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An L^p-Version of Hardy's Theorem for the q-Weinstein Fourier Transform

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ABSTRACT: This paper uses some basic notions and results related to the q-Weinstein harmonic analysis to study some theorems of the uncertainty principle. More precisely, we give a q-version of Hardy's theorem for the q-Weinstein Fourier transform.

Key Words: q-Weinstein operator, q-Weinstein Fourier transform, Hardy's theorem, uncertainty principle.

Contents

1. Introduction

The classical uncertainty principle asserts that a non-zero integrable function f and its Fourier transform $\widehat{f}$, defined for $\lambda \in \mathbb{R}$ by

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\lambda t} dt,$$

cannot both be sharply localized.

In the language of quantum mechanics, this principle says that an observer cannot simultaneously and precisely determine the values of position and momentum of a quantum particle. A mathematical formulation of these physical ideas was first developed by Heisenberg [16] in 1927. Later, precisely in 1933, Hardy [15] has obtained a theorem concerning the decay of a measurable function f on $\mathbb{R}$ and its Fourier transform $\widehat{f}$ at infinity. This theorem yields that if a measurable function f on $\mathbb{R}$ satisfies

$$|f(x)| \leq Ce^{-ax^2} \quad \text{and} \quad |\widehat{f}(y)| \leq Ce^{-by^2},$$

for some constants $a, b > 0$, $C > 0$ and $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

Several generalizations of Hardy's theorem have appeared since then, most notably among them being the following result of Cowling-Price (cf. [3]) which consists in replacing L^∞ estimates by L^p estimates.

Theorem 1.1 *Let $1 \leq p, q \leq \infty$ be such that $\min(p, q) < \infty$, $a, b \in \mathbb{R}_*^+$ and f a measurable function on $\mathbb{R}$ such that:*

(A) $\|e^{ax^2} f\|_{L^p(\mathbb{R})} < +\infty$.

(B) $\|e^{by^2} \widehat{f}\|_{L^q(\mathbb{R})} < +\infty$.

If $ab \geq \frac{1}{4}$, then $f = 0$ almost everywhere. If $ab < \frac{1}{4}$, then there are infinitely many linearly independent functions satisfying (A) and (B).

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Considerable attention has been devoted to discovering generalizations to new contexts for Theorem 1.1. This theorem has been proved for some types of Lie groups [25]. We emphasize that this theorem have been generalized in [9] for the Cartan motion group. Generalizations of this result to the Heisenberg group and the motion group have been proved in [10,24]. Recently, it has also been extended in [18] for the Dunkl transform in the space $L^p(\mathbb{R}^d, w_k(x)dx)$, where w_k is a weight function invariant under the action of an associated reflection group. An extension of these theorem using different differential operators has been given, where considering generalized Fourier transforms: Bessel-Struve transform [14], Jacobi–Dunkl transform [4], q-Dunkl transform [12] and Dunkl-Bessel transform [21].

In recent years, the q-theory, called also in some literature "quantum calculus", began to arise. Interest in this theory is grown at an explosive rate by both physicists and mathematicians due to a large number of its application domains. For instance, a lot of work has been carried out while developing some q-analogues of Fourier analysis using elements of quantum calculus (see [1,5,7,8,23] and references therein).

In [1], Bettaibi and Ben Mohamed introduced and studied a q-analogue of the classical Weinstein transform. In particular they provided, for this transform, a Plancheral formula and proved an inversion theorem. In this paper, we shall prove an L^p version of the Hardy theorem for the q-Weinstein Fourier transform, which is similar to that of Cowling and Price.

The outline of this paper is arranged as follows.

In Section 2, we state some basic notions and results from the q-harmonic analysis related to q-Weinstein Fourier transform that will be needed throughout this paper.

In Section 3, we prove a new estimate of the q-Weinstein kernel in terms of the classical exponential and we study an analogue of Theorem 1.1 associated with q-Weinstein Fourier transform.

2. Preliminaries

The harmonic analysis associated with the q-analogue of Weinstein operator is studied by Bettaibi and Ben Mohamed in [1]. In this section, we collect some results related to the q-Weinstein operator and related q-harmonic analysis. The references [1,2] are devoted to the q-Weinstein Fourier analysis.

2.1. Background and q-symbols

We recall some usual notions and notations used in the q-theory (see [19] and [20]). We refer to the book by G. Gasper and M. Rahman [19] for the definitions, notations and properties of the q-shifted factorials. Throughout this paper, we assume $0 < q < 1$, $\alpha \geq -1/2$ and we denote

- $\mathbb{R}_q = \{\pm q^n,\, n \in \mathbb{Z}\}$ and $\mathbb{R}_q^+ = \{q^n,\, n \in \mathbb{Z}\}$.
- $\mathbb{R}^2_{q,+} = \mathbb{R}_q \times \mathbb{R}_{q,+}$.
- $x = (x_1, x_2) \in \mathbb{R}^2_{q,+}$.
- $-x = (-x_1, x_2)$.
- $\|x\| = \sqrt{x_1^2 + x_2^2}$.

For complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1, \;\; (a;q)_n = \prod_{l=0}^{n-1}(1 - aq^l), \; n = 1, 2, ..., \;\; (a;q)_\infty = \prod_{l=0}^{\infty}(1 - aq^l). \tag{2.1}$$

We also denote for all $x \in \mathbb{C}$ and $n \in \mathbb{N}$

$$[x]_q = \frac{1 - q^x}{1 - q}, \qquad [n]_q! = [1]_q \times [2]_q ... \times [n]_q = \frac{(q;q)_n}{(1-q)^n}.$$

The q-Gamma function is given by (see [13])

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ... \tag{2.2}$$

It satisfies the following relations

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x) = [x]_q!, \quad \Gamma_q(1) = 1 \text{ and } \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x), \ \mathrm{Re}(x) > 0.$$

The q-Jackson integrals from 0 to a, from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined by (see [13])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{+\infty} q^n f(aq^n),$$

$$\int_0^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n f(q^n),$$

$$\int_{-\infty}^{+\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{+\infty} q^n\left[f(q^n) + f(-q^n)\right],$$

provided the sums converge absolutely. In particular, for $a \in \mathbb{R}_q^+$,

$$\int_a^{+\infty} f(x)d_qx = (1-q)a\sum_{n=-\infty}^{-1} q^n f(aq^n).$$

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [13])

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

The Rubin's q^2-differential operator is defined in [22,23] by

$$\partial_q f(x) = \begin{cases} \dfrac{f(q^{-1}x) + f(-q^{-1}x) - f(qx) + f(-qx) - 2f(-x)}{2(1-q)x} & \text{if } x \neq 0, \\ \lim\limits_{x\to 0}\partial_q f(x), \qquad (\text{in } \mathbb{R}_q) & \text{if } x = 0. \end{cases}$$

Remark that if f is differentiable at x, then $\partial_q f(x)$ tends to $f^{'}(x)$ as q tends to 1^-.
A repeated application of the Rubin's q^2-differential operator n times is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

For $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, we use the notation

$$\mathcal{D}_q^\beta = \partial_{x_1,q}^{\beta_1}\partial_{x_2,q}^{\beta_2}.$$

The q-analogue of the Laplace operator or the q-Laplacian operator is given by

$$\Delta_q = \partial_{x_1,q}^2 + \partial_{x_2,q}^2.$$

The third q-Bessel function is defined as follows (see [1,17])

$$J_\alpha(x;q^2) = \frac{(q^{2\alpha+2};q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}.$$

$J_\alpha(\cdot;q^2)$ has the normalized form

$$j_\alpha(x;q^2) = \sum_{n=0}^{+\infty}(-1)^n\frac{\Gamma_{q^2}(\alpha+1)q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}\left(\frac{x}{1+q}\right)^{2n}. \tag{2.3}$$

The q-trigonometric functions q-cosine and q-sine are defined by (see [22,23])

$$\cos(x;q^2) = j_{-1/2}(x;q^2) \quad \text{and} \quad \sin(x;q^2) = xj_{1/2}(x;q^2), \tag{2.4}$$

and the q-analogue exponential function is given by

$$e(x;q^2) = \cos(-ix;q^2) + i\sin(-ix;q^2). \tag{2.5}$$

These three functions are absolutely convergent for all x in the plane and when q tends to 1^- they tend to the corresponding classical ones pointwise and uniformly on compacts. Note that we have for all $x \in \mathbb{R}_q$

$$|j_\alpha(x;q^2)| \leq \frac{1}{(q;q)_\infty} \quad \text{and} \quad |e(ix;q^2)| \leq \frac{2}{(q;q)_\infty}.$$

By the use of the q^2-analogue differential operator ∂_q, we note:

- $\mathcal{S}_q(\mathbb{R}^2_{q,+})$, the space of functions f defined on $\mathbb{R}^2_{q,+}$ satisfying

$$P_{n,q}(f) = \sup_{x\in\mathbb{R}^2_{q,+}} \sup_{|\beta|\leq n} |\mathcal{D}^\beta_q(\|x\|^{2n}f(x))| < +\infty, \quad \forall n \in \mathbb{N},$$

  and

$$\lim_{x\to(0,0)} \mathcal{D}^\beta_q f(x) \quad (\text{in } \mathbb{R}^2_{q,+}) \quad \text{exists.}$$

- $\mathcal{S}_{*,q}(\mathbb{R}^2_{q,+})$, the space of functions in $\mathcal{S}_q(\mathbb{R}^2_{q,+})$, even with respect to the last variable.
- $\mathcal{D}_{*,q}(\mathbb{R}^2_{q,+})$, the space of infinity q-differentiable functions on $\mathbb{R}^2_{q,+}$ with compact supports, even with respect to the last variable.
- $L^p_{q,\alpha}(\mathbb{R}^2_{q,+})$, $p \in [1,+\infty]$, the Lebesgue space constituted of measurable functions on $\mathbb{R}^2_{q,+}$ such that

$$\|f\|_{\mu_{q,\alpha},p} = \begin{cases} \left(\int_{\mathbb{R}^2_{q,+}} |f(x)|^p d\mu_{q,\alpha}(x)\right)^{1/p} < \infty & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^2_{q,+}} |f(x)| < \infty & \text{if } \ p = \infty. \end{cases}$$

  where $d\mu_{q,\alpha}$ is the measure on $\mathbb{R}^2_{q,+}$ given by

$$d\mu_{q,\alpha}(x) = \frac{(1+q)^{\frac{1}{2}-\alpha}}{2\Gamma_{q^2}(1/2)\Gamma_{q^2}(\alpha+1)} x_2^{2\alpha+1} d_qx_2 d_qx_1. \tag{2.6}$$

2.2. q-Weinstein operator and q-Weinstein transform

In [1], a q-analogue of the Weinstein operator and its associated Fourier transform are introduced and investigated. In this Subsection, we collect some of their basic properties.

The q-Weinstein operator is given by

$$\Delta_{q,\alpha} = \partial^2_{q,x_1} + \frac{1}{|x_2|^{2\alpha+1}} \partial_{q,x_2}(|x_2|^{2\alpha+1}\partial_{q,x_2}) = \partial^2_{q,x_1} + \mathcal{B}_{q,\alpha}, \quad \alpha \geq -1/2,$$

where $\mathcal{B}_{q,\alpha}$ is the q-Bessel operator defined in [8,6].

It satisfies the following relations:

- For $\alpha = -1/2$, $\Delta_{q,\alpha}$ reduces to the q-Laplacian operator Δ_q.
- $\Delta_{q,\alpha}$ lives $\mathcal{S}_{*,q}(\mathbb{R}^2_{q,+})$ invariant.

- For all $f,g\in\mathcal{S}_{*,q}(\mathbb{R}^2_{q,+})$, $\Delta_{q,\alpha}$ is self-adjoint, that is

$$\int_{\mathbb{R}^2_{q,+}}\Delta_{q,\alpha}f(x)g(x)d\mu_{q,\alpha}(x)=\int_{\mathbb{R}^2_{q,+}}f(x)\Delta_{q,\alpha}g(x)d\mu_{q,\alpha}(x),$$

if the integrals exist.

It was shown in [1] that for all $\lambda=(\lambda_1,\lambda_2)\in\mathbb{C}^2$, the function

$$x\mapsto\Lambda_q^\alpha(\lambda,x)=e(-i\lambda_1x_1;q^2)j_\alpha(\lambda_2x_2;q^2),\quad\forall x\in\mathbb{R}^2_{q,+},\tag{2.7}$$

is the analytic solution, even with respect to the second variable of the q-differential-difference equation:

$$\begin{cases}\mathcal{B}_{q,\alpha}\varphi(x)=-\lambda_2^2\varphi(x),\\ \partial_{q,x_1}\varphi(x)=-\lambda_1^2\varphi(x),\end{cases}$$

with the following initial conditions

$$\varphi(0,0)=0,\ \ \partial_{q,x_1}\varphi(0,0)=-i\lambda_1,\ \ \partial_{q,x_2}\varphi(0,0)=0.$$

The function $\Lambda_q^\alpha(\lambda,\cdot)$ called q-Weinstein kernel has a unique extension to $\mathbb{C}^2$. In the following result, we summarise some of its properties:

Proposition 2.1 *The following properties are checked:*

(i) For all $\lambda,x\in\mathbb{R}^2_{q,+}$ and $a\in\mathbb{C}$, we have

$$\Lambda_q^\alpha(\lambda,x)=\Lambda_q^\alpha(x,\lambda),\ \ \Lambda_q^\alpha(a\lambda,x)=\Lambda_q^\alpha(\lambda,ax),\ \ \overline{\Lambda_q^\alpha(\lambda,x)}=\Lambda_q^\alpha(-\lambda,x).$$

(ii) For all $\lambda\in\mathbb{R}^2_{q,+}$, $\Lambda_q^\alpha(\lambda,\cdot)$ is bounded on $\mathbb{R}^2_{q,+}$ and we have

$$\Lambda_{q,\alpha}(x,\lambda)\le\frac{4}{(q,q)^2_\infty}.$$

(iii) For $\alpha>-1/2$, the function $\Lambda_q^\alpha(\lambda,\cdot)$ has also the following q-integral representation of Mehler type:

$$\Lambda_q^\alpha(\lambda,x)=\frac{(1+q)\Gamma_{q^2}(\alpha+1)}{2\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})}e(i\lambda_1x_1;q^2)\int_{-1}^1W_\alpha(t,q^2)e(-i\lambda_2x_2;q^2)d_qt.$$

Proof: See [1, Proposition 3.3]. □

Definition 2.1 [1] *The q-Weinstein Fourier transform $\mathcal{F}_W^{q,\alpha}$ associated with the q-Weinstein operator $\Delta_{q,\alpha}$ is defined for every function f in $L^1_{q,\alpha}(\mathbb{R}^2_{q,+})$ by*

$$\mathcal{F}_W^{q,\alpha}(f)(\lambda)=\int_{\mathbb{R}^2_{q,+}}f(x)\Lambda_q^\alpha(\lambda,x)d\mu_{q,\alpha}(x),\tag{2.8}$$

for all $\lambda\in\mathbb{R}^2_{q,+}$.

Remark 2.1 *Letting $q\to1$ subject to the condition*

$$\frac{\mathrm{Log}(1-q)}{\mathrm{Log}(q)}\in2\mathbb{Z},$$

gives, at least formally, the classical Weinstein transform on $\mathbb{R}^2_+=\mathbb{R}\times\mathbb{R}^+$.

The q-Weinstein transform satisfies the following properties:

- *$L^1 - L^\infty$-boundedness:*
  For all $f \in L^1_{q,\alpha}(\mathbb{R}^2_{q,+})$, we have $\mathcal{F}^{q,\alpha}_W(f) \in L^\infty_{q,\alpha}(\mathbb{R}^2_{q,+})$ and we get

$$\|\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},\infty} \leq \frac{4}{(q,q)^2_\infty}\|f\|_{\mu_{q,\alpha},1}. \tag{2.9}$$

- *Riemann-Lebesque Lemma:*
  If $f \in L^1_{q,\alpha}(\mathbb{R}^2_{q,+})$, then

$$\lim_{\substack{\|\lambda\|\to\infty \\ \lambda\in\mathbb{R}^2_{q,+}}} \mathcal{F}^{q,\alpha}_W(f)(\lambda) = 0.$$

- *q-Plancherel formula:*
  The q-Weinstein transform $\mathcal{F}^{q,\alpha}_W$ is an isomorphism from $\mathcal{S}_{*,q}(\mathbb{R}^2_{q,+})$ onto itself and extends uniquely to an isometric isomorphism on $L^2_{q,\alpha}(\mathbb{R}^2_{q,+})$ with:

$$\|\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},2} = \|f\|_{\mu_{q,\alpha},2}. \tag{2.10}$$

- *q-Inversion formula:*
  If $f \in L^1_{q,\alpha}(\mathbb{R}^2_{q,+})$ such that $\mathcal{F}^{q,\alpha}_W(f) \in L^1_{q,\alpha}(\mathbb{R}^2_{q,+})$, then the q-inversion formula holds and we have

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^2_{q,+}} \mathcal{F}^{q,\alpha}_W(f)(\lambda)\Lambda^\alpha_q(\lambda,x)d\mu_{q,\alpha}(\lambda) \\ &= \overline{\mathcal{F}^{q,\alpha}_W(\overline{\mathcal{F}^{q,\alpha}_W}(f))(x)}. \end{aligned} \tag{2.11}$$

Proposition 2.2 *(q-hausdorff inequality) If $f \in L^p_{q,\alpha}(\mathbb{R}^2_{q,+})$, with $p \in [1,+\infty]$, then $\mathcal{F}^{q,\alpha}_D(f) \in L^{p'}_{q,\alpha}(\mathbb{R}_q)$. Moreover, if $1 \leq p \leq 2$, hence we have*

$$\|\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},p'} \leq \left(\frac{4}{(q,q)^2_\infty}\right)^{\frac{2}{p}-1}\|f\|_{\mu_{q,\alpha},p}, \tag{2.12}$$

where the numbers p and p' above are conjugate exponents:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: This is an immediate consequence of (2.9), q-Plancherel formula (2.10), q-inversion formula (2.11) and the Riesz-Thorin Theorem. □

3. An L^p Version of Hardy's Theorem in q-Weinstein Harmonic Analysis

In this Section, we shall state an L^p-version of Hardy's theorem for the q-Weinstein transform. We first begin with auxiliary results.

Lemma 3.1 *For all $\alpha \geq -\frac{1}{2}$ and $z \in \mathbb{C}$. There exists a constant $A_{q,\alpha}$ such that*

$$|j_\alpha(z;q^2)| \leq A_{q,\alpha}e^{|z|}. \tag{3.1}$$

Proof: It follows from (2.3) that

$$|j_\alpha(z;q^2)| \leq \Gamma_{q^2}(\alpha+1)\sum_{n=0}^{+\infty}\frac{(1+q)^{-2n}q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}|z|^{2n}.$$

Now, by relations (2.1) and (2.2), we get

$$\frac{\Gamma_{q^2}(\alpha+1)(1+q)^{-2n}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} = \frac{(1-q)^{2n}}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_\infty}.$$

Thus,

$$|j_\alpha(z;q^2)| \leq \frac{1}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_\infty} \sum_{n=0}^{+\infty} q^{n(n+1)} |z|^{2n}, \tag{3.2}$$

by virtue of $0 < (1-q)^{2n} < 1$. Now by using the Stirling formula

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n},$$

we have immediately

$$(2n)! q^{n(n+1)} \to 0 \quad \text{as} \quad n \to +\infty.$$

As a consequence, there is an integer $N > 0$ such that

$$q^{n(n+1)} \leq \frac{1}{(2n)!}, \quad \forall n \geq N.$$

From this and (3.2), we obtain

$$\Gamma_{q^2}(\alpha+1) \sum_{n=N}^{+\infty} \frac{(1+q)^{-2n} q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} |z|^{2n} \leq \frac{1}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_\infty} e^{|z|}.$$

On the other hand, by the continuity of the exponential function, there exists a constant C such that

$$\Gamma_{q^2}(\alpha+1) \sum_{n=0}^{N-1} \frac{(1+q)^{-2n} q^{n(n+1)}}{\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)} |z|^{2n} \leq C e^{|z|}.$$

By combining the two inequalities above, we get the desired result. □

Lemma 3.2 *For all $z \in \mathbb{C}$, the following inequality*

$$|e(iz;q^2)| \leq \frac{1}{(q;q)_\infty} e^{2|z|} \tag{3.3}$$

holds.

Proof: In view of relations (2.4) and (2.5), we get

$$|e(iz;q^2)| \leq |j_{-1/2}(z;q^2)| + |z||j_{1/2}(z;q^2)|.$$

Now, by using Lemma 3.1, we obtain

$$|e(iz;q^2)| \leq A_{q,-1/2} e^{|z|} + A_{q,1/2} |z| e^{|z|},$$

where

$$A_{q,-1/2} = \frac{1}{(q;q^2)_\infty (q^2;q^2)_\infty} \quad \text{and} \quad A_{q,1/2} = \frac{1}{(q^3;q^2)_\infty (q^2;q^2)_\infty}.$$

Since,

$$(q;q)_\infty = (q;q^2)_\infty (q^2;q^2)_\infty,$$
$$(q;q)_\infty = (1-q)(q^3;q^2)_\infty (q^2;q^2)_\infty.$$

Then, we have

$$\begin{aligned} |e(iz;q^2)| &\le \frac{1}{(q;q)_\infty} e^{|z|}(1+|z|) \\ &\le \frac{1}{(q;q)_\infty} e^{2|z|}, \end{aligned}$$

by virtue of $e^{|z|} \ge |z|+1$. This completes the proof. □

Corollary 3.1 *Let $\alpha \ge -1/2$, $z=(z_1,z_2)\in\mathbb{C}^2$ and $x\in\mathbb{R}^2_{q,+}$. Then we have the following inequality*

$$|\Lambda_{q,\alpha}(x,z)| \le B_{q,\alpha} e^{2|z_1x_1|+|z_2x_2|}, \tag{3.4}$$

where

$$B_{q,\alpha} = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty (q^{2\alpha+2};q^2)_\infty}.$$

In the rest of this paper, we need the following lemma of Phragmén-Lindelöf type that we get using the same technique as in [18]. We mention only one difference that we work here in the case where $d=2$.

Lemma 3.3 *Let $p\in[1,+\infty]$ and h be an entire function on $\mathbb{C}^2$ such that*

(1) $\forall z\in\mathbb{C}^2, \quad |h(z)| \le M e^{a\left[(\operatorname{Re} z_1)^2+(\operatorname{Re} z_2)^2\right]}.$

(2) $\forall x\in\mathbb{R}^2_{q,+}, \quad \|h\|_{\mu_{q,\alpha},p} < +\infty,$

for some positive constants a and M. Then $h=0$. Moreover, if $p=\infty$, then h is constant on $\mathbb{C}^2$.

Lemma 3.4 *Let $p\in[1,+\infty]$ and let f be a measurable function on $\mathbb{R}^2_{q,+}$ such that $\|e^{a\|x\|^2}f(x)\|_{\mu_{q,\alpha},p} < +\infty$, for some $a>0$. Then the function defined on $\mathbb{C}^2$ by*

$$\mathcal{F}_W^{q,\alpha}(f)(z) = \int_{\mathbb{R}^2_{q,+}} f(x)\Lambda_q^\alpha(z,x)d\mu_{q,\alpha}(x), \tag{3.5}$$

is well defined and entire on $\mathbb{C}^2$. Moreover, for all $b\in]0,a[$, there exists a positive constant C such that

$$\forall z\in\mathbb{C}^2, \quad |\mathcal{F}_W^{q,\alpha}(f)(z)| \le C e^{\frac{\|z\|^2}{4b}}, \tag{3.6}$$

Proof: From (3.4), Hölder's inequality and the analyticity theorem under the integral sign, we deduce that the function defined on $\mathbb{C}^2$ by (3.5) is well defined and entire on $\mathbb{C}^2$.

We will now prove (3.6). It follows from (3.4) and Hölder's inequality that

$$\begin{aligned} &|\mathcal{F}_W^{q,\alpha}(f)(z)| \le \int_{\mathbb{R}^2_{q,+}} |f(x)||\Lambda_q^\alpha(z,x)|d\mu_{q,\alpha}(x) \\ &\le B_{q,\alpha}\int_{\mathbb{R}^2_{q,+}} |f(x)|e^{2|z_1x_1|+|z_2x_2|}d\mu_{q,\alpha}(x) \\ &\le B_{q,\alpha}\int_{\mathbb{R}^2_{q,+}} e^{a\|x\|^2}|f(x)|e^{2|z_1x_1|+|z_2x_2|-a(x_1^2+x_2^2)}d\mu_{q,\alpha}(x) \\ &\le K_{q,\alpha}\|e^{a\|x\|^2}f(x)\|_{\mu_{q,\alpha},p}\left(\int_{-\infty}^{+\infty} e^{p'(2|z_1x_1|-ax_1^2)}\left(\int_0^{+\infty} e^{p'(|z_2x_2|-ax_2^2)}x_2^{2\alpha+1}d_qx_2\right)d_qx_1\right)^{1/p'}, \end{aligned}$$

where $K_{q,\alpha}$ is a positive constant and p and p' are conjugate numbers.
Now, for $b \in]0, a[$, we have

$$\begin{aligned}
&\int_0^{+\infty} e^{p'(|z_2 x_2| - a x_2^2)} x_2^{2\alpha+1} d_q x_2 \\
&= \int_0^{+\infty} e^{p'(|z_2| x_2 - b x_2^2)} e^{-p'(a-b)x_2^2} x_2^{2\alpha+1} d_q x_2 \\
&\leq \sup_{x \in \mathbb{R}_q^+} \left(e^{p'(|z_2| x_2 - b x_2^2)} \right) \int_0^{+\infty} e^{-p'(a-b)x_2^2} x_2^{2\alpha+1} d_q x_2 \\
&= C_1 e^{\frac{p'|z_2|^2}{4b}}.
\end{aligned}$$

with

$$C_1 = \int_0^{+\infty} e^{-p'(a-b)x_2^2} x_2^{2\alpha+1} d_q x_2.$$

We proceed with the same technique as above, we find that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{p'(2|z_1 x_1| - a x_1^2)} d_q x_1 \\
&= \int_{-\infty}^{+\infty} e^{p'(2|z_1 x_1| - b x_1^2)} e^{-p'(a-b)x_1^2} d_q x_1 \\
&\leq \sup_{x \in \mathbb{R}_q} \left(e^{p'(2|z_1 x_1| - b x_1^2)} \right) \int_{-\infty}^{+\infty} e^{-p'(a-b)x_1^2} d_q x_1 \\
&= C_2 e^{\frac{p'|z_1|^2}{b}}
\end{aligned}$$

with

$$C_2 = \int_{-\infty}^{+\infty} e^{-p'(a-b)x_1^1} d_q x_1.$$

Hence, we have

$$|\mathcal{F}_W^{q,\alpha}(f)(z)| \leq C e^{\frac{|z_1|^2}{b}} e^{\frac{|z_2|^2}{4b}} \leq C e^{\frac{\|z\|^2}{4b}}.$$

This completes the proof. □

At this point, we are in a position to prove the main result of the paper.

Theorem 3.1 *Let f be a measurable function on $\mathbb{R}^2_{q,+}$ such that*

$$\|e^{a\|x\|^2} f\|_{\mu_{q,\alpha},p} < +\infty, \tag{3.7}$$

and

$$\|e^{b\|y\|^2} \mathcal{F}_W^{q,\alpha}(f)\|_{\mu_{q,\alpha},p'} < +\infty, \tag{3.8}$$

almost everywhere for $x, y \in \mathbb{R}^2_{q,+}$, for some constants $a > 0$, $b > 0$, $1 \leq p, p' \leq +\infty$ and at least one of p and p' is finite. If $ab > \frac{1}{4}$, then f= 0 almost everywhere.

Proof: Let $a, b > 0$ satisfying the conditions of the theorem such that $ab > \frac{1}{4}$ and we take $a_1 \in]\frac{1}{4}, a[$.

Consider the function h defined on $\mathbb{C}^2$ by

$$h(z) = \left(\prod_{j=1}^{2} e^{\frac{z_j^2}{4a_1}} \right) \mathcal{F}_W^{q,\alpha}(f)(z). \tag{3.9}$$

This function is entire on $\mathbb{C}^2$ and using (3.6), we obtain

$$|h(z)| \leq Ce^{\frac{1}{2a_1}\left[(\operatorname{Re} z_1)^2+(\operatorname{Re} z_2)^2\right]}. \tag{3.10}$$

In the following, we distinguish two cases for the number p'.

First case: If $p' < +\infty$, we have

$$\begin{aligned} \|h\|^{p'}_{\mu_{q,\alpha},p'} &= \int_{\mathbb{R}^2_{q,+}} |e^{\|x\|^2/4a_1}\mathcal{F}^{q,\alpha}_W(f)(x)|^{p'} d\mu_{q,\alpha}(x) \\ &= \int_{\mathbb{R}^2_{q,+}} |e^{b\|x\|^2}\mathcal{F}^{q,\alpha}_W(f)(x)|^{p'} e^{p'((1/4a_1)-b)\|x\|^2} d\mu_{q,\alpha}(x). \end{aligned}$$

Using the fact that $a_1 b > \frac{1}{4}$ and the hypothesis (3.8), we obtain

$$\|h\|_{\mu_{q,\alpha},p'} \leq \|e^{b\|x\|^2}\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},p'} < +\infty. \tag{3.11}$$

From relations (3.10) and (3.11), it follows from Lemma 3.3 that $h(z) = 0$ for all $z \in \mathbb{C}^2$. Thus $\mathcal{F}^{q,\alpha}_W(f)(x) = 0$ for all $x \in \mathbb{R}^2_{q,+}$. The injectivity of $\mathcal{F}^{q,\alpha}_W$ then implies the result of the theorem in this case.

Second case: If $p' = +\infty$, we have

$$\begin{aligned} \|h\|_{\mu_{q,\alpha},\infty} &= \|e^{\|x\|^2/4a_1}\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},\infty} \\ &\leq \|e^{b\|x\|^2}\mathcal{F}^{q,\alpha}_W(f)\|_{\mu_{q,\alpha},\infty} < +\infty, \end{aligned}$$

by virtue of $a_1 b > \frac{1}{4}$.

Using relations (3.10), (3.12) and Lemma 3.3, there exists a positive constant K such that for all $x \in \mathbb{R}^2_{q,+}$, $h(x) = K$. On the other hand, from (3.9), we have

$$\mathcal{F}^{q,\alpha}_W(f)(x) = Ke^{-\|x\|^2/4a_1}, \quad \forall x \in \mathbb{R}^2_{q,+}. \tag{3.12}$$

But the assumption on $\mathcal{F}^{q,\alpha}_W(f)$ is expressed as

$$|\mathcal{F}^{q,\alpha}_W(f)(x)| \leq Me^{-b\|x\|^2} \quad \text{a.e} \quad x \in \mathbb{R}^2_{q,+}, \tag{3.13}$$

for some constant $M > 0$. The continuity of $\mathcal{F}^{q,\alpha}_W(f)$ on $\mathbb{R}^2_{q,+}$ shows that the inequality (3.13) holds everywhere. Then we must have

$$Ke^{(b-(1/4a_1))\|x\|^2} \leq M,$$

everywhere by (3.12) and (3.13). This is impossible since $a_1 b > 1/4$, unless $K = 0$. Thus $\mathcal{F}^{q,\alpha}_W(f)(x) = 0$ everywhere and then $f = 0$ almost everywhere on $\mathbb{R}^2_{q,+}$. □

In the following corollary, we determine the functions f satisfying (3.7) and (3.8) in the particular case $p = p' = +\infty$. The result we obtain is an analogue for the q-Weinstein transform of the Hardy's classical theorem.

Corollary 3.2 *Let f be a measurable function on $\mathbb{R}^2_{q,+}$ such that*

$$|f(x)| \leq Me^{-a\|x\|^2} \quad and \quad |\mathcal{F}_W^{q,\alpha}(f)(\lambda)| \leq Me^{-b\|\lambda\|^2}, \tag{3.14}$$

almost everywhere for $x, \lambda \in \mathbb{R}^2_{q,+}$ and for some constants $a > 0$, $b > 0$ and $M > 0$. If $ab > \frac{1}{4}$, then $f = 0$ almost everywhere.

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Conflict of Interests

The authors declares that they have no conflict of interest.

Data Availability Statement

The datasets generated during the current study are available from the corresponding author on reasonable request.

References

1. Bettaibi, Y., Ben Mohamed, H., *Generalized Weinstein transform in quantum calculus*, MathLAB J. 3, 50-65 (2019).
2. Bettaibi, Y., Ben Mohamed, H., *Real Paley-Wiener Theorem for the Generalized Weinstein transform in quantum calculus*, arXiv:2005.00489v1 (2020).
3. Cowling, M. G., Price, J. F., *Generalizations of Heisenberg inequality*, Lecture Notes in Mathematics, 992, Springer, Berlin, 443–449, (1983).
4. Daher, R., Hamad, S. L., *L^p–L^q-version of Morgan's theorem for the Jacobi–Dunkl transform*, Integr. Trans. Spec. Funct. 19(3), 165-169, (2008).
5. Daher, R., Tyr, O., *An analog of Titchmarsh's theorem for the q-Dunkl transform in the space $L^2_{q,\alpha}(\mathbb{R}_q)$*. J. Pseudo-Differ. Oper. Appl. 11, 1933-1949, (2020).
6. Dhaouadi, L., Fitouhi, A., El Kamel, J., *Inequalities in q-Fourier analysis*. J. Inequal. Pure Appl. Math. 7(5), Article 171, (2006).
7. Dhaouadi, L., *On the q-Bessel Fourier transform*, Bull. Math. Anal. Appl., 5(2), 42-60, (2013).
8. Dhaouadi, L., *Positivity of the generalized translation associated with the q-Hankel transform and applications*, Integr. Trans. Spec. Funct. 26(2), 102–117, (2015).
9. Eguchi, M., Koizumi, S., Kumahara, K., *An analogue of the Hardy theorem for the Cartan motion group*, Proc. Japan Acad. 74, 149-151, (1998).
10. Eguchi, M., Koizumi, S., Kumahara, K., *An L^p version of the Hardy theorem for the motion group*, J. Austral. Math. Soc. (Series A) 68, 55-67, (2000).
11. Fitouhi, A., Hamza, M., Bouzeffour, F., *The q-j_α-Bessel function*, J. Approx. Theory. 115, 144–166, (2002).
12. Fitouhi, A., Nouri, F., Safraoui, A., *An L^p Version of Hardy Uncertainty Principle for the q-Dunkl Transform on the Real Line*, Int. Journal of Math. Analysis, 4(5), 249-257, (2010).
13. Jackson, F. H., *On a q-Definite Integrals*, Journal of Pure and Applied Mathematics 41, 193-203, (1910).
14. Hamem, S., Kamoun, L., Negzaoui, S., *Cowling-Price type theorem related to Bessel-Struve transform*. Arab. J. Math. Sci. 19(2), 187–198, (2013).
15. Hardy, G. H., *A theorem concerning Fourier transforms*, J. Lond. Math. Soc. 8 (1933).
16. Heisenberg, W., *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Z. f. Physik. 43, 172–198, (1927).
17. Ismail, M. E. H., *The zeros of basic Bessel functions, the Function $J_{v+ax}(x)$, and associated orthogonal polynomials*. J. Math. Anal. Appl. 86, 1-19, (1982).
18. Gallardo, L., Trimèche, K., *An L^p version of Hardy's theorem for the Dunkl transform*. J. Aust. Math. Soc. 77, 371–385, (2004).
19. Gasper, G., Rahman, M., *Basic Hypergeometric Series, Encyclopedia of Mathematics and its Applications*, 35. Cambridge Univ. Press, Cambridge (1990).

20. Kac, V. G., Cheung, P., *Quantum Calculus*, Universitext, Springer-Verlag, NewYork (2002).

21. Mejjaoli, H., Trimèche, K.: An analogue of Hardy's theorem and its L^p version for the Dunkl–Bessel transform, J. Concr. Appl. Math. **2**(4), 397–417 (2004)

22. Rubin, R. L., Duhamel, *Solutions of non-Homogenous q^2-Analogue Wave Equations*. Proc. of Amer. Math. Soc. 135(3), 777-785, (2007).

23. Rubin, R. L., *A q^2-Analogue Operator for q^2-analogue Fourier Analysis*, J. Math. analys. App. 212, 571-582, (1997).

24. Sitaram, A., Sundari, M., Thangavalu, S., *Uncertainty principles on certain Lie groups*, Proc. Indian Acad. Sci. (Math. Sci.) 105, 135-151, (1995).

25. Sundari, M., *Hardy's theorem for the n-dimensional Euclidean motion group*, Proc. Amer. Math. Soc. 126, 1199-1204, (1998).

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