{aligned}$$

Since $m(x) < p^*(x)$ (respectively $t(x) < p^\partial(x)$), it follows by Proposition 2.3 that $E \hookrightarrow L^{m(x)}(\Omega)$ (respectively $E \hookrightarrow L^{t(x)}(\partial\Omega)$) is continuous embeddings. From this we conclude that

$$|u|_{L^{m(x)}(\Omega)} \leq c_0 \|u\|, \tag{3.1}$$

and

$$|u|_{L^{t(x)}(\partial\Omega)} \leq C_0\|u\|. \tag{3.2}$$

By (3.1), (3.2) and Lemma 2.1, we obtain

$$|J(u)| \leq 2c_1\frac{c}{\beta^-}|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\left(\|u\|^{\beta^-}+\|u\|^{\beta^+}\right)+2C_1\frac{C}{\delta^-}|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\left(\|u\|^{\delta^-}+\|u\|^{\delta^+}\right),$$

or

$$|J(u)| \leq 2c_2\frac{c}{\beta^-}|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\|u\|^{\tilde{\beta}}+2C_2\frac{C}{\delta^-}|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\|u\|^{\tilde{\delta}}.$$

Consequently, J is well defined. We proceed to show that $J \in C^1(E,\mathbb{R})$. Let us first prove that J is Gâteaux differentiable. We have

$$\begin{aligned} DJ(u,v) &= \lim_{h\to 0}\frac{J(u+hv)-J(u)}{h} \\ &= \lim_{h\to 0}\left[\int_\Omega \frac{\lambda(x)\big(F(x,u+hv)-F(x,u)\big)}{h}dx+\int_{\partial\Omega}\frac{\mu(x)\big(G(x,u+hv)-G(x,u)\big)}{h}d\sigma\right]. \end{aligned}$$

Using the mean value theorem, we see that

$$DJ(u,v)=\lim_{h\to 0}\left[\int_\Omega \lambda(x)f(x,u+h\vartheta v)v(x)dx+\int_{\partial\Omega}\mu(x)g(x,u+h\vartheta v)v(x)d\sigma\right], \quad \vartheta\in(0,1).$$

For $|h|\leq 1$, the condition (F_1) and Young's inequality shows that

$$\begin{aligned} |\lambda(x)f(x,u+h\vartheta v)v(x)| \leq & c|\lambda(x)||u+h\vartheta v|^{\beta(x)-1}|v| \\ \leq & \frac{c(m(x)-\beta(x))}{m(x)}|\lambda(x)|^{\frac{m(x)}{m(x)-\beta(x)}} \\ & +\frac{c(\beta(x)-1)}{m(x)}\left[|u+h\vartheta v|^{\beta(x)-1}\right]^{\frac{m(x)}{\beta(x)-1}}+\frac{c}{m(x)}|v|^{m(x)}. \end{aligned}$$

Since the function $\psi : s \to |s|^p$ is convex for $p \geq 1$, it follows that

$$\begin{aligned} |\lambda(x)f(x,u+h\vartheta v)v(x)| \leq & \frac{c(m(x)-\beta(x))}{m(x)}|\lambda(x)|^{\frac{m(x)}{m(x)-\beta(x)}} \\ & +\frac{c(\beta(x)-1)}{m(x)}2^{m(x)-1}\left[|u|^{m(x)}+|v|^{m(x)}\right]+\frac{c}{m(x)}|v|^{m(x)}. \end{aligned} \tag{3.3}$$

We now apply the same argument again, with (F_1) replaced by (G_1), to obtain

$$\begin{aligned} |\mu(x)g(x,u+h\vartheta v)v(x)| \leq & \frac{C(t(x)-\delta(x))}{t(x)}|\mu(x)|^{\frac{t(x)}{t(x)-\delta(x)}} \\ & +\frac{C(\delta(x)-1)}{t(x)}2^{t(x)-1}\left[|u|^{t(x)}+|v|^{t(x)}\right]+\frac{C}{t(x)}|v|^{t(x)}. \end{aligned} \tag{3.4}$$

Note that the last expression to the right in (3.3) (respectively (3.4)) is independent of h and is in $L^1(\Omega)$ (respectively $L^1(\partial\Omega)$). Therefore, by the dominated convergence theorem, it may be concluded that

$$DJ(u,v)=\int_\Omega \lambda(x)f(x,u)v(x)dx+\int_{\partial\Omega}\mu(x)g(x,u)v(x)d\sigma. \tag{3.5}$$

It is known that the Nemytskii operators

$$\begin{aligned} \mathcal{F} : L^{m(x)}(\Omega) &\to L^{\frac{m(x)}{\beta(x)-1}}(\Omega) \\ u &\mapsto f(x,u) \end{aligned}$$

and

$$\begin{array}{rcl} \mathcal{G} : L^{t(x)}(\partial\Omega) & \to & L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega) \\ u & \mapsto & g(x,u) \end{array}$$

are bounded and continuous operators. Using Propositions 2.1 and 2.3, we obtain

$$\begin{array}{rcl} DJ(u,v) & \leq & 3|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}|f(x,u)|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}|v|_{L^{m(x)}(\Omega)} \\ & & +3|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}|g(x,u)|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}|v|_{L^{t(x)}(\partial\Omega)} \\ & \leq & 3c_0|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}|f(x,u)|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}\|v\| \\ & & +3C_0|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}|g(x,u)|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}\|v\|. \end{array}$$

Therefore, the functional $DJ(u,v)$ is linear and bounded. Consequently, is the Gâteaux derivative of J. Show that $DJ(u,v)$ is continuous. For $u, u', v \in E$, by (3.5) and Proposition 2.1, we have

$$\begin{array}{rcl} |\langle DJ(u)-DJ(u'),v\rangle| & \leq & 3|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}|f(x,u)-f(x,u')|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}|v|_{L^{m(x)}(\Omega)} \\ & & +3|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}|g(x,u)-g(x,u')|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}|v|_{L^{t(x)}(\partial\Omega)} \\ & \leq & c_3|f(x,u)-f(x,u')|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}\|v\| \\ & & +C_3|g(x,u)-g(x,u')|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}\|v\|, \end{array}$$

thus

$$\|DJ(u)-DJ(u')\|_{E^*} \leq c_3|f(x,u)-f(x,u')|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)} + C_3|g(x,u)-g(x,u')|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}.$$

Hence, $DJ(u)$ is a continuous operator, therefore J is Fréchet differentiable and $J \in C^1(E,\mathbb{R})$ with $J'(u) = DJ(u)$.

(b) Let $(u_k) \in E$ be a sequence such that $u_k \rightharpoonup u$. Show that $J(u_k) \to J(u)$.
Assume by contradiction that $J(u_k) \nrightarrow J(u)$, then there exists $\epsilon > 0$ and a subsequence (u_k) such that

$$\epsilon \leq |J(u_k) - J(u)|.$$

Again, by the mean value theorem, for $0 < \vartheta_k < 1$, we have

$$0 < \epsilon \leq |\langle J'(u + \vartheta_k(u_k - u)), u_k - u\rangle|.$$

Put $w_k = u + \vartheta_k(u_k - u)$. As $J'(u)(w) = \int_\Omega \lambda(x)f(x,u)wdx + \int_{\partial\Omega}\mu(x)g(x,u)wd\sigma$, using Proposition 2.1, (F_1) and (G_1), we have

$$\begin{array}{rcl} \langle J'(w_k), u_k-u\rangle & = & \int_\Omega \lambda(x)f(x,w_k)(u_k-u)dx + \int_{\partial\Omega}\mu(x)g(x,w_k)(u_k-u)d\sigma \\ & \leq & \int_\Omega |\lambda(x)||f(x,w_k)||u_k-u|dx + \int_{\partial\Omega}|\mu(x)||g(x,w_k)||u_k-u|d\sigma \\ & \leq & \int_\Omega c|\lambda(x)||w_k|^{\beta(x)-1}|u_k-u|dx + \int_{\partial\Omega}C|\mu(x)||w_k|^{\delta(x)-1}|u_k-u|d\sigma \\ & \leq & 3c|\lambda(x)|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\left||w_k|^{\beta(x)-1}\right|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}|u_k-u|_{L^{m(x)}(\Omega)} \\ & & +3C|\mu(x)|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\left||w_k|^{\delta(x)-1}\right|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}|u_k-u|_{L^{t(x)}(\partial\Omega)}. \end{array}$$

Since $\lim_{k\to+\infty}|w_k|_{L^{m(x)}(\Omega)} \neq \infty$ and $\lim_{k\to+\infty}|w_k|_{L^{t(x)}(\partial\Omega)} \neq \infty$, by Lemma 2.1, it follows immediately that

$$\lim_{k\to+\infty}\left||w_k|^{\beta(x)-1}\right|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)} \neq \infty \quad \text{and} \quad \lim_{k\to+\infty}\left||w_k|^{\delta(x)-1}\right|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)} \neq \infty.$$

As $E \hookrightarrow L^{m(x)}(\Omega)$ and $E \hookrightarrow L^{t(x)}(\partial\Omega)$ are compact embeddings, letting $k \to +\infty$ in the above inequality, it follows that the right-hand side converges to 0, therefore, J is completely continuous. Finally, show that J' is completely continuous. Let us recall that

$$\langle J'(u), v\rangle = \int_\Omega \lambda(x) f(x,u) v dx + \int_{\partial\Omega} \mu(x) g(x,u) v d\sigma.$$

For $(u_k) \subset E$ such that $u_k \rightharpoonup u$, we have (u_k) is bounded. Now, (2.1) makes it obvious that

$$\begin{aligned} |\langle J'(u_k) - J'(u), v\rangle| &\leq \int_\Omega |\lambda(x)||f(x,u_k) - f(x,u)||v| dx + \int_{\partial\Omega} |\mu(x)||g(x,u_k) - g(x,u)||v| d\sigma \\ &\leq 3|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} |f(x,u_k) - f(x,u)|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)} |v|_{L^{m(x)}(\Omega)} \\ &\quad + 3|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)} |g(x,u_k) - g(x,u)|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)} |v|_{L^{t(x)}(\partial\Omega)}. \end{aligned}$$

The compact embedding $E \hookrightarrow L^{m(x)}(\Omega)$ (respectively $E \hookrightarrow L^{t(x)}(\partial\Omega)$) guarantees the existence of a subsequence (u_k) such that $u_k \to u$ in $L^{m(x)}(\Omega)$ (respectively in $L^{t(x)}(\partial\Omega)$). Using the fact that the operators $\mathcal{F}$ and $\mathcal{G}$ are continuous, it is clear that J' is completely continuous, which proves the lemma.

□

To apply mountain pass theorem, We define the functional $\Phi : E \to \mathbb{R}$ by

$$\begin{aligned} \Phi(u) = &\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} + \frac{|\nabla u|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u|^{p(x)}}{p(x)} + \frac{|u|^{q(x)}}{q(x)} \right) d\sigma \\ &- \int_\Omega \lambda(x) F(x,u) dx - \int_{\partial\Omega} \mu(x) G(x,u) d\sigma, \end{aligned} \tag{3.6}$$

Under (F_1) and (G_1), we have Φ is of class $C^1(E, \mathbb{R})$, therefore, a weak solution of problem (1.1) can be defined as follows.

Definition 3.1 *We say that $u \in E$ is a weak solution to the problem (1.1) if :*

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2} \right) \nabla u \nabla v dx + \int_{\partial\Omega} \left(|u|^{p(x)-2} + |u|^{q(x)-2} \right) u v d\sigma \\ &\quad - \int_\Omega \lambda(x) f(x,u) v dx - \int_{\partial\Omega} \mu(x) g(x,u) v d\sigma = 0, \quad \textit{for every } v \in E. \end{aligned}$$

Lemma 3.2 *The functional Φ satisfies the Palais-Smale (PS) condition.*

Proof: Let $M > 0$ and $(u_k) \subset E$ be such that $|\Phi(u_k)| \leq M$ and $\Phi'(u_k) \to 0$ in E'. Let us prove that (u_k) is bounded.

Assume by contradiction that $\|u_k\| \to \infty$. Then, using Proposition 2.4, for large enough k we have

$$\begin{aligned}
M+1+\|u_k\| \geq & \Phi(u_k) - \frac{1}{\theta}\langle \Phi'(u_k), u_k\rangle \\
\geq & \int_\Omega \left(\frac{|\nabla u_k|^{p(x)}}{p(x)} + \frac{|\nabla u_k|^{q(x)}}{q(x)} \right) dx + \int_{\partial\Omega} \left(\frac{|u_k|^{p(x)}}{p(x)} + \frac{|u_k|^{q(x)}}{q(x)} \right) d\sigma - \int_\Omega \lambda(x) F(x,u_k) dx \\
& - \int_{\partial\Omega} \mu(x) G(x,u_k) d\sigma - \frac{1}{\theta}\int_\Omega \left(|\nabla u|^{p(x)} + |\nabla u|^{q(x)} \right) dx - \frac{1}{\theta}\int_{\partial\Omega} \left(|u|^{p(x)} + |u|^{q(x)} \right) d\sigma \\
& + \frac{1}{\theta}\int_\Omega \lambda(x) f(x,u_k) u_k dx + \frac{1}{\theta}\int_{\partial\Omega} \mu(x) g(x,u_k) u_k d\sigma \\
\geq & \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \rho_p(u_k) + \left(\frac{1}{q^+} - \frac{1}{\theta} \right) \rho_q(u_k) + \int_\Omega \lambda(x) \left(\frac{1}{\theta} f(x,u_k) u_k - F(x,u_k) \right) dx \\
& + \int_{\partial\Omega} \mu(x) \left(\frac{1}{\theta} g(x,u_k) u_k - G(x,u_k) \right) d\sigma \\
\geq & \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_k\|^{p^-} + \int_\Omega \lambda(x) \left(\frac{1}{\theta} f(x,u_k) u_k - F(x,u_k) \right) dx \\
& + \int_{\partial\Omega} \mu(x) \left(\frac{1}{\theta} g(x,u_k) u_k - G(x,u_k) \right) d\sigma.
\end{aligned}$$

For $\mu(x) > 0$ a.e. $x \in \partial\Omega$, by (F_1) and (G_2), we obtain

$$1+M+\|u_k\| \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_k\|^{p^-} - c\left(\frac{1}{\theta} + \frac{1}{\beta^-} \right) \int_\Omega |\lambda(x)||u_k|^{\beta(x)} dx. \tag{3.7}$$

Moreover, by Proposition 2.1 and Lemma 2.1, we have

$$\begin{aligned}
\int_\Omega |\lambda(x)||u_k|^{\beta(x)} dx & \leq 2|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \left||u_k|^{\beta(x)}\right|_{L^{\frac{m(x)}{\beta(x)}}(\Omega)} \\
& \leq 2|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \left(|u_k|^{\beta^-}_{L^{m(x)}(\Omega)} + |u_k|^{\beta^+}_{L^{m(x)}(\Omega)} \right) \\
& \leq c_4 |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \left(\|u_k\|^{\beta^-} + \|u_k\|^{\beta^+} \right),
\end{aligned} \tag{3.8}$$

or

$$\int_\Omega |\lambda(x)||u_k|^{\beta(x)} dx \leq c_5 |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \|u_k\|^{\tilde{\beta}}, \tag{3.9}$$

Substituting 3.7 (respectively 3.8) into 3.9, we obtain

$$\begin{aligned}
1+M+\|u_k\| & \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_k\|^{p^-} - c_6 |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \left(\|u_k\|^{\beta^-} + \|u_k\|^{\beta^+} \right) \\
& \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_k\|^{p^-} - 2c_6 |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \|u_k\|^{\beta^+}
\end{aligned}$$

or

$$1+M+\|u_k\| \geq \left(\frac{1}{p^+} - \frac{1}{\theta} \right) \|u_k\|^{p^-} - c_7 |\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} \|u_k\|^{\tilde{\beta}},$$

Since $\theta > p^+$ and $p^- > \beta^+ \geq \tilde{\beta}$, this leads to a contradiction. Consequently, (u_k) is bounded in E. As E is a reflexive Banach space, we infer that, up to a subsequence $(_k)$, we have $u_k \rightharpoonup u$ in E. According to the fact that $\Phi'(u_k) \to 0$, it follows that

$$\lim_{k\to\infty} \langle \Phi'(u_k), u_k - u\rangle = 0.$$

More specifically

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k-\nabla u)dx+\int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k-\nabla u)dx\\ +\int_{\partial\Omega}|u_k|^{p(x)-2}u_k(u_k-u)d\sigma+\int_{\partial\Omega}|u_k|^{q(x)-2}u_k(u_k-u)d\sigma\\ -\int_\Omega \lambda(x)f(x,u_k)(u_k-u)dx-\int_{\partial\Omega}\mu(x)g(x,u_k)(u_k-u)d\sigma=0.\end{aligned} \tag{3.10}$$

By (F_1), (G_1) and the Hölder type inequality (2.1), it may be concluded that

$$\left|\int_\Omega \lambda(x)f(x,u_k)(u_k-u)dx\right|\le 3c|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\left||u_k|^{\beta(x)-1}\right|_{L^{\frac{m(x)}{\beta(x)-1}}(\Omega)}|u_k-u|_{L^{m(x)}(\Omega)}.$$
and
$$\left|\int_{\partial\Omega}\mu(x)g(x,u_k)(u_k-u)d\sigma\right|\le 3C|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\left||u_k|^{\delta(x)-1}\right|_{L^{\frac{t(x)}{\delta(x)-1}}(\partial\Omega)}|u_k-u|_{L^{t(x)}(\partial\Omega)}.$$

On the other hand, since $p(x)$, $q(x)$ and $t(x)$ (respectively $m(x)$) fulfills the condition (3) (respectively (2)) of Proposition (2.3), it follows that $u_k\to u$ in $L^{p(x)}(\partial\Omega)$, $L^{q(x)}(\partial\Omega)$ and $L^{t(x)}(\partial\Omega)$ (respectively $L^{m(x)}(\Omega)$). Therefore, taking into account the above considerations, relation (3.10) is reduced to

$$\lim_{k\to\infty}\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k-\nabla u)dx+\int_\Omega |\nabla u_k|^{q(x)-2}\nabla u_k(\nabla u_k-\nabla u)dx=0.$$

The fact that the two terms of the above limit have the same sign, implies that

$$\lim_{k\to\infty}\int_\Omega |\nabla u_k|^{p(x)-2}\nabla u_k(\nabla u_k-\nabla u)dx=0.$$

We infer that the operator $\Delta_{p(x)}$ is of type (S_+), see for example [6], hence $u_k\to u$ in E, and the lemma follows. □

A second auxiliary lemma is required to apply Theorem 2.1.

Lemma 3.3 *1. There exist λ^*, R, $\rho>0$ such that for every $|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\in(0,\lambda^*)$, we have $\Phi(u)\ge\rho$ if $\|u\|=R$.*

2. There exist $e\in E$ with $\|e\|>R$ such that $\Phi(e)<0$.

Proof:
1. we have

$$\begin{aligned}\Phi(u)=&\int_\Omega\left(\frac{|\nabla u|^{p(x)}}{p(x)}+\frac{|\nabla u|^{q(x)}}{q(x)}\right)dx+\int_{\partial\Omega}\left(\frac{|u|^{p(x)}}{p(x)}+\frac{|u|^{q(x)}}{q(x)}\right)d\sigma\\ &-\int_\Omega\lambda(x)F(x,u)dx-\int_{\partial\Omega}\mu(x)G(x,u)d\sigma.\end{aligned}$$

Using Propositions 2.1, 2.4 and Lemmas 2.1, 3.1, for $\|u\|<1$, we obtain

$$\begin{aligned}\Phi(u)\ge&\frac{1}{p^+}\|u\|^{p^+}-c\int_\Omega\frac{\lambda(x)}{\beta(x)}|u|^{\beta(x)}dx-C\int_{\partial\Omega}\frac{\mu(x)}{\delta(x)}|u|^{\delta(x)}d\sigma\\ \ge&\frac{1}{p^+}\|u\|^{p^+}-c_8|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\left(\|u\|^{\beta^-}+\|u\|^{\beta^+}\right)-C_8|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\left(\|u\|^{\delta^-}+\|u\|^{\delta^+}\right)\\ \ge&\frac{1}{p^+}\|u\|^{p^+}-2c_8|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\|u\|^{\beta^-}-2C_8|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\|u\|^{\delta^-},\end{aligned}$$

or

$$\Phi(u) \geq \frac{1}{p^+}\|u\|^{p^+} - c_9|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\|u\|^{\tilde{\beta}} - C_9|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\|u\|^{\tilde{\delta}}.$$

The above inequalities can be written as

$$\Phi(u) \geq \left(\frac{1}{2p^+} - 2c_8|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\|u\|^{\beta^- - p^+}\right)\|u\|^{p^+} + \left(\frac{1}{2p^+} - 2C_8|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\|u\|^{\delta^- - p^+}\right)\|u\|^{p^+}, \tag{3.11}$$

or

$$\Phi(u) \geq \left(\frac{1}{2p^+} - c_9|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)}\|u\|^{\tilde{\beta} - p^+}\right)\|u\|^{p^+} + \left(\frac{1}{2p^+} - C_9|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)}\|u\|^{\tilde{\delta} - p^+}\right)\|u\|^{p^+}. \tag{3.12}$$

Since $\tilde{\delta} \geq \delta^- > p^+$, it follows that the functions $k,\ k' : [0,1] \to \mathbb{R}$ defined respectively by

$$k(s) = \frac{1}{2p^+} - 2C_8|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)} s^{\delta^- - p^+}$$

and

$$k'(s) = \frac{1}{2p^+} - C_9|\mu|_{L^{\frac{t(x)}{t(x)-\delta(x)}}(\partial\Omega)} s^{\tilde{\delta} - p^+},$$

are positive on the neighborhood of the origin. Therefore, there exist $R \in (0,1)$ such that $k(R),\ k'(R) > 0$. For $\|u\| = R$, we can choose

$$\lambda^* = \min\left\{1, \frac{R^{p^+ - \beta^-}}{8c_8 p^+}, \frac{R^{p^+ - \tilde{\beta}}}{4c_9 p^+}\right\}.$$

Thus for all $|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} < \lambda^*$, there exist $\rho = \frac{R^{p^+}}{4p^+}$ such that $\Phi(u) \geq \rho > 0$ for every $u \in E$ with $\|u\| = R$.

2. From assumption (G_1), we can deduce that

$$G(x,s) \geq C_{10}|s|^{\theta} - C_{11}, \quad \forall s \in \mathbb{R},\ a.e.\, x \in \partial\Omega,$$

where $C_{10},\ C_{11} > 0$. Let $\varphi \in C_0^\infty(\Omega)$ and $h > 1$. Using (F_1) and (G_1), we obtain

$$\begin{aligned}
\Phi(h\varphi) &= \int_\Omega \left(\frac{h^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)} + \frac{h^{q(x)}}{q(x)}|\nabla\varphi|^{q(x)}\right)dx + \int_{\partial\Omega}\left(\frac{h^{p(x)}}{p(x)}|\varphi|^{p(x)} + \frac{h^{q(x)}}{q(x)}|\varphi|^{q(x)}\right)d\sigma \\
&- \int_\Omega \lambda(x)F(x,h\varphi)dx - \int_{\partial\Omega}\mu(x)G(x,h\varphi)d\sigma \\
&\leq \frac{h^{p^+}}{p^-}\rho_p(\varphi) + \frac{h^{q^+}}{q^-}\rho_q(\varphi) + \int_\Omega \lambda(x)|F(x,h\varphi)|dx - \int_{\partial\Omega}\mu(x)G(x,h\varphi)d\sigma \\
&\leq \frac{h^{p^+}}{p^-}\rho_p(\varphi) + \frac{h^{q^+}}{q^-}\rho_q(\varphi) + \frac{c}{\beta^-}h^{\beta^+}\int_\Omega \lambda(x)|\varphi|^{\beta(x)}dx - C_{10}h^{\theta}\int_{\partial\Omega}\mu(x)|\varphi|^{\theta}d\sigma \\
&+ C_{11}\int_{\partial\Omega}\mu(x)d\sigma.
\end{aligned}$$

Since $\mu(x)|\varphi|^\theta$ is positive, $\theta > p^+ > q^+$ and $p^+ > \beta^+$ it follows that $\lim_{h\to+\infty}\Phi(h\varphi) = -\infty$. Therefore for $\varphi \not\equiv 0$ and large enough h, we may choose $e = h\varphi$ such that $\Phi(e) < 0$ and $\|e\| > R$.

□

In this moment, proving the existence result is more of a formality. It is obvious that $\Phi(0) = 0 < \rho$. Using Lemmas 3.2-3.3 and Theorem 2.1, We deduce that Φ has a critical point u_1 satisfies $\Phi(u_1) := c_{10} > 0$. Thus, the problem (1.1) admits a nontrivial weak solution.

The task now is to show the existence of a second weak solution. We start with the lemma below.

Lemma 3.4 *There exists $\varphi \in E\backslash\{0\}$, $\varphi \geq 0$ such that for all $h > 0$ small enough, we have $\Phi(h\varphi) < 0$.*

Proof: Let $\varphi \in C_0^\infty(\overline{\Omega})$, $\varphi \geq 0$, $\varphi \not\equiv 0$. For $h \in (0, 1)$, using (F_2), we have

$$\begin{aligned}
\Phi(h\varphi) &= \int_\Omega \left(\frac{h^{p(x)}}{p(x)}|\nabla\varphi|^{p(x)} + \frac{h^{q(x)}}{q(x)}|\nabla\varphi|^{q(x)}\right)dx + \int_{\partial\Omega}\left(\frac{h^{p(x)}}{p(x)}|\varphi|^{p(x)} + \frac{h^{q(x)}}{q(x)}|\varphi|^{q(x)}\right)d\sigma \\
&- \int_\Omega \lambda(x)F(x, h\varphi)dx - \int_{\partial\Omega}\mu(x)G(x, h\varphi)d\sigma \\
&\leq \frac{h^{p^-}}{p^-}\rho_p(\varphi) + \frac{h^{q^-}}{q^-}\rho_q(\varphi) - h^{\beta_0}\int_\Omega \lambda(x)k_0(x)|\varphi|^{\beta_0}dx - \int_{\partial\Omega}\mu(x)G(x, h\varphi)d\sigma \\
&\leq h^{q^-}\left(\frac{\rho_p(\varphi)}{p^-} + \frac{\rho_q(\varphi)}{q^-}\right) - h^{\beta_0}\int_\Omega \lambda(x)k_0(x)|\varphi|^{\beta_0}dx < 0.
\end{aligned}$$

Since $\varphi \not\equiv 0$, it follows that $\rho_p(\varphi)$, $\rho_q(\varphi) > 0$. Let τ be a real number such that

$$0 < \tau < \min\left\{1, \frac{\int_\Omega \lambda(x)k_0(x)|\varphi|^{\beta_0}dx}{\frac{\rho_p(\varphi)}{p^-} + \frac{\rho_q(\varphi)}{q^-}}\right\}.$$

It is easily seen that $\Phi(h\varphi) < 0$ For all $h < \tau^{\frac{1}{q^- - \beta_0}}$, and this is precisely the assertion of the lemma. □

Now, assume that $|\lambda|_{L^{\frac{m(x)}{m(x)-\beta(x)}}(\Omega)} < \lambda^*$ and set $B_R(0) = \{\varphi \in E;\ \|\varphi\| < R\}$. Using Lemma 3.3, we conclude that

$$\inf_{\partial B_R(0)} \Phi(u) \geq \rho > 0.$$

Moreover, Lemma 3.4 provide the existence of $\varphi \in E$ satisfies $\Phi(h\varphi) < 0$ with $h > 0$ small enough.

By (3.11) and (3.12)), it is obvious that Φ is bounded from below in $B_R(0)$. Therefore

$$-\infty < c_{10} := \inf_{\overline{B_R(0)}} \Phi(u) < 0,$$

for $u \in \overline{B_R(0)}$. Take $0 < \epsilon < \inf_{\partial B_R(0)} \Phi(u) - \inf_{\overline{B_R(0)}} \Phi(u)$ and Apply the Ekeland's variational principle [9] to the functional $\Phi : \overline{B_R(0)} \to \mathbb{R}$, we deduce that there exists $u_\epsilon \in \overline{B_R(0)}$ such that

$$\begin{aligned}
\Phi(u_\epsilon) &< \inf_{\overline{B_R(0)}} \Phi + \epsilon \\
&< \Phi(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.
\end{aligned}$$

As

$$\Phi(u_\epsilon) \leq \inf_{\overline{B_R(0)}} \Phi + \epsilon \leq \inf_{B_R(0)} \Phi + \epsilon < \inf_{\partial B_R(0)} \Phi.$$

Thus $u_\epsilon \in B_R(0)$. Obviously, u_ϵ is a minimum of the functional I defined on $\overline{B_R(0)}$ by $I(u) = \Phi(u) + \epsilon\|u - u_\epsilon\|$. Hence

$$\frac{I(u_\epsilon + h\varphi) - I(u_\epsilon)}{t} \geq 0,$$

for $h > 0$ small and $\varphi \in B_R(0)$. We deduce that

$$\frac{\Phi(u_\epsilon + h\varphi) - \Phi(u_\epsilon)}{h} + \epsilon\|\varphi\| \geq 0,$$

Letting $h \to 0$, the inequality above becomes $\langle \Phi'(u_\epsilon), \varphi\rangle + \epsilon\|\varphi\| \geq 0$, we infer that $\|\Phi'(u_\epsilon)\|_{X^*} \leq \epsilon$. Therefore, there exists $(\varphi_n) \subset B_R(0)$ such that

$$\Phi(\varphi_n) \to c_{10} \;\text{ and }\; \Phi'(\varphi_n) \to 0. \tag{3.13}$$

Obviously, (φ_n) is a bounded sequence in E. As E is a reflexive space, there exist a subsequence, still denoted (φ_n) such that $\varphi_n \rightharpoonup u_2$ in E. Using Proposition 2.1 and Lemma 3.1, we obtain $\varphi_n \to u_2$ in E. Consequently, by relation (3.13)

$$\Phi(u_2) = c_{10} \;\text{ and }\; \Phi'(u_2) = 0.$$

Thus u_2 is a nontrivial solution problem (1.1).

What is left is to show that $u_2 \neq u_1$. Since

$$\Phi(u_1) = c_7 > 0 > c_{10} = \Phi(u_2),$$

it follows that $u_2 \neq u_1$, which completes the proof.

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Mostafa Rahmani,
University First Mohamed of Oujda, Oriental Applied Mathematics Laboratory (LaMAO), Morocco.
E-mail address: ma.rahmani@ump.ac.ma

and

Belhadj Karim,
University Moulay Ismail of Meknes, FST Errachidia, LMIMA Laboratory, ROLALI Group, Morocco.
E-mail address: karembelf@gmail.com

and

Abdessamad Lakhdi,
University Moulay Ismail of Meknes, FST Errachidia, Morocco.
E-mail address: lakhdi.abdessamad@gmail.com

and

Abdellah ZEROUALI,
Regional Centre of Trades Education and Training, Oujda, Morocco.
E-mail address: abdellahzerouali@yahoo.fr